Korteweg–de Vries Equation and Generalizations. I. A Remarkable Explicit Nonlinear Transformation

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An explicit nonlinear transformation relating solutions of the Korteweg–de Vries equation and a similar nonlinear equation is presented. This transformation is generalized to solutions of a one-parameter family of similar nonlinear equations. A transformation is given which relates solutions of a “forced” Korteweg–de Vries equation to those of the Korteweg–de Vries equation.

1. INTRODUCTION

Interest in nonlinear dispersive wave equations has focused recently on the simplest model equation of this type, namely,

\[ u_t + uu_x + u_{xxx} = 0, \tag{1} \]

where subscripts denote partial differentiation. Korteweg and de Vries\(^1\) first derived (1) (the KdV equation) in their study of long water waves in a (relatively shallow) channel. Recently, this equation has been derived in plasma physics\(^2,3\) and in studies of anharmonic (nonlinear) lattices.\(^4,5\) Existence and uniqueness of solutions of the KdV equation for appropriate initial and boundary conditions have recently been proved by Sjöberg.\(^6\) The simplest modification of the nonlinear term in (1) leads to a similar equation

\[ v_t + v^2v_x + v_{xxx} = 0, \tag{2} \]

which also arises in the study of anharmonic lattices.\(^5\)

The present paper is the first in a prospective series of works on properties and solutions of the KdV equation and its generalizations.\(^7\)

I am privileged to write this first paper in the series, which presents a remarkable explicit nonlinear transformation between solutions of (1) and (2). Also, a transformation to an accelerating coordinate system is presented which relates solutions of (1) and a “forced” KdV equation. The second paper in the series\(^8\) will discuss the existence of conservation laws and constants of motion for these equations. Also, it will show how the nonlinear transformation leads to associated eigenvalue problems. The third paper will show that the KdV equation governs small but finite perturbations from homogeneous equilibrium for a wide class of nonlinear dispersive systems. The fourth will show how the KdV equation and some generalizations can be viewed as Hamiltonian systems. The fifth paper will give a detailed discussion of polynomial conservation laws, including uniqueness and nonexistence proofs. The sixth paper in this series will consider the associated eigenvalue problems and will show how a study of them leads to exact general solution of the KdV equation. These papers will be referred to as I, II, III, IV, V, and VI.

2. TRANSFORMATION RELATING EQUATIONS (1) AND (2)

Equations (1) and (2) are particularly interesting, since they are exceptional among equations of the form

\[ u_t + u^pu_x + u_{xxx} = 0, \quad p = 1, 2, 3, \ldots, \tag{3} \]

as the only ones possessing more than three “polynomial conservation laws” (not trivially equivalent; see II, Sec. 2. This result will be proved in V).

The similarity between (1) and (2), both in form and in possession of many polynomial conservation laws (see II), suggested that their solutions might be intimately related. A detailed comparison of these laws led to the discovery that if \( v \) satisfies (2), then \( u \), defined by

\[ u \equiv v^9 \pm (-6)^{1/9}v_x, \tag{4} \]

was satisfied by

\[ \text{J. Math. Phys. 9, 1204 (1968), following paper.} \]
satisfies (1). By explicit calculation, in fact,
\[ u_t + uu_x + u_{xxx} = \left(2v \pm (-6)^{1/2} \frac{\partial}{\partial x}\right)(v, v_x, v_{xx}). \] (5)
The presence of the operator \(2v \pm (-6)^{1/2}\frac{\partial}{\partial x}\) hinders us from concluding inversely that if \(u\) satisfies (1), then any solution \(v\) of the Riccati equation (4) is a solution of (2).

The reader need not be concerned about the occurrence of the imaginary coefficient in (4). It is an historic accident that we chose to study (1) and (2) with the signs of the terms as given. For (1), the particular choice of signs is unimportant, since appropriate changes of sign of the variables yield transformations between any two possibilities. (See V for transformation properties of the KdV equation.) However, for (2), the relative sign of the last two terms is invariant to such transformations, but can be reversed by the substitution \(v \to iv\). We could have confined our discussion here to real solutions by considering two versions of (2), one with like and one with unlike signs.

The transformation takes (2) with cubic nonlinearity into the quadratically nonlinear KdV equation (1). It is rare and surprising to find a transformation between two simple nonlinear partial differential equations of independent interest. One is reminded of the Hopf–Cole transformation of the quadratically nonlinear Burgers equation into the linear heat conduction (diffusion) equation. A number of investigators (including us) have attempted unsuccessfully to find a similar simple linearizing transformation for the KdV equation, but a complicated one will be given in VI.

