1 Chapter 5.6: Inverse Trigonometric Functions

Inverse Functions

- An inverse function "reverses" a function. If f is a function that takes in input x and spits out output y, then the inverse function, f^{-1}, takes in y and outputs x.
- For a function to have an inverse, it must be a one-to-one function. That is, for all y in the range of the function, there must be exactly one x in the domain such that $f(x) = y$
- Let $f(x)$ and $g(x)$ be one-to-one functions. If $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, then $f(x)$ and $g(x)$ are inverse functions

Inverse Sine

- Sine is not a one-to-one, however, if we truncate it to the domain $[-\pi/2, \pi/2]$ then it becomes one-to-one and invertible. The inverse of $f(x) = \sin x,\ -\pi/2 \le x \le \pi/2$ is called inverse sine or arcsine and is denoted by $f^{-1}(x) = \sin^{-1} x$ or $f^{-1}(x) = \arcsin x$
- You can obtain the graph for $f^{-1}(x) = \arcsin x$ by taking the graph of $f(x) = \sin x,\ -\pi/2 \le x \le \pi/2$ and reflecting it over the line $y = x$

Example 5.6.1: Calculate $\arcsin(-1)$, $\sin^{-1}(-1/2)$, $\sin^{-1}(0)$, and $\arcsin(3/2)$.
Solution: One helpful way to calculate these values is to actually first give it a variable name, then apply the function to both sides of the equation, and finally use our knowledge of the unit circle to find the angle. For the first angle, let:

$$\begin{aligned}
y = \arcsin(-1) &\implies \sin y = -1 \implies y = -\frac{\pi}{2}, \\
y = \sin^{-1}\left(-\frac{1}{2}\right) &\implies \sin y = -\frac{1}{2} \implies y = -\frac{\pi}{6}, \\
y = \sin^{-1}(0) &\implies \sin y = 0 \implies y = 0, \\
y = \arcsin\left(\frac{3}{2}\right) &\implies \sin y = \frac{3}{2} \implies \text{Undefined},
\end{aligned}$$

Inverse Cosine

- Similar to sine, cosine is not a one-to-one, however, if we truncate it to the domain $[0, \pi]$ then it becomes one-to-one and invertible. The inverse of $f(x) = \cos x,\ 0 \le x \le \pi$ is called inverse cosine or arccosine and is denoted by $f^{-1}(x) = \cos^{-1} x$ or $f^{-1}(x) = \arccos x$
- You can obtain the graph for $f^{-1}(x) = \arccos x$ by taking the graph of $f(x) = \cos x,\ 0 \le x \le \pi$ and reflecting it over the line $y = x$
- Note that if we had truncated the cosine function over the same interval we used to truncate the sine function, then we wouldn't obtain the full range of $\cos x$ – to see this look at a graph of $\cos x$!!

Inverse Tangent

- The inverse of $f(x) = \tan x,\ -\pi/2 \le x \le \pi/2$ is called inverse tangent or arctangent and is denoted by $f^{-1}(x) = \tan^{-1} x$ or $f^{-1}(x) = \arctan x$
- You can obtain the graph for $f^{-1}(x) = \arctan x$ by taking the graph of $f(x) = \tan x,\ -\pi/2 \le x \le \pi/2$ and reflecting it over the line $y = x$

Example 5.6.2: Calculate the exact values of $\arccos\left(\frac{1}{2}\right)$, $\arctan(1)$, $\text{arcsec}(2)$, $\text{arccsc}\left(\frac{2}{\sqrt{3}}\right)$.
Solution: We follow the same approach as in the first example:

$$\begin{aligned}
y = \arccos\left(\frac{1}{2}\right) &\implies \cos y = \frac{1}{2} \implies y = \frac{\pi}{3}, \\
y = \arctan(1) &\implies \tan y = 1 \implies y = \frac{\pi}{4}, \\
y = \text{arcsec}(2) \implies \sec y = 2 &\implies \cos y = \frac{1}{2} \implies y = \frac{\pi}{3}, \\
y = \text{arccsc}\left(\frac{2}{\sqrt{3}}\right) \implies \csc y = \frac{2}{\sqrt{3}} &\implies \sin y = \frac{\sqrt{3}}{2} \implies y = \frac{\pi}{3}
\end{aligned}$$

Other Inverse Trigonometric Functions

- The inverse of $f(x) = \sec x,\ 0 \leq x \leq \pi$ is called <u>inverse secant</u> or <u>arcsecant</u> and is denoted by $f^{-1}(x) = \sec^{-1} x$ or $f^{-1}(x) = \operatorname{arcsec} x$
- The inverse of $f(x) = \csc x,\ -\pi/2 \leq x \leq \pi/2$ is called <u>inverse cosecant</u> or <u>arccosecant</u> and is denoted by $f^{-1}(x) = \csc^{-1} x$ or $f^{-1}(x) = \operatorname{arccsc} x$
- The inverse of $f(x) = \cot x,\ 0 \leq x \leq \pi$ is called <u>inverse cotangent</u> or <u>arccotangent</u> and is denoted by $f^{-1}(x) = \cot^{-1} x$ or $f^{-1}(x) = \operatorname{arccot} x$

Example 5.6.3: Find the inverse of $f(x) = 3\cos(x-1) + 2,\ 1 \leq x \leq 1 + \pi$.
Solution: We start by swapping x and y and then trying to solve the equation for y again:

$$\begin{aligned} x &= 3\cos(y-1) + 2, \\ \frac{x-2}{3} &= \cos(y-1) \\ \arccos\left(\frac{x-2}{3}\right) + 1 &= y = f^{-1}(x) \end{aligned}$$

Remark: If $f(x)$ is an odd function, then its inverse function is an odd function (sine, cosecant, tangent). **Remark**: A consequence of having to truncate the original trig functions to make them invertible is that sometimes we must be careful when evaluating compositions of trig and inverse trig functions. The next example illustrates exactly why.
Example 5.6.4: Calculate $\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right)$ and $\cos(\arccos(2))$
Solution: To calculate the first expression we start by noting that $\sin\left(\frac{3\pi}{2}\right) = -1$. Then we let

$$y = \sin^{-1}(-1) \implies \sin y = -1 \implies y = -\frac{\pi}{2}$$

Then it follows that

$$\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right) = -\frac{\pi}{2}$$

To find the second expression we first note

$$y = \arccos(2) \implies \cos y = 2 \implies \text{Undefined}$$