1 Chapter 5.1: Angles and Their Measure

Definitions

- A ray is a part of a line made up of an endpoint and all of the points on one side of the endpoint
- An angle is formed by rotating a ray about its endpoint
  - The angle's initial side is the ray's original position
  - The angle's terminal side is the ray's position after the rotation
- An angle is positive if the rotation is counterclockwise
- An angle is negative if the rotation is clockwise
- Two angles that have the same initial and terminal sides are coterminal angles
- An angle is in standard position if its vertex is at the origin and its initial side lies on the positive x-axis
- An angle in standard position is quadrantal if its terminal side lies on a coordinate axis

Measuring Angles in Degrees

- A common way to measure angles is in degrees, where 1/360 of a rotation counterclockwise is measured as one degree (1°)
  - An acute angle has a measure between (but not equal to) 0° and 90°
  - A right angle has a measure of 90°
  - An obtuse angle has a measure between (but not equal to) 90° and 180°
  - A straight angle has a measure of 180°
    - Note that there are angles that don't fit any of these categories.
- It is also common to further divide degrees into fractional parts using minutes and seconds which are defined by
  - One minute $1'$, is defined as $\frac{1}{60}(1^\circ)$
  - One second $1''$, is defined as $\frac{1}{60}(1') = \frac{1}{3600}(1^\circ)$

Example 5.1.1: Convert the following to decimal degree notation: $24^\circ 8' 15''$. Then convert the following to degree, minute, second notation: 67.526.
Solution: (a) We just need to write the minutes and seconds in terms of degrees using their definitions:

$$24^\circ 8' 15'' = 24^\circ + 8\left[\frac{1}{60}(1^\circ)\right] + 15\left[\frac{1}{3600}(1^\circ)\right] \approx 24.14^\circ$$

(b) Converting the other way, we need to systematically work our way down to the smallest unit:

$$\begin{aligned} 67.526 &= 67^\circ + 0.526^\circ \cdot \frac{60'}{1^\circ} \\ &= 67^\circ + 31.56' \\ &= 67^\circ + 31' + .56' \cdot \frac{60''}{1'} \\ &= 67^\circ + 31' + 33.6'' \approx 67^\circ 31' 34'' \end{aligned}$$

Measuring Angles in Radians

- Angles can also be measured in radians
- Think of angles as central angles inside a circle of radius r – then a positive central angle that intercepts an arc of the circle of length equal to the radius of the circle is said to have measure 1 radian
  - The circumference of a circle is known to be $2\pi r$ which by the above definition means one full rotation is 2π radians

- The radian measure θ of a central angle that intercepts an arc of length s on a circle of radius r is given by $\theta = s/r$ radians

Relationship Between Radians and Degrees
Note that one full rotation is 2π radians which is equal to $360°$, or one-half rotation is π radians which is equal to $180°$. We will use this fact to establish our conversion ratios

- If given $\theta°$, then to rewrite the angle in radians take

$$\theta = \theta° \cdot \frac{\pi}{180°} \text{ radians}$$

- If given θ radians, then to rewrite the angle in degrees take

$$\theta° = \theta \cdot \frac{180°}{\pi}$$

- Notice that in either case, you can "cancel" the units as if they were an algebraic variable.

Example 5.1.2: Convert the following angles between degrees and radians: $30°$, $-220°$, $(3\pi)/2$ radians, 1 radian
Solution: We make use of the conversion ratios above to calculate:

$$\begin{aligned} 30° &= 30° \cdot \frac{\pi}{180°} = \frac{\pi}{6} \text{ rad}, \\ -220° &= -220° \cdot \frac{\pi}{180°} = -\frac{11\pi}{9} \text{ rad}, \\ \frac{3\pi}{2} &= \frac{3\pi}{2} \cdot \frac{180°}{\pi} = 270°, \\ 1 &= 1 \cdot \frac{180°}{\pi} \approx 57.3° \end{aligned}$$

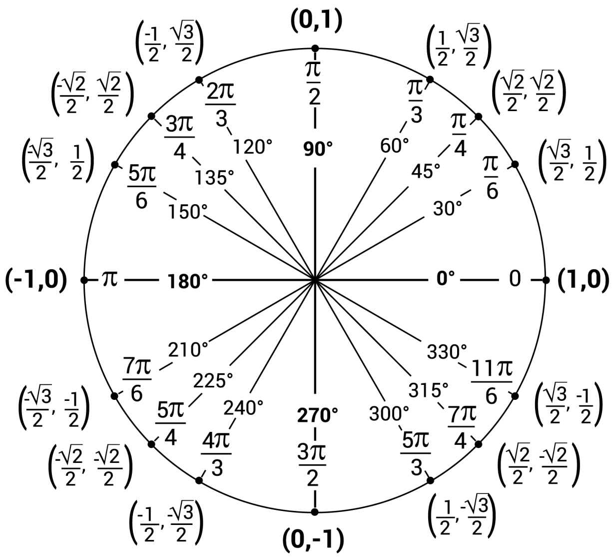

Figure 1: Shown is the unit circle. Identified are special angles, given in both degrees and radians, that give exact trigonometric values. The points on the unit circle associated with a given angle can be used to calculate the trigonometric values as $\cos\theta = x$ and $\sin\theta = y$ for an angle θ and point (x, y).

