1 Chapter 7.6: Polar Coordinates

- Generally when we talk about coordinate systems our default setting is to use the Cartesian coordinates (also called rectangular coordinates) where a point in the plane is descibed in terms of the horizontal and vertical distances from the origin
  - In Cartesian (rectangular) coordinates, the point $(1, 1)$ is 1 unit to the right and 1 unit up of the origin
- Today we are going to talk about Polar coordinates which describe a point in the plane using a distance r and angle θ
  - For example $P = (r, \theta)$ is a point in Polar coordinates that is distance r from the origin in the direction of the angle θ
- In the Polar coordinate system, the horizontal axis is called the polar axis
- r is the "directed distance" from the pole O (the origin) to the point P
- θ is a directed angle from the polar axis to the line segment $\overline{OP}$ in counterclockwise direction (the angle can be measured in degrees or radians)
  - Notice the pole O has infinite representation $(0, \theta)$ where θ can be any real number positive or negative
  - Similarly, notice (r, θ) and $(r, \theta + 2\pi n)$ will represent the same points for θ measured in radians
  - r is allowed to be negative in this context (why we call it a "directed distance") so (r, θ) and $(-r, \theta + \pi)$ will represent the same point

Converting Between Polar and Cartesian Coordinates
We consider a point in the plane which we want to represent in both coordinate systems, shown in Figure 1.

- Consider we know the Cartesian representation, so x_p and y_p are known. Then it follows that there is a right triangle formed with side lengths x_p, y_p, and r where r is the hypotenuse. Hence we can use Pythagorean's Theorem to determine
$$r^2 = x_p^2 + y_p^2 \tag{1}$$
- Further, from our trig definitions we can see that
$$\tan\theta' = \frac{y_p}{x_p} \implies \theta' = \tan^{-1}\left(\left|\frac{y_p}{x_p}\right|\right) \tag{2}$$
where θ' is the reference angle for θ. Hence, given a point $P = (x_p, y_p)$ we can use the above equations to calculate $P = (r, \theta)$, the polar representation of the point.
- Now consider we know the Polar representation of the point, so r and θ are known but not x_p and y_p.
- Then we can use our trigonometric definitions to see that
$$x = r\cos\theta, \qquad y = r\sin\theta \tag{3}$$

Example 7.6.1: Convert to Cartesian (rectangular) coordinates: $(-3, 60^\circ)$ and $(2, -\frac{\pi}{4})$.
Solution: For the first point, we simply calculate:
$$x = r\cos\theta = -3\cos(60^\circ) = -3\left(\frac{1}{2}\right) = -\frac{3}{2},$$
$$y = r\sin\theta = -3\sin(60^\circ) = -3\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2},$$

to find that the point $(-3, 60^\circ)$ in the Polar coordinate plane can be represented by $(-\frac{3}{2}, -\frac{3\sqrt{3}}{2})$ in the Cartesian coordinate plane. Similarly for the second point, we calculate
$$x = r\cos\theta = 2\cos\left(-\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2},$$
$$y = r\sin\theta = 2\sin\left(-\frac{\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2},$$

to find that the point $(2, -\frac{\pi}{4})$ in the Polar coordinate plane can be represented by $(\sqrt{2}, -\sqrt{2})$ in the Cartesian coordinate plane.

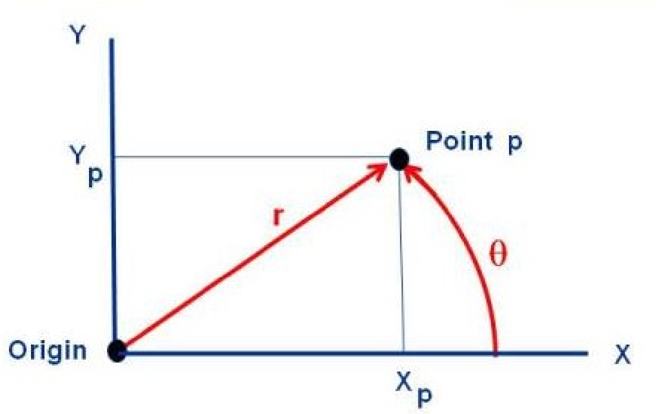

Figure 1: An image of a point $P = (x_p, y_p) = (r, \theta)$ shown in both Cartesian and Polar coordinates

Example 7.6.2: Convert the point in Cartesian coordinates, $(1, -1)$, to Polar coordinates with $r > 0$ and $0 \leq \theta \leq 2\pi$.
Solution: We start by noting that the point lies in quadrant III with $x = 1$, $y = -1$. Hence, we can find

$$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}, \qquad \theta' = \tan^{-1}\left|\frac{-1}{1}\right| = \frac{\pi}{4}$$

Hence the point $(1, -1)$ in Cartesian coordinates can be represented by $\left(\sqrt{2}, \frac{5\pi}{4}\right)$.
Remark: We can also represent equations in Polar coordinates. We can translate equations between coordinates using the above techniques as well.
Example 7.6.3: Convert $(x-1)^2 + (y+1)^2 = 2$ to polar form.
Solution: Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Then we can write:

$$\begin{aligned}
\Big(x - 1\Big)^2 + \Big(y + 1\Big)^2 &= 2, \\
\Big(r\cos(\theta) - 1\Big)^2 + \Big(r\sin(\theta) + 1\Big)^2 &= 2, \\
r^2\Big(\cos^2(\theta) + \sin^2(\theta)\Big) + r\Big(2\sin(\theta) - 2\cos(\theta)\Big) + 2 &= 2, \\
r^2 + r\Big(2\sin(\theta) - 2\cos(\theta)\Big) &= 0, \\
r\Big(r + 2\sin(\theta) - 2\cos(\theta)\Big) &= 0, \\
r = 2\cos(\theta) - 2\sin(\theta)
\end{aligned}$$

Notice that to get the final equation we actually utilized the zero-product property. In this instance, we can then only take one factor because the factor $r = 0$ (which is just the origin) is contained in our final equation.