1 Chapter 8.3: Partial Fraction Decomposition

- In the past we have been finding the least common denominator (LCD) in order to simplify two rational fractions into a single fraction, i.e.

$$\frac{2}{x+3}+\frac{3}{x-1}=\frac{2(x-1)}{(x+3)(x-1)}+\frac{3(x+3)}{(x-1)(x+3)}=\frac{2x-2}{(x+3)(x-1)}+\frac{3x+9}{(x+3)(x-1)}=\frac{5x+7}{(x+3)(x-1)}$$

- In this section, we are going to be reversing this process. In the above equation, the two fractions on the leftmost side are called partial fractions and their sum is called the partial fraction decomposition of the rational expression on the rightmost side of the above equation.
- To find the partial fraction decomposition of a rational expression, we factor the denominator and then use the method of undetermined coefficients to solve for the correct factors. This is illustrated through examples below

Example 8.3.1: Find the partial fraction decomposition of

$$\frac{3x^2+4x+3}{x^3-x}$$

Solution: We first note that the denominator can be factored as $x^3-x=x(x^2-1)=x(x+1)(x-1)$. We then use the method of undetermined coefficients to guess the correct factors:

$$\begin{aligned}\frac{3x^2+4x+3}{x^3-x}&=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1},\\ 3x^2+4x+3&=A(x+1)(x-1)+B(x)(x-1)+C(x)(x-1),\\ 3x^2+4x+3&=(A+B+C)x^2+(C-B)x+(-A\end{aligned}$$

We now just equate the coefficients on both sides. So we have a system of three equations for three variables (A,B,C):

$$-A=3,\qquad C-B=4,\qquad A+B+C=3$$

The solution to this is given by $(A,B,C)=(-3,1,5)$ so that the partial fraction decomposition is

$$\frac{3x^2+4x+3}{x^3-x}=-\frac{3}{x}+\frac{1}{x+1}+\frac{5}{x-1}$$

Remark: If the denominator contains a repeated root, then you must use successive powers of that root in the method of undetermined coefficients. This is illustrated in the example below.
Example 8.3.2: Find the partial fraction decomposition of

$$\frac{x+5}{x(x-1)^2}$$

Solution: We have a repeated root. Hence in the method of undetermined coefficients we have to make the guess

$$\begin{aligned}\frac{x+5}{x(x-1)^2}&=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2},\\ x+5&=A(x-1)^2+B(x)(x-1)+Cx,\\ x+5&=(A+B)x^2+(-2A-B+C)x+A\end{aligned}$$

Equating coefficients and solving the linear system of equations yields $(A,B,C)=(5,-5,6)$. Hence the partial fraction decomposition is

$$\frac{x+5}{x(x-1)^2}=\frac{5}{x}-\frac{5}{x-1}+\frac{6}{(x-1)^2}$$

Remark: If the denominator contains an irreducible quadratic root, then you must use a linear factor in the method of undetermined coefficients. This is illustrated in the example below.
Example 8.3.2: Find the partial fraction decomposition of

$$\frac{3x^2+5x-2}{x(x^2+2)}$$

Solution: The denominator is already factored. We set up the method of undetermined coefficients as follows:

$$\begin{aligned}\frac{3x^2+5x-2}{x(x^2+2)} &= \frac{Ax+B}{x^2+2}+\frac{C}{x},\\ 3x^2+5x-2 &= (Ax+B)(x)+C(x^2+2),\\ 3x^2+5x-2 &= (A+C)x^2+Bx+2C,\end{aligned}$$

Equating coefficients yields a system of coefficients for our unknown constants A, B, C. Solving this system yields $(A, B, C) = (4, 5, -1)$ so the partial fraction decomposition is

$$\frac{3x^2+5x-2}{x(x^2+2)} = \frac{4x+5}{x^2+2} - \frac{1}{x}$$