1 Chapter 7.1: Law of Sines

<u>**Oblique Triangles**</u>

Recall that an <u>oblique triangle</u> is any triangle without a right angle. Consider the oblique triangle $\triangle ABC$. Notice that we can break these into two cases: (i) where $\angle B$ is acute or $\angle B$ is obtuse. These are sketched in Figure 1

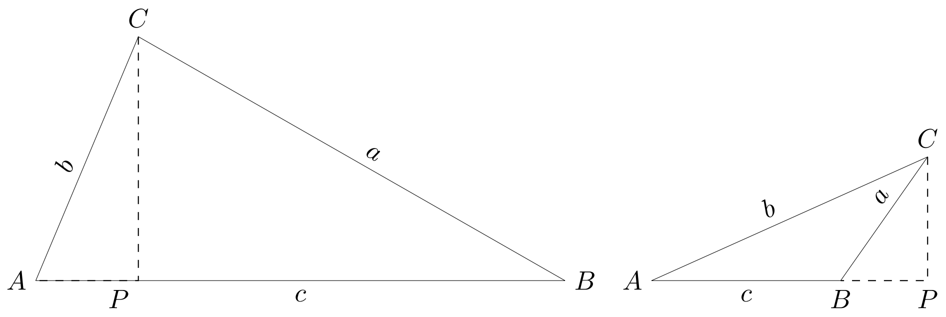

Figure 1: On the left is the case of acute $\angle B$ while on the right is the case of obtude $\angle B$. In the notes notice we adopt the notation $\overline{CP} = h$

Take the case where $\angle B$ is acute, seen on the left in Figure 1. We want to calculate $h = \overline{CP}$. Looking at the two right triangles, we see that:

$$\sin A = \frac{h}{b}, \quad \sin B = \frac{h}{a}$$

Lets now look at the case where $\angle B$ is obtude, seen on the right in Figure 1. We again want to calculate $h = \overline{CP}$. Looking at the two right triangles, we see that:

$$\sin A = \frac{h}{b}, \quad \sin(180^\circ - B) = \sin B = \frac{h}{a}$$

where we used the symmetry of sine in the second equation

<u>**Law of Sines**</u>

The Law of sines utilizes the facts discussed above to make a statement true for all angles and side lengths in oblique triangles. Notice that in either of the cases above, we can rearrange the equations so that $h = b \sin A = a \sin B$. We now divide this equation by the product ab to find that:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

We could now take the exact same triangles and draw altitudes from the points A and B to find the same relation between the other side lengths and angles in the triangle. The combination of these is the <u>Law of Sines</u> which plainly states that for any oblique triangle we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1}$$

<u>**Example 7.1.1**</u>: Find side length c in an oblique triangle given that $a = 6$, $A = 30^\circ$, and $C = 72^\circ$.
Solution: From the Law of Sines we have that

$$\frac{\sin A}{a} = \frac{\sin C}{c} \implies \frac{\sin(30^\circ)}{6} = \frac{\sin(72^\circ)}{c} \implies c = 12\sin(72^\circ) \approx 11.41$$

<u>**Example 7.1.2**</u>: Solve oblique triangle $\triangle ABC$ with $a = 16$, $A = 65^\circ$, and $b = 30^\circ$.
Solution: From the Law of Sines, it follows that

$$\frac{\sin(65^\circ}{16} = \frac{\sin B}{30} \implies \sin B = \frac{30}{16}\sin(65^\circ) \approx 1.69$$

Hence there are no solutions because the range of sine is $[-1, 1]$ and 1.69 is outside this range.

<u>**Example 7.1.3**</u>: Solve oblique triangle $\triangle ABC$ with $a = 30$, $c = 50$, and $C = 60^\circ$.
Solution: From the Law of Sines, it follows that

$$\frac{\sin(60^\circ)}{50} = \frac{\sin A}{30} \implies \sin A = \frac{30}{50}\sin(60^\circ) \implies A \approx 31.3^\circ$$

We can now solve for angle B with the knowledge that three angles in a triangle must sum to 180°. Hence we find $B = 88.7^\circ$. We now use the Law of Sines again to solve for side length b since

$$\frac{\sin(60^\circ)}{50} = \frac{\sin(88.7^\circ)}{b} \implies b = \frac{50\sin(88.7^\circ)}{\sin(60^\circ)} \approx 57.7$$