Coterminal Angles:
An angle θ has infinitely many coterminal angles given by:

$$\theta + 2\pi n, \qquad \text{for integer } n \text{ and } \theta \text{ in radians}, \tag{1}$$
$$\theta° + 360° n, \qquad \text{for integer } n \text{ and } \theta \text{ in degrees} \tag{2}$$

Example: 5.1.3: Find the angle between 0 and 2π radians that is coterminal with each of the given angles: $\frac{19\pi}{4}$ and $\frac{-17\pi}{6}$.
Solution: All we need to do is successively add or subtract 2π until we are in the desired range. Hence we calculate:

$$\frac{19\pi}{4} - 2\pi = \frac{11\pi}{4},$$
$$\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4},$$

which is the coterminal angle we want. Now we repeat this process for the second angle:

$$\frac{-17\pi}{6} + 2\pi = \frac{-5\pi}{6}$$
$$\frac{-5\pi}{6} + 2\pi = \frac{7\pi}{6}$$

which is the coterminal angle we want.

Complements and Supplements

- Two angles are complements (or complimentary angles) if the sum of their measures is $90°$ (or $\pi/2$ radians)
- Two angles are supplements (or supplimentary angles) if the sum of their measures is $180°$ (or π radians)
  - Both angles have to be positive for these terms to apply

Example: 5.1.4: Find the compliment and suppliment of the given angles: $73°$ and $(3\pi)/4$.
Solution: We call the unknown angles θ_c and θ_s and then setup the following equations:

$$\theta_c + 74° = 90, \qquad \theta_s + 73° = 180$$

to find $\theta_c = 17°$ and $\theta_s = 107°$. For the second angle, we similarly have:

$$\theta_c + \frac{3\pi}{4} = \frac{\pi}{2}, \qquad \theta_s + \frac{3\pi}{4} = \pi$$

to find θ_c does not exist and $\theta_s = \frac{\pi}{4}$.

Arc Length Formula

- Recall we defined radians by the formula $\theta = s/r$ where s was arc length, θ was an angle in radians, and r is the radius of the circle. This can be used to solve for arc length and later extended to solve for distances travelled in certain problems.

Example: 5.1.5: A circle has radius of 18 inches. Find the length of the arc intercepted by a central angle with measure $210°$.
Solution: We sinply make use of the formula, however, we have to be careful to convert the angle to radians:

$$s = r\theta = (18 \text{ in})\left[210° \cdot \frac{\pi}{180°}\right] = (18 \text{ in})\left[\frac{7\pi}{6}\right] = 21\pi \text{ in} \approx 65.97 \text{ in}$$

Area of a Sector

- We want to find the area of an arc sector, A, knowing the central angle θ that defines the arc and the radius of the circle r. To find a formula for this, we note that the following proportion must hold

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{length of arc}}{\text{circumference of circle}}$$
$$\frac{A}{\pi r^2} = \frac{\theta r}{2\pi r}$$
$$\implies A = \frac{1}{2}r^2\theta \tag{3}$$

  To see why the proportion must be true, think about the extreme cases when the sector shrinks to nothing and expands to be the entire circle.

Example: 5.1.6: Find the area of a sector of a circle with $r = 10$ in and $\theta = 60°$.
Solution: We simply use the formula derived in Eq. (3)

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(100)\left(60° \cdot \frac{\pi}{180°}\right) = \frac{1}{2}(100)\left(\frac{\pi}{3}\right) \approx 52.4 \text{ in}^2$$

Linear and Angular Speed

- Suppose an object travels around a circle of radius r. If the object travels through a central angle of θ radians and an arc of length s in time t, then
  1. $v = s/t$ is the average linear speed of the object
  2. $\omega = \theta/t$ is the average angular speed of the object
  3. $v = r\omega$

Example: 5.1.7: A gear with radius of R inches rotates at 15 revolutions per minute. Find the angular speed and the linear speed of the tip of the gear.
Solution: Note that 1 revolution= 2π radians, so that 15 revolutions is 30π radians. Hence, the angular speed is just 30π radians per minute. The linear speed can then be calculated as $v = r\omega = (12)(30\pi) = 1131$ inches per minute