1 Chapter 5.2: Right Triangle Trigonometry

Last class we discussed angles and their relation to circles, but they also naturally arise in triangles. Consider ΔABC with angle θ as given by Figure 1. Because the sum of all three angles must equal $180°$ (or π rads), $0 < \theta < 90°$ (or $0 < \theta < \pi/2$).

Trigonometric Function Definitions
For ΔABC with angle θ as given by Figure 1, the following holds

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{a}, \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{a}{b}, \tag{1}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{c}{a}, \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{a}{c}, \tag{2}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{b}{c}, \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{c}{b}, \tag{3}$$

Remark: Note that the trigonometric functions are defined as ratios of side lengths of a right triangle. Therefore, angles that come from similar triangles have equal trigonometric function values.

Example 5.2.1: Find the exact trigonometric function values of θ as given by Figure 1 with $a = 3$ and $b = \sqrt{7}$.

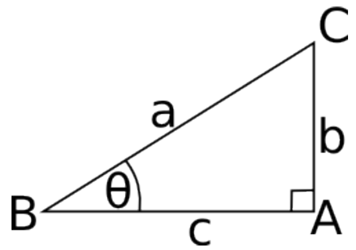

Figure 1: Shown is ΔABC with angle θ.

Solution: We need to start by finding c. The Pythagorean Theorem relates all three side lengths in one equation and thus can help us here. It states that $a^2 = b^2 + c^2$ as in the notation of the figure. Plugging in the given values yields $c = 4$. We then use the definitions to find

$$\sin\theta = \frac{3}{4}, \qquad \csc\theta = \frac{4}{3},$$
$$\cos\theta = \frac{\sqrt{7}}{4}, \qquad \sec\theta = \frac{4\sqrt{7}}{7},$$
$$\tan\theta = \frac{3\sqrt{7}}{7}, \qquad \cot\theta = \frac{\sqrt{7}}{3},$$

Reciprocal Functions: Notice that the trigonometric functions are related via reciprocals

$$\begin{aligned} \csc\theta = \frac{1}{\sin\theta} &\iff \sin\theta = \frac{1}{\csc\theta} \iff (\sin\theta)(\csc\theta) = 1, \\ \sec\theta = \frac{1}{\cos\theta} &\iff \cos\theta = \frac{1}{\sec\theta} \iff (\cos\theta)(\sec\theta) = 1, \\ \cot\theta = \frac{1}{\tan\theta} &\iff \tan\theta = \frac{1}{\cot\theta} \iff (\tan\theta)(\cot\theta) = 1, \end{aligned} \tag{4}$$

Example 5.2.2: Let $\sin\theta = 5/13$ and $\cos\theta = 12/13$. Find the other trig function values for θ.
Solution: Plugging these three side lengths into Pythagorean's Theorem confirms this is a right triangle. Then based on the definitions of the trig functions, we can identify that the opposite side is of length 5, the adjacent side is of length 12, and the hypotenuse is of 13. The trig function values for this are

$$\tan\theta = \frac{5}{12}, \quad \cot\theta = \frac{12}{5}, \quad \csc\theta = \frac{13}{5}, \quad \sec\theta = \frac{13}{12}$$

Example 5.2.3: Find trig function values given that θ is an angle of a right triangle with $\sin\theta = 2/5$.
Solution: The first step is to solve for the missing side length using Pythagorean's Theorem which yields:

$$x^2 + 2^2 = 5^2 \implies x^2 = 21 \implies x = \sqrt{21}$$

Then it follows that

$$\cos\theta = \frac{\sqrt{21}}{5}, \quad \tan\theta = \frac{2\sqrt{21}}{21}, \quad \csc\theta = \frac{5}{2}, \quad \sec\theta = \frac{5\sqrt{21}}{21}, \quad \cot\theta = \frac{\sqrt{21}}{2}$$

Two Special Triangles

1. **45-45-90 Triangle**: Consider a right triangle as seen in Figure 1 where $b = c = 1$ and denote the top angle by α. Then notice that all of the trig functions are equal for θ and α. This implies that the two angles have to be equal and since the sum of all angles in a triangle is 180°, it must be true that $\theta = \alpha = 45°$. Further, the hypotenuse must be given by $a = \sqrt{2}$ so that the trig functions take the values:

$$\begin{aligned} \sin\theta &= \frac{\sqrt{2}}{2}, & \csc\theta &= \sqrt{2}, \\ \cos\theta &= \frac{\sqrt{2}}{2}, & \sec\theta &= \sqrt{2}, \\ \tan\theta &= 1, & \cot\theta &= 1, \end{aligned}$$

2. **30-60-90 Triangle**: This special triangle most commonly arises when considering an equilateral triangle of side length 2 and then splitting in half as is done in Figure 2. The resulting side lengths are given in the figure as well as the angles. Notice that the trig values can be exactly expressed from the leftmost image and the definitions.

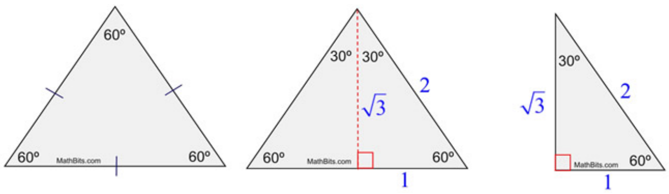

Figure 2: Natural way of encountering the 30-60-90 Triangle in real-world applications.

Cofunction Identities
For θ measured in radians ($90° = \frac{\pi}{2}$), it follows that:

$$\begin{aligned} \sin\theta &= \cos\left(\frac{\pi}{2} - \theta\right), & \cos\theta &= \sin\left(\frac{\pi}{2} - \theta\right), \\ \sec\theta &= \csc\left(\frac{\pi}{2} - \theta\right), & \csc\theta &= \sec\left(\frac{\pi}{2} - \theta\right), \\ \tan\theta &= \cot\left(\frac{\pi}{2} - \theta\right), & \cot\theta &= \tan\left(\frac{\pi}{2} - \theta\right) \end{aligned}$$

This can be verified for the case of the special 30-60-90 Triangle seen in Figure 2.

Example 5.2.4: Given $\cot 68° \approx .4040$, find $\tan 22°$. Find the exact value of $\cos 72° / \sin 18°$.
Solution: For the first part, notice that the angles 68° and 22° are complementary angles. Hence using the cofunction identities

$$\tan 22° = \cot(90° - 22°) = \cot 68° \approx .4040$$

The same idea hold for the second part where we have

$$\cos 72° = \sin(90° - 72°) = \sin 18° \implies \frac{\cos 72°}{\sin 18°} = 1$$