1 Chapter 6.1: Trigonometric Identities

Pythagorean Theorem

- Let $P = (x, y)$ be any point on the terinal side of an angle θ. Then by definition, we have

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}$$

Then by Pythagorean's Theorem, we have

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (r\cos\theta)^2 + (r\sin\theta)^2 &= r^2 \end{aligned}$$

$$\cos^2\theta + \sin^2\theta = 1 \tag{1}$$

This is the first Pythagorean Identity. The other two verisions can be derived by dividing this equation by either $\cos^2\theta$ or $\sin^2\theta$. The result of this is, respectively, given as

$$1 + \tan^2\theta = \sec^2\theta, \tag{2}$$

$$\cot^2\theta + 1 = \csc^2\theta. \tag{3}$$

Fundamental Identities

- Reciprocal Identities

$$\begin{aligned} \csc\theta = \frac{1}{\sin\theta} &\iff \sin\theta = \frac{1}{\csc\theta} \iff \quad (\sin\theta)(\csc\theta) = 1, \\ \sec\theta = \frac{1}{\cos\theta} &\iff \cos\theta = \frac{1}{\sec\theta} \iff \quad (\cos\theta)(\sec\theta) = 1, \\ \cot\theta = \frac{1}{\tan\theta} &\iff \tan\theta = \frac{1}{\cot\theta} \iff \quad (\tan\theta)(\cot\theta) = 1, \end{aligned} \tag{4}$$

- Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \tag{5}$$

- Even/Odd Identities

$$\begin{aligned} &\sin(-\theta) = -\sin(\theta), \quad \cos(-\theta) = \cos(\theta), \qquad \tan(-x) = -\tan(x) \\ &\csc(-\theta) = -\csc(\theta), \quad \sec(-\theta) = \sec(\theta), \qquad \cot(-x) = -\cot(x) \end{aligned} \tag{6}$$

Example 6.1.1: If $\sin(\theta) = 4/5$ and $\pi/2 < \theta < \pi$, find $\cot(\theta)$.
Solution: Start by noticing that $\csc(\theta) = 5/4$. Further, by the Pythagorean Identities we have $\cot^2(\theta) + 1 = \csc^2(\theta)$. Now we know θ is in quadrant II which means $\cot(\theta) < 0$. Hence from the Pythagorean Identity, we have:

$$\cot(\theta) = -\sqrt{\csc^2\theta - 1} = -\sqrt{\frac{25}{16} - 1} = -\frac{3}{4}$$

Example 6.1.2: Rewrite in terms of sines and cosines; then simplify.

$$\frac{\tan(\theta)}{\sec(\theta) + 1} + \frac{\tan(\theta)}{\sec(\theta) - 1}$$

Solution: We first convert to sines and cosines:

$$
\begin{aligned}
&= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}+\frac{\cos\theta}{\cos\theta}} + \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}-\frac{\cos\theta}{\cos\theta}}, \\
&= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1+\cos\theta}{\cos\theta}} + \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1-\cos\theta}{\cos\theta}}, \\
&= \frac{\sin\theta}{1+\cos\theta} + \frac{\sin\theta}{1+\cos\theta}, \\
&= \frac{\sin\theta\,[1-\cos\theta]}{1-\cos^2\theta} + \frac{\sin\theta\,[1+\cos\theta]}{1-\cos^2\theta}, \\
&= \frac{1-\cos\theta+1+\cos\theta}{\sin\theta}, \\
&= \frac{2}{\sin\theta}
\end{aligned}
$$

Verifying Identities

- We do this by tranforming one side of an equation into the other side using a sequence of steps each of which produces an identity
- Methods and/or Tips:
  1. Start with the more complicated side
  2. Stay focused on the final expression
  3. Option: convert to sines and cosines
  4. Option: work on both sides
  5. Option: use conjugates

Example 6.1.3: Verify the following identity

$$\frac{1-\sin^2 x}{\cos x} = \cos x$$

Notice that we can apply the first Pythagorean Identity to rewrite the numerator. Doing this we find

$$\frac{1-\sin^2 x}{\cos x} = \frac{\cos^2 x}{\cos x} = \cos x$$

Hence, the two sides of the equation are always equal. **Example 6.1.4**: Verify the following identity

$$\tan^4 x = \tan^2 x \sec^2 x - \sec^2 x + 1$$

We choose to work with the RHS. To start, we apply the second form of Pythagorean's Identity to get

$$\tan^2 x \sec^2 x - \sec^2 x + 1 = \tan^2 x \sec^2 x - \tan^2 x = \tan^2 x(\sec^2 x - 1) = \tan^4 x$$