1 Chapter 6.2: Sum and Difference Identities

Cosine Sum and Difference Identities

Imagine there are two angles u and v with associated points on the unit circle $P_u = (\cos u, \sin u)$ and $P_v = (\cos v, \sin v)$. We can similarly imagine looking at the angle $u - v$ with associated point on the unit circle $Q = (\cos(u-v), \sin(u-v))$ which is in standard position. I claim that the triangle that forms from the origin, P_u, P_v must be congruent to the triangle that forms between the origin, $Q_1 = (1, 0)$, and Q [to see this choose values for the angles u and v and draw this out]. If the triangles are congruent, then the side lengths must all be equal. Particularly:

$$d(P_u, P_v) = d(Q_1, Q),$$
$$d(P_u, P_v)^2 = d(Q_1, Q)^2.$$

We know what the distance formula is and we know what these points are, so we can find that

$$d(P_u, P_v)^2 = 2 - 2(\cos u \cos v + \sin u \sin v)$$

while

$$d(Q_1, Q)^2 = 2 - 2\cos(u - v)$$

Hence, it must be true that

$$\cos(u - v) = \cos u \cos v + \sin u \sin v \tag{1}$$

To get the cosine addition formula, simply let $v = -v$ in the previous equation to get

$$\cos(u + v) = \cos u \cos v - \sin u \sin v \tag{2}$$

Sine Sum and Difference Identities

Recall the cofunction identities

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right), \qquad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right),$$
$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right), \qquad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right),$$
$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right), \qquad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

Use the very first cofunction identity listed to see that

$$\sin(u - v) = \cos\left(\frac{\pi}{2} - (u - v)\right)$$
$$= \cos\left(\left(\frac{\pi}{2} - u\right) + v\right)$$

Now apply the cosine addition formula to find

$$\sin(u - v) = \sin u \cos v - \cos u \sin v \tag{3}$$

Then let $v = -v$ above to find

$$\sin(u + v) = \sin u \cos v + \cos u \sin v \tag{4}$$

Self Check: Use these formulas to work out the derivation for the tangent sum and difference identities.

$$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}, \qquad \tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \tag{5}$$

Example 6.2.1: Find the exact value of $\sec\left(\frac{\pi}{12}\right)$.

Solution: First note that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. Then we have that

$$\sec\left(\frac{\pi}{12}\right) = \sec\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}$$

Using the cosine difference formula and the unit circle, we calculate

$$\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2} + \sqrt{6}}{4}$$

Hence we find

$$\sec\left(\frac{\pi}{12}\right) = \frac{1}{\frac{\sqrt{2}+\sqrt{6}}{4}} = \sqrt{6} - \sqrt{2}$$