1 Chapter 6.4: Product-to-Sum and Sum-to-Product Formulas

Product-to-Sum Formulas

To find a product-to-sum formula for cosine, we start by recalling the sum and addition formulas for cosine:

$$\cos(x-y) = \cos x \cos y + \sin x \sin y, \tag{1}$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{2}$$

Now add these two equations to see that

$$\cos(x-y) + \cos(x+y) = 2\cos x \cos y \implies \boxed{\cos x \cos y = \frac{1}{2}\Big(\cos(x-y) + \cos(x+y)\Big)} \tag{3}$$

We can derive a similar formula for sine by taking the above boxed formula and subtracting $\cos(x+y)$ from both sides. We then find:

$$\cos x \cos y - \cos(x+y) = \frac{1}{2}\Big(\cos(x-y) + \cos(x+y)\Big) - \cos(x+y), \tag{4}$$
$$\cos x \cos y - \Big(\cos x \cos y - \sin x \sin y\Big) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y) - \cos(x+y), \tag{5}$$
$$\boxed{\sin x \sin y = \frac{1}{2}\Big(\cos(x-y) - \cos(x+y)\Big)} \tag{6}$$

Now to derive mixed product-to-sum formulas we instead start from the sum and difference formulas for the sine function. Recall that these are given by:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \tag{7}$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y \tag{8}$$

By simply adding these two formulas, we immediately find:

$$\boxed{\sin x \cos y = \frac{1}{2}\Big(\sin(x+y) + \sin(x-y)\Big)} \tag{9}$$

Similarly, by subtracting the sum and difference formulas for sine, we find:

$$\boxed{\cos x \sin y = \frac{1}{2}\Big(\sin(x+y) - \sin(x-y)\Big)} \tag{10}$$

Sum-to-Product Formulas

We will not investigate the derivation of each of these formulas explicitly, however, note that you can verify the sum-to-product formulas to come by simply applying the product-to-sum formulas we just derived.
The Sum-to-Product formulas are given by

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \tag{11}$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right), \tag{12}$$
$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \tag{13}$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right). \tag{14}$$

Example 6.4.1: Simplify $\sin 43° + \sin 17°$ using identities and the unit circle (you do not need a calculator).
Using the sum-to-product formula for addition of two sine functions gives us

$$\sin 43° + \sin 17° = 2\sin\left(\frac{43° + 17°}{2}\right)\cos\left(\frac{43° - 17°}{2}\right) = 2\sin 30° \cos 13° = \cos 13° \tag{15}$$