1 Chapter 5.3: Trigonometric Functions of Any Angle and The Unit Circle

Trigonometric Functions of any Angle

Let $P = (x, y)$ be any point on a terminal ray of an angle θ in standard position and let $r = \sqrt{x^2 + y^2}$. Then we have the following definitions

$$\sin\theta = \frac{y}{r}, \qquad \csc\theta = \frac{r}{y}, \quad (y \neq 0) \tag{1}$$

$$\cos\theta = \frac{x}{r}, \qquad \sec\theta = \frac{r}{x}, \quad (x \neq 0) \tag{2}$$

$$\tan\theta = \frac{y}{x}, \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y}, \quad (y \neq 0) \tag{3}$$

Self-Check: Pick a point in quadrant I to confirm that the more narrow definitions from the last section agree with this more general form.

Example 5.3.1: Suppose θ is an angle whose terminal side contains $P = (1, 3)$. Find the exact values for the trigonometric functions of θ.
Solution: We have that $x = 1, y = 3$. We can then calculate $r = \sqrt{10}$. Using the definitions above, we have:

$$\sin\theta = \frac{3\sqrt{10}}{10}, \qquad \csc\theta = \frac{\sqrt{10}}{3},$$

$$\cos\theta = -\frac{\sqrt{10}}{10}, \qquad \sec\theta = -\sqrt{10},$$

$$\tan\theta = -3, \qquad \cot\theta = -\frac{1}{3},$$

Quadrantal Angles

- Notice that for these angles either x or y is 0. Hence some trig functions may be undefined for these angles.

Example 5.3.2: Find the values of the six trig functions for $\theta = \pi, (3\pi)/2$.
Solution: For $\theta = \pi$, one such point we can use is $P = (-1, 0)$. Hence we have

$$\sin\pi = 0, \qquad \csc\pi = \text{Und},$$

$$\cos\pi = -1, \qquad \sec\pi = -1,$$

$$\tan\pi = 0, \qquad \cot\pi = \text{Und},$$

For $\theta = (3\pi)/2$, one such point we can use is $P = (0, -1)$. Hence we have

$$\sin\frac{3\pi}{2} = -1, \qquad \csc\frac{3\pi}{2} = -1,$$

$$\cos\frac{3\pi}{2} = 0, \qquad \sec\frac{3\pi}{2} = \text{Und},$$

$$\tan\frac{3\pi}{2} = \text{Und}, \qquad \cot\frac{3\pi}{2} = 0,$$

Coterminal Angles

- Because the value of each trig function of an angle in standard position is completely determined by the position of the terminal side, the following statements are true:
  1. Coterminal angles are assigned identical values by the six trig functions
  2. The signs of the values of the trig functions are determined by the quadrant containing the terminal side of the angle
  3. For any integer n, $\theta + 2\pi n$ are coterminal angles and hence:

$$\sin\theta = \sin(\theta + 2\pi n), \qquad \csc\theta = \csc(\theta + 2\pi n), \tag{4}$$

$$\cos\theta = \cos(\theta + 2\pi n), \qquad \sec\theta = \sec(\theta + 2\pi n), \tag{5}$$

$$\tan\theta = \tan(\theta + 2\pi n), \qquad \cot\theta = \cot(\theta + 2\pi n) \tag{6}$$

Example 5.3.3: Find exact values for $\cos 1830°$ and $\sin(31\pi)/3$.
Solution: Our first idea should be to find the coterminal angles that lie in $[0, 2\pi]$ or $[0°, 360°]$ because we know some values in this range already. Subtracting 5 full revolutions off our first angle tells us that 1830° is a coterminal angle to 30°. Similarly, we can find that $(31\pi)/3$ is a coterminal angle to $\pi/3$. Hence:

$$\cos 1830° = \cos 30° = \frac{\sqrt{3}}{2}, \qquad \sin\left(\frac{31\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

where we have used our knowledge of the special 30-60-90 Triangle previously discussed.

Signs of the Trigonometric Functions

- In Quadrant I:
  - All trig functions are positive
- In Quadrant II:
  - Only sine and cosecant are positive
- In Quadrant III:
  - Only tangent and cotangent are positive
- In Quadrant IV:
  - Only cosine and secant are positive

Example 5.3.4: If $\sin\theta > 0$ and $\cos\theta < 0$, in which quadrant does θ lie?
Solution: Note that r is always positive. Hence $\sin\theta > 0 \implies y > 0$, and $\cos\theta < 0 \implies x < 0$. Thus our angle must lie in Quadrant II.

