# 1 Chapter 6.3: Double-Angle and Half-Angle Formulas

**Recall**: The addition formulas for sine, cosine, and tangent are given by

$$\sin(u+v) = \sin u \cos v + \cos u \sin v, \tag{1}$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v, \tag{2}$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \tag{3}$$

**Double Angle Formulas**
To derive the double angle formulas for the above trig functions, simply set $v = u = x$. Then we find:

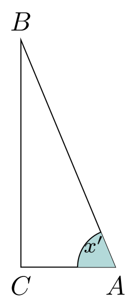
$$\sin(2x) = 2\sin x \cos x, \tag{4}$$
$$\cos(2x) = \cos^2 x - \sin^2 x, \tag{5}$$
$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x} \tag{6}$$

Recall that we can use the Pythagorean Identities to rewrite $\cos^2 x$ and $\sin^2 x$ in the double-angle formula for cosine. Doing this, yields the alternate formulas:

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1 \tag{7}$$

**Example 6.3.1**: If $\sin x = 12/13$ and the angle $x$ lies in quadrant II, find exactly $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$.
We want to draw a triangle with all three side lengths labeled and the reference angle for $x$ inside the triangle. We can do this because we know $x$ is in quadrant II. Hence, we draw a triangle with a positive vertical leg, a negative horizontal leg, and a positive hypotenuse (always positive regardless of quadrant). Then from the valeu of sine, we can determine that the vertical leg must have length 12 while the hypotenuse must have length 13. Now use Pythagorean's Theorem to find that the magnitude of the unknown side length is 5. Once we know all side lengths, it becomes trivial to calculate the trigonometric function values needed. In the picture below, note that we have $\overline{AC} = -5, \overline{CB} = 12$ and $\overline{BA} = 13$. Since we have given the side lengths their correct signs, we now just find sine, cosine, and tangent of $x$ by using "SohCahToa" and the figure below. Hence we have:

$$\sin(2x) = 2\sin x \cos X = 2\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right) = -\frac{120}{169}, \tag{8}$$
$$\cos(2x) = \cos^2 x - \sin^2 x = \left(\frac{-5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = -\frac{119}{169}, \tag{9}$$
$$\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{120}{119} \tag{10}$$




**Example 6.3.2**: Verify $\cos(3x) = 4\cos^3 x - 3\cos x$.
We start utilizing the addition formula for cosine with $u = x$ and $v = 2x$. We then have

$$\begin{aligned}
\cos(3x) = \cos(x+2x) &= \cos(2x)\cos x - \sin(2x)\sin x, && (11)\\
&= \left[2\cos^2 x - 1\right]\cos x - [2\sin x \cos x]\sin x, && (12)\\
&= 2\cos^3 x - \cos x - 2\sin^2 x \cos x, && (13)\\
&= 2\cos^3 x - \cos x - 2\left[1 - \cos^2 x\right]\cos x, && (14)\\
&= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x, && (15)\\
&= 4\cos^3 x - 3\cos x && (16)
\end{aligned}$$

Note that we started by using the double angle formulas for cosine and sine. We then used Pythagorean's Identity for sine squared.

**Power Reducing Formulas**
Note that one form of our double angle formula for cosine reads

$$\cos(2x) = 1 - 2\sin^2 x \implies \boxed{\sin^2 x = \frac{1-\cos(2x)}{2}} \tag{17}$$

and similarly using the other formula we can find

$$\cos(2x) = 2\cos^2(x) - 1 \implies \boxed{\cos^2 x = \frac{\cos(2x)+1}{2}} \tag{18}$$

Finally, we can determine a similar formula for tangent by combinin the above results. That is:

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos(2x)}{\cos(2x)+1} \tag{19}$$

**Example 6.3.3**: Write a formula for $\sin^4 x$ that contains only first powers of cosines and sines.
We start by noting that

$$\begin{aligned}
\sin^4 x = (\sin^2 x)^2 &= \left(\frac{1-\cos(2x)}{2}\right)^2, && (20)\\
&= \frac{1}{4}\left(1 - 2\cos(2x) + \cos^2(2x)\right), && (21)\\
&= \frac{1}{4}\left(1 - 2\cos(2x) + \left[\frac{1}{2}\cos(4x) + \frac{1}{2}\right]\right), && (22)\\
&= \frac{1}{4} = \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x) + \frac{1}{8}, && (23)\\
&= \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x) && (24)
\end{aligned}$$

Note that above we used the power reducing formula for sine. In the next line we used the power reducing formula for cosine. Then we just simplified the expression combining like terms as usual.

**Half-Angle Formulas**
To derive the half-angle formulas, we simply take the power reducing formulas, substitute $x \to x/2$, and solve for the left-hand-side to find:

$$\begin{aligned}
\sin\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1-\cos x}{2}}, && (25)\\
\cos\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1+\cos x}{2}}, && (26)\\
\tan\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1-\cos x}{1+\cos x}} && (27)
\end{aligned}$$

Note that we can determine the correct sign in the above formulas by simply figuring out which quadrant the angle $x/2$ lies in.
**Example 6.3.4**: Find exactly $\sin\left(\frac{\pi}{12}\right)$.
We first note that if $x = \pi/6$, then $x/2 = \pi/12$. Hence, we can use the half angle formula for sine with $x = \pi/6$. We also note that the angle $\pi/12$ is in the first quadrant where sine is positive and so we take the positive square root in the half-angle formula. Hence we find:

$$\sin\left(\frac{\pi}{12}\right) = \sqrt{\frac{1-\cos(\pi/6)}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \cdots = \frac{\sqrt{2-\sqrt{3}}}{4} \tag{28}$$