3. A GENERALIZATION

A generalization\(^{11}\) of the transformation (4), which in II is used to prove the existence of infinitely many conservation laws, is that one-parameter family of nonlinear equations similar to (1) and (2), but containing both types of nonlinear terms simultaneously, can be transformed into (1). Noting that (1) is invariant to Galilean transformation (again, see V), whereas (2) is not, we define
\[ t' \equiv t, \quad x' \equiv x - \frac{3}{2\epsilon^2} t, \] (6)
\[ u(x, t) \equiv u(x', t') + \frac{3}{2\epsilon^2}, \] (7)
\[ u(x, t) \equiv \frac{\epsilon}{\sqrt{6}} w(x', t') + \frac{\sqrt{6}}{2\epsilon}, \] (8)
where the specific dependence on the arbitrary parameter \(\epsilon\) has been chosen for convenience in II. Then (1) remains invariant, of course, but (2) (dropping the primes) becomes
\[ w_t + (w + 1/4\sqrt{6}\epsilon^2)w_x + w_{xxx} = 0, \] (9)
and (4) (with the plus sign) becomes
\[ u \equiv w + i\epsilon w_x + \frac{1}{4}\epsilon^2 w^2. \] (10)
We observe that (9) reduces to (1) for \(\epsilon = 0\), and after the rescaling \(w' \equiv (\epsilon/\sqrt{6})w\) it reduces to (2) for \(\epsilon \to \infty\).

4. TRANSFORMATION TO ACCELERATING COORDINATE SYSTEM

The KdV equation (1) can be generalized by adding a time-dependent "forcing term," and for convenience we write it as
\[ u_t + uu_x + u_{xxx} = y_{tt}, \] (11)
where we assume that \(y_{tt} = y_{tt}(t)\) is a known function. This equation arises in the study of ion-acoustic waves.\(^ {5,12}\) It also arises in a study of the propagation of electrostatic waves through an ion sheath where \(y_{tt} = 1.\)\(^ {13}\)

We now give a transformation which reduces the "forced" KdV equation (11) to the KdV equation (1). This transformation is also applicable to more general equations (e.g., the Burgers equation) where \(u_{xxx}\) is replaced by any arbitrary function of \(x\) derivatives of \(u\) not depending on either \(u\) itself or explicitly on \(x\) or \(t\). Define new variables
\[ t' \equiv t, \quad x' \equiv x - y(t), \] (12)
\[ u(x, t) \equiv u'(x', t') + y_{t}(t'), \] (13)
Direct substitution of this transformation into (11) shows that the KdV equation (1) is indeed obtained for the primed variables. We note a strong similarity to the Galilean transformation (6) and (7).

The physical interpretation of this transformation is clear. The quantity \(y(t)\) represents a time-dependent translation of the \(x\) axis, and, therefore, the forcing

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\(^{10}\) J. D. Cole, Quart. Appl. Math. 9, 225 (1951).
\(^{11}\) I am grateful to C. S. Gardner for this generalization.
\(^{12}\) S. H. Lam and C. Berman, Department of Aerospace and Mechanical Sciences, Princeton University (private communication).
term in (11) may be interpreted as due solely to an acceleration of the $x$ axis.

Since we have assumed only that $y(t)$ is known, we have the freedom to set $y(0) = y_{1}(0) = 0$. With this information the transformation becomes particularly useful, since the initial values for the two equations are identical:

$$u(x, 0) = u'(x', 0) = u'(x, 0).$$ (14)

Therefore, if the solution of the KdV equation (1) is known, then the full solution is obtained from (12) and (13) with only two simple quadratures to obtain $y(t)$ and $y(t)$.

Note added in proof: The transformation (12) and (13) was used by Moore$^{14}$ for studying the viscous boundary layer on an accelerating semi-infinite flat plate. I wish to thank H.-H. Chiu of the Department of Aeronautics and Astronautics at New York University for this reference.

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Korteweg–de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion*

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With extensive use of the nonlinear transformations presented in Paper I of the series, a variety of conservation laws and constants of motion are derived for the Korteweg–de Vries and related equations. A striking connection with the Sturm–Liouville eigenvalue problem is exploited.

1. INTRODUCTION

In this second paper of the series on the properties and solutions of the KdV equation, $u_t + uu_x + u_{xx} = 0$ and its generalizations, we present our current body of knowledge on the existence of conservation laws and of constants of motion (i.e., "temporal invariants") for the KdV equation (1.1) and two similar nonlinear equations (1.2) and (1.9) given in the first paper of this series,$^1$ referred to as I. The present Paper II is meant to be read in conjunction with I, where nonlinear transformations relating solutions of (1.1) to those of (1.2) and (1.9) are given. (References to physical applications are also given there.) Most of the results on the conservation laws and the constants of motion are based on, or are in some way related to, these transformations.

A conservation law associated with an equation such as (1.1) is expressed by an equation of the form

$$T_t + X_x = 0,$$ (1)

where $T$, the conserved density, and $-X$, the flux of $T$, are functionals of $u$. If $T$ is a local functional of $u$, i.e., if the value of $T$ at any $x$ depends only on the values of $u$ in an arbitrarily small neighborhood of $x$, then $T$ is a local conserved density; if $X$ is also local, then (1) is a local conservation law. In particular, if $T$ is a polynomial in $u$ and its $x$ derivatives and not dependent explicitly on $x$ or $t$, then we call $T$ a polynomial conserved density; if $X$ is also such a polynomial, we call (1) a polynomial conservation law.

[We need never allow for dependence on $t$ derivatives of $u$, since (1.1) permits them to be eliminated in favor of $x$ derivatives; similarly with the other such equations we deal with.] In Sec. 2 we present a number of polynomial conservation laws which have been found explicitly, and in Sec. 3 we prove that there are infinitely many of them for each of (1.1), (1.2), and (1.9).

There is a close relationship between constants (of motion) and conservation laws. For example, if one