1 Exponential and Logarithmic Equations and Inequalities

Exponential Equations
An exponential equation is an equation where the variable you want to solve for (typically x) is in an exponent. In this case, use the following procedure to solve:

1. Isolate the exponential term on one side
2. Take the logarithm of both sides
3. Use the rules of logarithms to simplify and solve

Example 4.4.1: Solve $5 \cdot 2^{x-3} = 17$
Solution: We follow the procedure above

$$\begin{aligned} 5 \cdot 2^{x-3} &= 17 \\ 2^{x-3} &= \frac{17}{5} \\ \ln\left(2^{x-3}\right) &= \ln\left(\frac{17}{5}\right) \\ (x-3)\ln 2 &= \ln\left(\frac{17}{5}\right) \\ x &= \frac{\ln\left(\frac{17}{5}\right)}{\ln 2} + 3 \end{aligned}$$

Example 4.4.2: Solve $5^{2x-3} = 3^{x+1}$
Solution: In this example we have multiple exponential terms and so we simply start with step 2 as shown below

$$\begin{aligned} 5^{2x-3} &= 3^{x+1} \\ \ln\left(5^{2x-3}\right) &= \ln\left(3^{x+1}\right) \\ (2x-3)\ln 5 &= (x+1)\ln 3 \\ x\Bigg(2\ln 5 - \ln 3\Bigg) &= \ln 3 + 3\ln 5 \\ x &= \frac{\ln 3 + 3\ln 5}{2\ln 5 - \ln 3} \end{aligned}$$

Example 4.4.3: Solve $3^x - 8 \cdot 3^{-x} = 2$
Solution: This is another special case. If we multiply this equation by a factor 3^x, we have

$$3^{2x} - 2 \cdot 3^x - 8 = 0$$

Now let $y = 3^x$ to see that the above equation is just the quadratic equation $y^2 - 2y - 8 = 0$ which has solutions $y = -2, 4$. We now have to plug these values into $y = 3^x$. However, notice for $y = -2$, we have $-2 = 3^x$ which cannot be true because the exponential function is always positive. We do, however, find a solution for $y = 4$ which gives $x = \ln 4 / \ln 3 \approx 1.26$.

Example 4.4.4: Use the table which gives information from 2010 and the population model $P(t) = P_0(1+r)^t$ to answer the questions. Assume the growth rate stays constant throughout.

1. Estimate the population of each country in 2020.
2. In what year will the population of the USA be 350 million?
3. In what year will the USA and Pakistan have the same population?

Country	Population	Annual Growth Rate
USA	308 million	0.9 %
Pakistan	185 million	2.1 %

Solution: We start by using the values from the table to setup the following population functions for the countries:

$$U(t) = 308(1.009)^t, \qquad P(t) = 185(1.021)^t$$

(a)To estimate the populations in 2020, 10 years after 2010, we let $t = 10$ in the above population functions to find that

$$U(10) \approx 336.87 \text{ million people}, \qquad P(10) \approx 227.73 \text{ million people}$$

(b) To find the year at which this happens, we set $U(t) = 350$ and solve for t

$$\begin{aligned} 350 &= 308(1.009)^t \\ \frac{350}{308} &= 1.009^t \\ \ln\frac{350}{308} &= t\ln 1.009 \\ t &= \frac{\ln\frac{350}{308}}{\ln 1.009} \approx 14.27 \text{ years after 2010} \end{aligned}$$

(c) To find the year at which this happens, we need to set $U(t) = P(t)$ and solve for t

$$\begin{aligned} 308(1.009)^t &= 185(1.021)^t \\ \frac{308}{185} &= \left(\frac{1.021}{1.009}\right)^t \\ t &= \frac{\ln\left(\frac{308}{185}\right)}{\ln\left(\frac{1.021}{1.009}\right)} \approx 43.12 \text{ years after 2010} \end{aligned}$$

Logarithmic Equations

An logarithmic equation is an equation where the variable you want to solve for (typically x) is in a logarithm. In this case, use the following procedure to solve:

1. Isolate all logarithmic terms on one side of the equation and write them in condensed form
2. Convert equation from logarithmic form to exponential form
3. Solve for variable
4. Check for extraneous solutions
   - Recall extraneous solutions are solutions found using a particular method that do not actually solve the original equation.

Example 4.4.5: Solve $\log_2(x-3) + \log_2(x-4) = 1$.

Solution: Follow the procedure above

$$\begin{aligned} \log_2(x-3) + \log_2(x-4) &= 1 \\ \log_2(x-3)(x-4) &= 1 \\ (x-3)(x-4) &= 2 \\ x^2 - 7x + 10 &= 0 \\ (x-5)(x-2) &= 0 \\ x &= 2, 5 \end{aligned}$$

However, upon checking these solutions, we see that $x = 2$ isn't in the domain of the equation and therefore is an extraneous solution that we can throw out. The only solution therefore is $x = 5$.

Example 4.4.6: Solve $\log_2(x+4) - \log_2(x+3) = 1$.

Solution: Follow the procedure above

$$\begin{aligned} \log_2(x+4) - \log_2(x+3) &= 1 \\ \log_2\frac{x+4}{x+3} &= 1 \\ \frac{x+4}{x+3} &= 2 \\ x + 4 &= 2x + 6 \\ x &= -2 \end{aligned}$$

which is a true solution to the equation.

Logistic Growth Model
The logistic growth model is a famous biological model for a population with plenty of food, space to grow, and no threat from predators which grows at a rate that is proportional to the population. A population with this type of growth is said to have a carrying capacity, or a maximum value for the population. The model is given by

$$P(t) = \frac{M}{1 + ae^{-kt}} \tag{1}$$

where $P(t)$ is the population at time t, M is the carrying capacity of the population, k is the growth rate of the population, and a is a positive constant.

Example 4.4.7: Suppose the carrying capacity of the human population is 35 billion. In 1987, the world population was 5 billion. Use the logistic growth model to calculate the average rate of growth, k, given that the population was 6 billion in 2003.
Solution: From the information given, we can discern that $P(0) = 5$ and $P(16) = 6$. Plugging the point at $t = 0$ into the logistic growth model with $M = 35$ gives:

$$P(0) = 5 \to \frac{35}{1+a} = 5 \implies 1 + a = 7 \implies a = 6$$

Now use this value of a in the second point given:

$$P(16) = 6 \to \frac{35}{1 + 6e^{-16k}} = 6 \implies 1 + 6e^{-16k} = \frac{35}{6} \implies ... \implies k = -\frac{1}{16} \ln\left(\frac{29}{36}\right) \approx .0135 = 1.35\ \%$$

Inequalities
The procedure remains the same for solving. Remember to flip direction of inequality when multiplying or dividing by a negative number.

Example 4.4.8: Solve $5(0.7)^x + 3 < 18$
Solution: We can solve this by following our usual procedure:

$$\begin{aligned} 5(0.7)^x + 3 &< 18 \\ 5(0.7)^x &< 15 \\ 0.7^x &< 3 \\ x \ln 0.7 &< \ln 3 \\ x &> \frac{\ln 3}{\ln 0.7} \end{aligned}$$

where notice we had to flip the inequality in the last step because $\ln 0.7 < 0$ (in general $\log_a c < 0$ if $c < 1, a > 1$).