1 Chapter 5.4: Graphs of Sine and Cosine Functions

Domain and Range

- By the definitions of our trig functions, we have:

$$\sin\theta = y/r, \qquad \cos\theta = x/r$$

- Because any real number represents an angle θ which has a corresponding point, $P = (x, y)$, on the unit circle, the domain of the sine and cosine functions are $(-\infty, \infty)$.
- Further note that $P = (x, y)$ is a point on the unit circle such that $r = 1$ and $|x|, |y| \leq 1$. Hence, the range of each function is $[-1, 1]$.

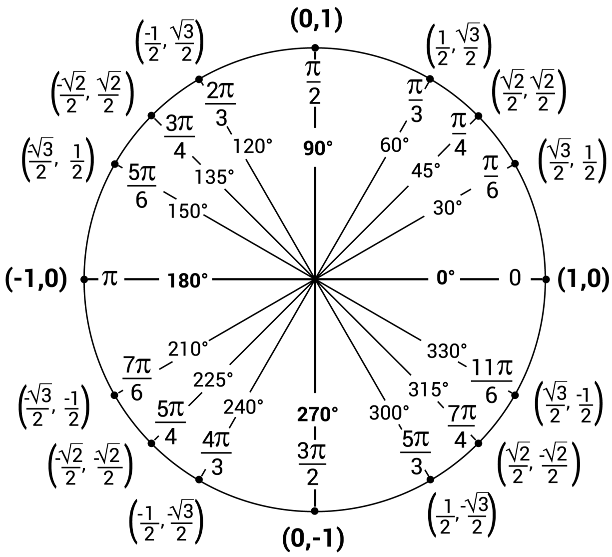

Figure 1: Shown is the unit circle. Identified are special angles, given in both degrees and radians, that give exact trigonometric values. The points on the unit circle associated with a given angle can be used to calculate the trigonometric values as $\cos\theta = x$ and $\sin\theta = y$ for an angle θ and point (x, y).

Zeros of Sine and Cosine

- When is $\cos\theta = 0$?
  - Looking at the unit circle in Figure 1, we can see that the x-coordinate is 0 at angles $\pi/2$ and $(3\pi)/2$ **and** all of their coterminal angles
  - Thus $\cos\theta = 0 \quad \text{at} \quad \theta = \pi/2 + n\pi$ for any integer n
- When is $\sin\theta = 0$?
  - Looking at the unit circle in Figure 1, we can see that the y-coordinate is 0 at angles 0 and π **and** all of their coterminal angles
  - Thus $\sin\theta = 0 \quad \text{at} \quad \theta = n\pi$ for any integer n

Even-Odd Properties

- For any angle θ, there is an associated point on the unit circle $P = (x, y)$; then angle $-\theta$ has the associated unit circle point $Q = (x, -y)$. Hence:

$$\cos\theta = x = \cos -\theta, \qquad \sin\theta = y = -\sin -\theta \tag{1}$$

  This property of $\cos\theta$ makes it an <u>even function</u>. Similarly, this property of $\sin\theta$ makes it an <u>odd function</u>.

Periodic Functions

- A function is said to be periodic if there is a positive number P such that $f(x+P) = f(x)$ for all x in the domain of f.
- The smallest value of P such that the above property is true is called the period of f. The graph of f over any interval of length P is called one cycle of the graph.
- For every real number θ,

$$\sin(\theta + 2\pi) = \sin\theta, \qquad \cos(\theta + 2\pi) = \cos\theta \tag{2}$$

  that is each of these functions has period 2π

$y = \sin x$	$y = \cos x$
Domain: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$
Range: $[-1, 1]$	Range: $[-1, 1]$
Zeros at $x = \pi n$	Zeros at $x = \frac{\pi}{2} + \pi n$
Max at $x = \frac{\pi}{2} + 2\pi n$	Max at $x = 2\pi n$
Min at $x = \frac{3\pi}{2} + 2\pi n$	Min at $x = \pi + 2\pi n$
Odd Function	Even Function
Period= 2π	Period= 2π

Table 1: Shown are some key properties that help in graphing sine and cosine functions

Five Key Points

It can help to start by plotting five key points in any given period of the parent function. These points for sine and cosine are listed in Table 2. **Amplitudes of Sine and Cosine**

$y = \sin x$	$y = \cos x$
$(0, 0)$	$(0, 1)$
$(\pi/2, 1)$	$(\pi/2, 0)$
$(\pi, 0)$	$(\pi, -1)$
$((3\pi)/2, -1)$	$((3\pi)/2, 0)$
$(2\pi, 0)$	$(2\pi, 1)$

Table 2: Listed are the five key points in the interval $[0, 2\pi]$.

- The functions $y = a\sin x$ and $y = a\cos x$ have amplitude $|a|$ and range $[-a, a]$. Note this represents a vertical stretch of the parent function.

Phase Shifts

- A phase shift of the parent function, or horizontal shift of the graph, is mathematically included via

$$y = \sin(x - c), \qquad y = \cos(x - c)$$

  where it is a shift right if $c > 0$ and a shift left if $c < 0$.
- More generally, for a function of the form $y = \sin b(x - c)$, the phase shift is defined as the value of x for which $b(x - c) = 0$.

