1 Chapter 6.6: Trigonometric Equations II

Solving Trigonometric Equations
In this section, we will study trigonometric equations with multiple angles and angles given as linear combinations of the solution variable. The procedure to solve is exactly the same. Remember to take into account symmetries of sine and cosine.

Example 6.6.1: Solve $\sin(2x) = \frac{1}{2}$ over the interval $[0, 2\pi)$.
Solution: We start by writing

$$\sin(2x) = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right)$$

Now setting the angles equal to each other in the above equation and including coterminal angles, we have:

$$2x = \frac{\pi}{6} + 2\pi n,$$
$$x = \frac{\pi}{12} + \pi n$$

and

$$2x = \frac{5\pi}{6} + 2\pi n,$$
$$x = \frac{5\pi}{12} + \pi n$$

Now we simply see which of the angles from these two solution sets are in the desired domain. Hence, the final solution is

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

Example 6.6.2: Solve $\sin(2x) = \sin\left(\frac{3\pi}{4} - x\right)$ over the interval $[0, 2\pi)$.
Solution: In this example, we note that the sine function is periodic with period $P = 2\pi$ and that $\sin x = \sin(\pi - x)$. Hence, we set up the following two equations:

$$2x = \left(\frac{3\pi}{4} - x\right) + 2\pi n, \qquad \text{and} \qquad 2x = \left[\pi - \left(\frac{3\pi}{4} - x\right)\right] + 2\pi n$$

Solving these two equations for x then yields:

$$x = \frac{\pi}{4} + \frac{2\pi n}{3}, \qquad \text{and} \qquad x = \frac{\pi}{4} + 2\pi n$$

Then, taking into account the given interval which we are supposed to solve over, the final solution is given by

$$x = \frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}$$

Example 6.6.3: Find all solutions of $\cot(3x) = 1$ over the interval $0° \leq x \leq 360°$.
Solution: We start by finding α such that

$$\cot(3x) = 1 = \cot\left(\frac{\pi}{4}\right)$$

Hence, we then have

$$3x = \frac{\pi}{4} + \pi n \implies x = \frac{\pi}{12} + \frac{\pi n}{3}$$

Hence, the final solution in the interval we were supposed to solve over is given by

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12} = 15°, 75°, 135°, 195°, 255°, 315°$$

Example 6.6.4: Solve $\sin(2x) + \sin(3x) = 0$ over $[0, 2\pi)$.
Solution: The first step is to use our sum-to-product formula. This may be hard to see at first, however, the idea behind doing this is that we want to then apply the zero-product property. Applying the sum-to-product formula allows us to rewrite the equation as

$$2\sin\left(\frac{5x}{2}\right)\cos\left(-\frac{x}{2}\right) = 0$$

We now set each factor to zero independently and use the unit circle to come up with our α's:

$$\sin\left(\frac{5x}{2}\right) = 0 = \sin(0) \qquad \text{and} \qquad \cos\left(-\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right) = 0 = \cos\left(\frac{\pi}{2}\right)$$

where we applied the even property of cosine in the second equation. From the first equation above, we then include symmetries and coterminal angles by writing

$$\frac{5x}{2} = 0 + 2\pi n, \qquad \frac{5x}{2} = \pi + 2\pi n,$$
$$x = \frac{4\pi n}{5}, \qquad x = \frac{2\pi}{5} + \frac{4\pi n}{5}$$

Similarly, from setting the second factor to zero we can write

$$\frac{x}{2} = \frac{\pi}{2} + 2\pi n, \qquad \frac{x}{2} = \frac{3\pi}{2} + 2\pi n,$$
$$x = \pi + 4\pi n, \qquad x = 3\pi + 4\pi n$$

Now we go through each of these four solution sets and pick out the angles that lie in the stated interval with which we were supposed to solve over. Doing this, we find that the final solution is

$$x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, \pi$$

Example 6.6.5: Solve $\frac{\pi}{4} + 3\sin^{-1}(x+1) = \frac{5\pi}{4}$.
Solution: We start by isolating the inverse trigonometric function. Doing this, we find:

$$\sin^{-1}(x+1) = \frac{\pi}{3}$$

We now take the sine of both sides of the equation. On the right hand side, arcsine and sine cancel each other returning just the argument, while on the left hand side, we can simplify the expression using the unit circle. Hence we find

$$x + 1 = \frac{\sqrt{3}}{2} \implies x = \frac{\sqrt{3}-2}{2}$$

Example 6.6.6: Solve $\sin(2x) + \cos(2x) = \sqrt{2}$ over $[0, 2\pi)$.
Solution: One way to solve this problem is to start by squaring both sides of the equation. One might think to do this for two reasons: (1) we know that $\sin^2\theta + \cos^2\theta = 1$ and (2) that $2\sin\theta\cos\theta = \sin 2\theta$. Utilizing these two facts, we find the following:

$$\begin{aligned} \sin(2x) + \cos(2x) &= \sqrt{2}, \\ \sin^2(2x) + 2\sin(2x)\cos(2x) + \cos^2(2x) &= 2, \\ 1 + \sin(4x) &= 2, \\ \sin(4x) &= 1 = \sin\left(\frac{\pi}{2}\right), \end{aligned}$$

where in the last line we found that $\alpha = \frac{\pi}{2}$. Hence we find that our solution is

$$4x = \frac{\pi}{2} + 2\pi n \implies x = \frac{\pi}{8} + \frac{\pi n}{2}$$

which implies the final solution is $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$. Checking for extraneous solutions, however, we find:

$$\begin{aligned} \text{x} = \frac{\pi}{8}: &\qquad \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}, \\ \text{x} = \frac{5\pi}{8}: &\qquad \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = -\sqrt{2}, \\ \text{x} = \frac{9\pi}{8}: &\qquad \sin\left(\frac{9\pi}{4}\right) + \cos\left(\frac{9\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}, \\ \text{x} = \frac{13\pi}{8}: &\qquad \sin\left(\frac{13\pi}{4}\right) + \cos\left(\frac{13\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}, \end{aligned}$$

Hence the final solution is $x = \frac{\pi}{8}, \frac{9\pi}{8}$.