1 Chapter 6.5: Trigonometric Equations I

Solving Trigonometric Equations

- An identity is an equation that is true for all values in the domain of the variable. Trigonometric equations, in general, are conditional equations that contain a trigonometric function with a variable that may or may not solvable.
- We will study 3 equations in this section:
  - Equations of the form:
    $$af(x-c) = k, \qquad \text{where } f(x) \text{ represents a trig function and } a, c, k \text{ are constants} \tag{1}$$
  - Equations that can be solved by factoring using the zero-product property (if $ab = 0$, then $a = 0$ or $b = 0$)
  - Equations that are equivalent to a polynomial equation in one trigonometric function

Equations of the form $af(x-c) = k$
Procedure for solving:

1. Find $\alpha = f^{-1}(k)$ and write $f(x-c) = f(\alpha)$
2. Use table provided or the unit circle to find all solutions of $f(x-c) = f(\alpha)$
3. Try integer values of n to find all solutions in requested interval

Remark: Note that from the difference formulas for sine and cosine, one can verify that:

$$\sin x = \sin(\pi - x), \qquad \text{and} \qquad \cos x = \cos(2\pi - x) \tag{2}$$

Note, however, that angles only have the same tangent value if they are coterminal.

Example 6.5.1: Find all solutions of $\sec x = 2/\sqrt{3}$.
Solution: Start by rewriting the equation in terms of cosine to see that we can equivalently write this as

$$\cos x = \frac{\sqrt{3}}{2}$$

Following our procedure, we start by noticing that $\alpha = \frac{\pi}{6} = \frac{11\pi}{6}$ satisfies the above equation. We then need to find all coterminal angles which is easily done by including the integer n in the solution as:

$$x = \frac{\pi}{6} + 2\pi n, \quad \frac{11\pi}{6} + 2\pi n$$

Example 6.5.2: Find all solutions in $[0, 2\pi)$ of: $3\sec\left(x - \frac{\pi}{6}\right) = \sec\left(x - \frac{\pi}{6}\right) + 3$.
Solution: We start by combining like terms to rewrite the equation as

$$\sec\left(x - \frac{\pi}{6}\right) = 2 \implies \cos\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$$

From here, we find that $\cos\alpha = \frac{1}{2}$ is satisfied for $\alpha = \frac{\pi}{3}, \frac{5\pi}{3}$. We now continue with step 2 of the procedure:

$$x - \frac{\pi}{6} = \frac{\pi}{3} \implies x = \frac{\pi}{2},$$
$$x - \frac{\pi}{6} = \frac{5\pi}{3} \implies x = \frac{11\pi}{6}$$

Now we need to also check for coterminal angles since all coterminal angles have equal trigonometric function values. However, we are asked for only solutions in $[0, 2\pi)$ so the solution is simply

$$x = \frac{\pi}{2}, \frac{11\pi}{6}$$

Example 6.5.3: Find all solutions of the equation:

$$\left(\sin x - 1\right)\left(\sqrt{3}\tan x + 1\right) = 0$$

Solution: We start by using the zero-product property which states that if the product of two factors is zero, then one of the factors must be zero. Hence, we set both factors to zero and solve. We start with

$$\sin x - 1 = 0 \implies \sin x = 1$$

This equation is satisfied by $\alpha = \frac{\pi}{2}$. Then this equation is satisfied by

$$x = \frac{\pi}{2} + 2\pi n$$

only (note that from the remark above we would expect that the angle $\pi - \frac{\pi}{2}$ has the same trig value but this actually gives the exact same angle). Further, we now set the other factor in the equation to zero and solve to find that:

$$\sqrt{3}\tan x + 1 = 0 \implies \tan x = -\frac{1}{\sqrt{3}}$$

Using the unit circle (or the right hand rule, or whatever other method you find easiest), we determine that this equation is satisfied by $\alpha = -\frac{\pi}{6}$. Now, here, we also need to account for coterminal angles as done before, but we note that even further tangent is actually periodic with period $P = \pi$. Hence it is easier to simply use the fact that the equation must then be satisfied by

$$x = -\frac{\pi}{6} + \pi n$$

Combining these, we have the equation is satisfied by all angles given by

$$x = \frac{\pi}{2} + 2\pi n, -\frac{\pi}{6} + \pi n$$

<u>**Example 6.5.4**</u>: Find all solutions of $2\cos^2 x + 3\sin x - 3 = 0$ in $[0, 2\pi)$.
We start by using a Pythagorean Identity to rewrite this equation solely in terms of sine functions. From this, we find

$$\begin{aligned} 2\cos^2 x + 3\sin x - 3 &= 0, \\ 2\left[1 - \sin^2 x\right] + 3\sin x - 3 &= 0, \\ -2\sin^2 x + 3\sin x - 1 &= 0 \end{aligned}$$

Now notice the above equation is a factorable quadratic equation in the variable $y = \sin x$. Hence, we can write this as

$$\Big(-2\sin x + 1 \Big)\Big(\sin x - 1 \Big) = 0$$

From here, we use the approach of the past example to solve. Note that setting the first factor to zero yields the solutions $x = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n$ while setting the second factor to zero yields the solutions $x = \frac{\pi}{2} + 2\pi n$. Combining these and taking only the solutions in the desired interval yields

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

<u>**Remark**</u>: Some trigonometric equations are most easily solved by squaring both sides of the equation and then using relevant identities to simplify. In these cases, one needs to be sure to check all answers found as certain operations applied to equations can lead to <u>extraneous</u> solutions that don't actually solve the original equation and/or aren't in the domain of the original equation (one such operation that leads to extraneous solutions is squaring both sides of an equation).