Vertical Shifts

- A vertical shift of the parent function, or horizontal shift of the graph, is mathematically included via

$$y = \sin x + d, \qquad y = \cos x + d$$

  where it is a shift up if $d > 0$ and a shift down if $d < 0$.
- More generally, for a function of the form $y = \sin b(x - c)$, the phase shift is defined as the value of x for which $b(x - c) = 0$.

Change of Period

- A change of period of the parent function, or horizontal stretch/compression of the graph, is mathematically included via

$$y = \sin bx, \qquad y = \cos bx$$

where this causes the period to become $P = (2\pi)/b$.

Simple Harmonic Motion

- Trigonometric functions can often describe or model motion caused by vibrations, rotations, or oscillations - e.g. sound waves, radio waves, electric current, guitar string, swing of a pendulum, etc.

- An object whose position relative to an equilibrium position at time t that can be described by either

$$y = a\sin(\omega t), \qquad \text{or} \qquad y = a\cos(\omega t) \quad (\omega > 0) \tag{3}$$

is said to be in simple harmonic motion.

- The amplitude, $|a|$, is the max distance the object travels from its equilibrium position. The period of the motion, $(2\pi)/\omega$, is the time it takes the object to complete one full cycle. The frequency of the motion is $\omega/(2\pi)$ and gives the number of cycles completed per unit of time.

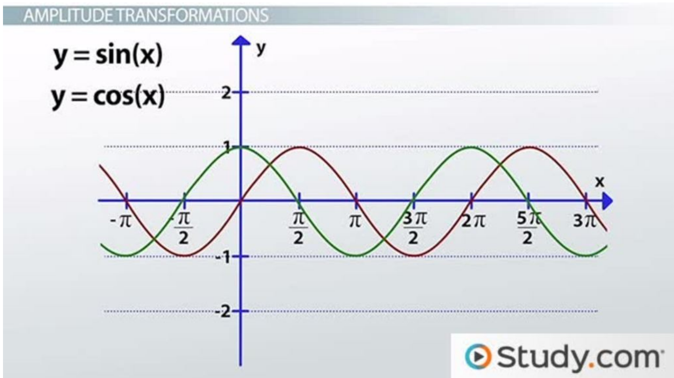

Figure 2: Shown is a graph of the parent functions for sine and cosine.

Example 5.4.1: Graph $y = 4\sin[3(x - \pi/9)] - 2$
Solution: I will graph this function by applying the transformations as described in chapter 4. First we identify the phase shift, or horizontal shift, as a right shift $\pi/9$ units. We also identify that we have a horizontal compression by a factor of 3 which shrinks the function's period to $P = (2\pi)/3$. Next we notice there is a vertical stretch by a factor of 4. Finally, there is a vertical shift down 2 units. When we go to graph, we start with a single period of the parent function and then apply these transformation in the order they were discussed. The results are seen in Figure 3.

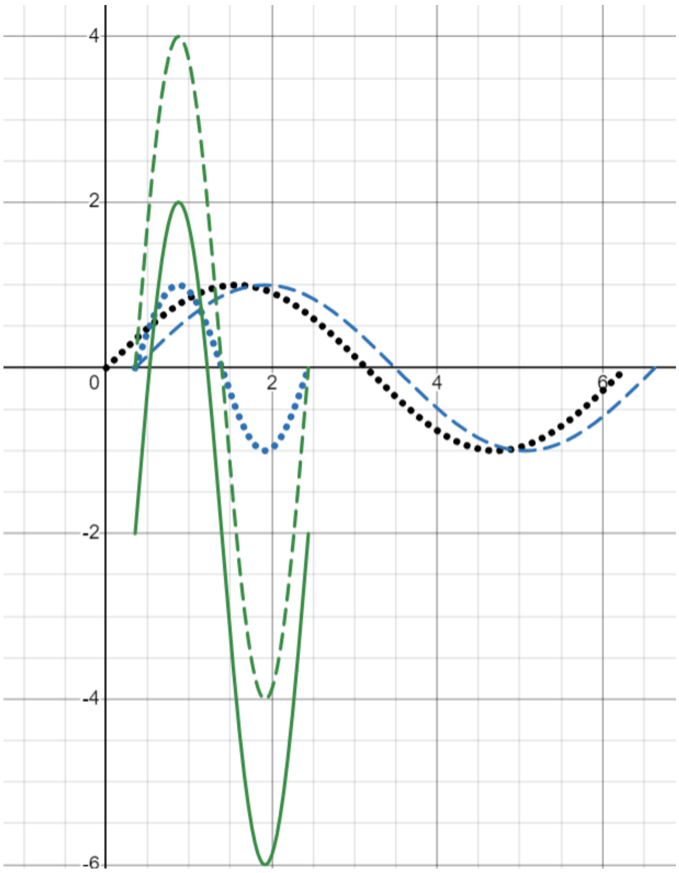

Figure 3: Shown is the graph to Example 5.4.1. The parent function is in black. In blue are the first two transformations (horizontal). In green are the last two transformations (vertical) with the solid green curve representing the final function.