1 Chapter 8.2: Systems of Linear Equations in Three Variables

Example 8.2.1: Solve the system

$$\begin{aligned} 2x + 5y &= 1, \\ x - 3y + 2z &= 1, \\ -x + 2y + z &= 7, \end{aligned}$$

Solution: We can add equation (3) to equation (2) and replace equation (2) with the result so that our system of equations now reads:

$$\begin{aligned} 2x + 5y &= 1, \\ -y + 3z &= 8, \\ -x + 2y + z &= 7, \end{aligned}$$

We can then add equation (1) to double equation (3) and replace equation (3) with the result to find:

$$\begin{aligned} 2x + 5y &= 1, \\ -y + 3z &= 8, \\ 9y + 2z &= 15, \end{aligned}$$

Next we add nine times equation (2) to equation (3) and replace equation (3) with the result to get:

$$\begin{aligned} 2x + 5y &= 1, \\ -y + 3z &= 8, \\ 29z &= 87, \end{aligned}$$

We can now solve the last equation for z and use a method called back substitution to find the remaining variables. From equation (3), we can find $z = 87/29 = 3$. Now substitute this value of z into equation (2) and solve for y to find that $y = 1$. Substitute both of these values for y, z into equation (1) and solve for x to find that $x = -2$. Hence the solution is $(-2, 1, 3)$.

Operations that Produce Equivalent Systems

1. Interchanging the position of any two equations
2. Multiplying any equation by a nonzero constant
3. Adding/Subtracting a multiple of one equation with another equation

Other Remarks

- If during the process to solve you find an equation of the form $0 = k$ where k is an nonzero number, then the system is inconsistent and there are no solutions
- If during the process to solve you find an equation of the form $0 = 0$, then the system is dependent and there are infinitely many solutions

Graphical Interpretation

- The graph of $ax + by = c$ is a line. Two lines either intersect at a point, at every point on the line, or not at all.
- The graph of $ax + by + cz = d$ is a plane. Three planes can intersect each other in the following ways
  - Intersect in a single point $\Longrightarrow$ the system has one solution
  - Intersect in one line $\Longrightarrow$ the system has infinitely many solutions
  - Coincide with each other $\Longrightarrow$ the system has infinitely many solution
  - Are all parallel $\Longrightarrow$ the system has no solution
  - Two are parallel and the third intersects both $\Longrightarrow$ the system has no solution
  - The three planes have no points in common $\Longrightarrow$ the system has no solution