1 Chapter 7.2: Law of Cosines

Law of Cosines
The Law of Cosines is a generalization of the Pythagorean Theorem that is true for all triangles, namely oblique triangles. It can be used to solve for triangles where two sides and an angle are known or where all three side lengths are known. We again consider the oblique triangle $\triangle ABC$ this time with the point A fixed at $(0,0)$. We again have only two cases illustrated by Figure 1

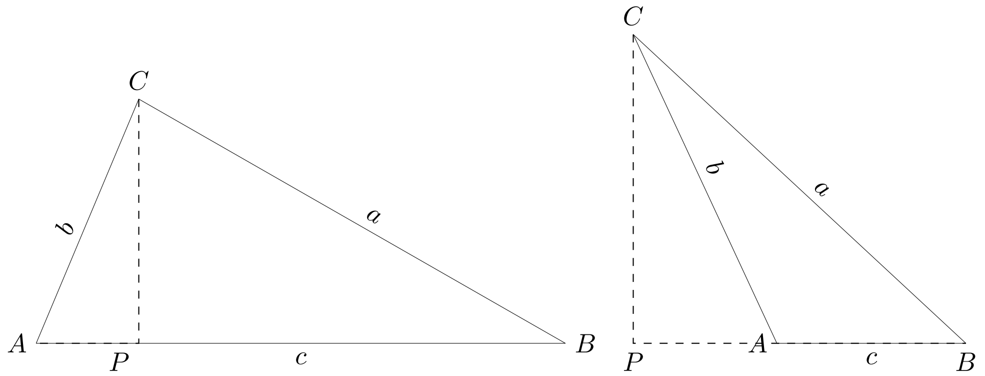

Figure 1: On the left is the case of acute $\angle B$ while on the right is the case of obtude $\angle B$. In the notes notice we adopt the notation $\overline{CP} = y$ and $\overline{AP} = x$.

Notice that if $A = (0,0)$ then $B = (c,0)$. Then by definitions of cosine and sine, we have

$$\cos A = \frac{x}{b}, \qquad \sin A = \frac{y}{b}$$
$$x = b\cos A, \qquad y = b\sin A$$

Hence, it follows that the other point in the triangle is given by $C = (b\cos A, b\sin A)$. We now apply the distance formula on the points B and C to find that:

$$\begin{aligned} a &= d(B,C), \\ a^2 &= d(B,C)^2, \\ &= (b\cos A - c)^2 + (b\sin A - 0)^2, \\ &= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A, \\ &= b^2\Big(\sin^2 A + \cos^2 A\Big) - 2bc\cos A + c^2, \\ a^2 &= b^2 + c^2 - 2bc\cos A \end{aligned}$$

This is one form of the Law of Cosines. The other forms are found by reworking this derivation with other vertices fixed at the origin.

Remark: Notice that if $A = 90°$, then a is the hypotenuse of the triangle and the Law of Cosines states that $a^2 = b^2 + c^2$ which is exactly Pythagorean's Theorem.
Law of Cosines In $\triangle ABC$ with sides a, b, c, we have that:

$$a^2 = b^2 + c^2 - 2bc\cos A, \tag{1}$$
$$b^2 = a^2 + c^2 - 2ac\cos B, \tag{2}$$
$$c^2 = a^2 + b^2 - 2ab\cos C, \tag{3}$$

or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \tag{4}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \tag{5}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}, \tag{6}$$

Example 7.2.1: In $\triangle ABC$, find c given that $a = 6$, $b = 3\sqrt{3}$, and $C = 30°$.
Solution: We simply plug into the Law of Cosines to find that:

$$c^2 = a^2 + b^2 - 2ab\cos C = (6)^2 + (3\sqrt{3})^2 - 2(6)(3\sqrt{3})\cos(30°) = 36 + 27 - 54 = 9 \implies c = 3$$

Example 7.2.2: Solve $\triangle ABC$ given that $a = 6$, $b = 3$, and $c = 4$.
Solution: We utilize the angle forms of the Law of Cosines to see that:

$$\begin{aligned} A &= \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) = \cos^{-1}\left(-\frac{11}{24}\right) \approx 117.3°, \\ B &= \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \cos^{-1}\left(\frac{43}{48}\right) \approx 26.4°, \\ C &= \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) = \cos^{-1}\left(\frac{29}{36}\right) \approx 36.3° \end{aligned}$$

Notice that we could have simplified our calculations by instead of applying the Law of Cosines a third time to find C but instead just rembering that all three angles in a triangle sum to $180°$, which implies $C = 180° - A - B = 36.3°$.

Example 7.2.3: Solve $\triangle ABC$ with $C = 60°$, $c = 50$, and $a = 30$.
Solution: Use the Law of Cosines to see that

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C, \\ 2500 &= 900 + b^2 - 30b, \\ 0 &= b^2 - 30b - 1600, \\ b &\approx -27.72, 57.72 \end{aligned}$$

We take the positive value as side lengths of a triangle have to be positive. From here, we can again apply the Law of Cosines to solve for the angle B:

$$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(0.0228) \approx 88.7°$$

We find the remaining angle by recalling all angles in a triangle have to sum to $180°$. Hence, we have that $A = 180° - B - C = 31.3°$.