Example 5.3.5: Given that $\tan\theta = -4/5$ and $\cos\theta > 0$, find the exact values of $\sin\theta$ and $\sec\theta$.
Solution: Note that $\cos\theta > 0 \implies x > 0$. Now from the value of $\tan\theta$ we can determine that $x = 5$, $y = -4$ and then use these to calculate $r = \sqrt{41}$. Finally we just use the definitions of the trig functions:

$$\sin\theta = \frac{y}{r} = -\frac{4\sqrt{41}}{41}, \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{41}}{5}$$

Reference Angle

- For any given angle θ there is a corresponding acute angle called its reference angle whose trig function values are all equal to those of θ except for possibly a sign difference
- Let θ be any angle in standard position that is not quadrantal. The reference angle for θ is the acute angle θ' formed by the terminal side of θ and the x-axis.

Example 5.3.6: Find reference angle θ' for each angle: 250°, $(3\pi)/5$, and 5.75.
Solution: Note that 250° is in quadrant III. Thus the angle between the terminal side and the x-axis is calculated as $\theta_1' = 250° - 180° = 70°$. Next we note $(3\pi)/5$ is in quadrant II. Thus the reference angle is calculated as $\theta_2' = \pi - (3\pi)/5 = (2\pi)/5$. Finally, we have the angle 5.75. We note that this angle is in quadrant IV because $5.74 > 4.71 \approx (3pi)/2$. Thus the reference angle is calculated as $\theta_3' = 2\pi - 5.75 \approx 0.53$.

Unit Circle

- Consider a circle of radius 1 centered at the origin as in Figure 1.
- Use the arclength formula with $r = 1$ to see that $s = r\theta = \theta$. Hence, for a unit circle we have

$$\sin s = \sin\theta = \sin\left(\frac{180° \cdot \theta}{\pi}\right)$$

- In this case, our trig function definitions simplify as well

$$\sin\theta = y, \qquad \csc\theta = \frac{1}{y}, \quad (y \neq 0) \tag{7}$$

$$\cos\theta = x, \qquad \sec\theta = \frac{1}{x}, \quad (x \neq 0) \tag{8}$$

$$\tan\theta = \frac{y}{x}, \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y}, \quad (y \neq 0) \tag{9}$$

- Any point in the (x, y) plane has an associated point P on the unit circle because similar triangles have equal trig function values.

Example 5.3.7: Find the exact value of $\sec(59\pi/6)$.
Solution: We will follow the given procedure: (i) find coterminal angle in $[0, 2\pi]$; (ii) Find the reference angle; (iii) Calculate the trig function values for the reference angle; (iv) Check the signs based on quadrant of the angle. Hence, we first find that $(59\pi/6)$ is coterminal to $(11\pi)/6$. This angle is in quadrant IV and so its reference angle is found via the formula $\theta' = 2\pi - (11\pi)/6 = \pi/6$. Again this is a value from the 30-60-90 Triangle and so we have $\sec(\pi/6) = \frac{1}{\cos(\pi/6)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Finally, we check that in quadrant IV, the x coordinate is positive and the y coordinate is negative which implies cosine (and secant) are positive. Thus we are done and

$$\sec\left(\frac{59\pi}{6}\right) \tag{10}$$

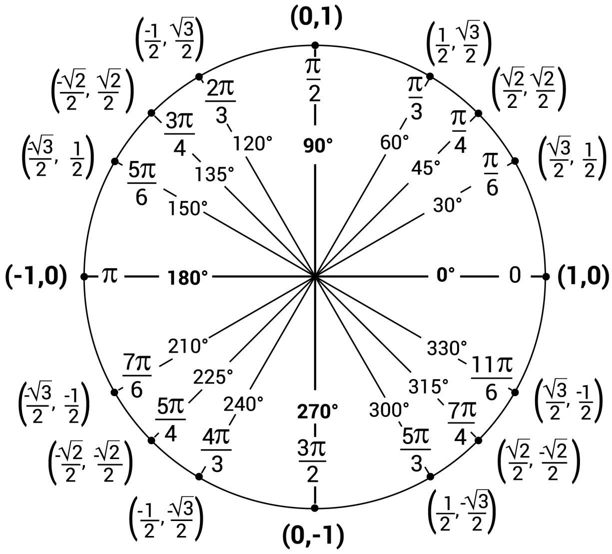

Figure 1: Shown is the unit circle. Identified are special angles, given in both degrees and radians, that give exact trigonometric values. The points on the unit circle associated with a given angle can be used to calculate the trigonometric values as $\cos\theta = x$ and $\sin\theta = y$ for an angle θ and point (x, y).