STRETCHING OF HEATED THREADS WITH TEMPERATURE-DEPENDENT VISCOSITY: ASYMPTOTIC ANALYSIS

P. D. Howell†, J. J. Wylie*, H. Huang† and R. M. Miura‡

†Centre for Industrial and Applied Mathematics, Mathematical Institute
University of Oxford, Oxford OX1 3LB, UK

*Department of Mathematics
City University of Hong Kong, Kowloon, Hong Kong

†Department of Mathematics and Statistics
York University, Toronto, Ontario M3J 1P3, CANADA

‡Department of Mathematical Sciences
New Jersey Institute of Technology, Newark, N.J. 07102, USA

Contributed in honor of Fred Wan on the occasion of his 70th birthday

ABSTRACT. We consider the stretching of a thin cylindrical thread with viscosity that depends on temperature. The thread is pulled with a prescribed force while receiving continuous heating from an external axially nonuniform heater. We use the canonical equations derived by Huang et al. (2007) and consider the limit of large dimensionless heating rate. We show that the asymptotic solution depends only on the local properties of the heating near its maximal heating value. We derive a uniformly valid asymptotic solution for the shape and the temperature profiles during the stretching process. We use a criterion to determine when breaking will occur and derive simple analytical expressions for the shape at breaking that clearly show the influence of heating strength and the degree of localization of the heating. The asymptotic shape profiles give good agreement with numerical simulations. These results are applied to the formation of glass microelectrodes.

1. Introduction. The stretching of viscous threads is important in a wide range of applications, e.g., the manufacture of optical fibers [2], the production of tapered optical microscopes [3], and the fabrication of glass microelectrodes [4, 5]. The materials used in these applications typically have viscosities that vary dramatically with temperature. Therefore, external heating of the thread can have significant effects on the stretching of threads and the resulting shapes. In some applications [3, 4, 5], the resulting thread shapes are crucial.

Previous studies of heated threads [4, 5] have shown that the stretching process typically results in two distinctive stages. Initially, the thread is heated, but the viscosity is sufficiently large that the extension is negligible. When the temperature
of the thread reaches a critical value, the viscosity is reduced significantly, and
the thread starts to extend rapidly. As the thread narrows, the local axial stress
increases. Typical materials can only support a finite stress before breaking, and
when the critical breaking stress is exceeded, the thread breaks near the location
where it is narrowest.

In this paper, we investigate how the heating rate and the heating profile affect
the final shape of the thread. We derive a uniformly valid asymptotic solution for
the thread shape when the heating is sufficiently strong. We then use this solution
to determine the shape of the thread when it breaks. At leading order, the thread
shape can be described by a simple analytical expression. Details of the heating
profile away from the maximal heating strength have negligible effect on the thread
shape. We show that the asymptotic solution is a good approximation by comparing
it with the results of numerical simulations. We discuss the importance of the design
of the heating profile for glass microelectrode pullers.

This paper is organized as follows. In Section 2, we give the general dimensionless
governing equations for an incompressible fluid thread with temperature-dependent
viscosity. Natural boundary and initial conditions are specified. This is followed by
the derivation of a uniformly valid asymptotic solution for large heating strength in
Section 3. The dynamics is broken down into three asymptotic phases in time. In
Section 4, we present a dimensionless breaking criterion and computational results
for large heating strength. Section 5 considers other boundary conditions for the
thread, including time-varying forces. We apply the results for threads to the for-
mation of glass microelectrodes in Section 6 and finish the paper with a discussion
in Section 7.

2. Model equations. Following [4, 5], a thread can be modeled as an incompress-
ible fluid with temperature-dependent viscosity. For typical applications, such as
microelectrode manufacturing and optical fiber tip production [3], the effects of
gravity, inertia, and surface tension are negligible. Furthermore, threads are long
and thin, i.e., they have small aspect ratio, so the long-wave approximation can
be used. In addition, the radial variation of the temperature (and thus viscosity)
can be neglected [5]. In the following, we will write down the governing equations
with all variables and parameters in dimensionless form. Equations of this type are
standard for thread pulling and detailed derivations and definitions of dimensionless
quantities can be found in [2, 4, 5] and will not be repeated here.

2.1. Governing equations. Let \( u \) and \( s \) be the dimensionless velocity and cross-
sectional area of the thread, respectively. For large aspect ratios, the momentum
equation [2] is given by

\[
\frac{\partial}{\partial x} \left( \mu s \frac{\partial u}{\partial x} \right) = 0
\]  

(2.1)

where \( \mu \) is the dimensionless viscosity and \( x \) is the dimensionless distance measured
in the axial direction. The mass conservation equation [2] is given by

\[
\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} + s \frac{\partial u}{\partial x} = 0
\]  

(2.2)

where \( t \) is the dimensionless time. The heating equation [5] is given by

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} = \mathcal{H}(x, \theta) \frac{H(x, \theta)}{s^{1/2}}
\]  

(2.3)
where \( \theta \) is the dimensionless temperature. The parameter \( H \) represents the dimensionless heating strength, which characterizes the typical amount of heat absorbed by the thread divided by the amount of heat required to significantly modify the viscosity. The exact definition of this parameter will vary depending on the nature of the heating and absorption properties of the thread. For example, for radiative heating, \( H \) is given by [5]

\[
H = \frac{6\mu_0 \sqrt{s_0 k_B \epsilon_h \alpha \theta_h^4 \sqrt{\pi}}}{\rho c_p \theta_a F_0 [1 - (1 - \alpha)(1 - \epsilon_h)]} \quad (2.4)
\]

where \( \mu_0 \) is the initial viscosity, \( s_0 \) is the initial area, \( k_B \) is the Boltzmann constant, \( \epsilon_h \) is the emissivity of the heater, \( \alpha \) is the absorptivity of the thread, \( \theta_h \) is the temperature of the heater, \( \rho \) is the thread density, \( c_p \) is the specific heat capacity, \( \theta_a \) is the ‘activation temperature change’ required to significantly change the viscosity, and \( F_0 \) is the applied force. The function \( H(x, \theta) \) is the dimensionless heating profile that characterizes the axial variation in the heating intensity. It is normalized to unity at its maximum. The details of \( H(x, \theta) \) depend on the particular design of the heating profile. If we neglect the heat loss to the background and assume that the temperature of the thread is much lower than the heater temperature, then the dimensionless heating profile is a function of \( x \) only [5].

In [4] and [5], examples of detailed heating profiles were considered. Although heating profiles vary widely, we will show that only the local curvature of \( H(x) \) at its maximal value plays a significant role in determining the thread shape in the large \( H \) limit. Therefore, we assume that

\[
H(x) \sim 1 - \frac{\kappa}{2}(x - x^*)^2 \quad \text{as} \quad x \to x^* \quad (2.5)
\]

where the heating profile has a single maximal value at \( x^* \in (0, 1) \) and \( \kappa \) is the local curvature of the heating profile at this maximum.

We will adopt the simple dimensionless viscosity law

\[
\mu = e^{-\theta}, \quad (2.6)
\]

which can accurately characterize viscosity variations over wide temperature ranges for many polymeric and glass materials.

We assume that the thread is fixed at one end and pulled with a fixed applied force at the free end. Therefore, the boundary conditions are given by

\[
u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad \mu s u_x = 1 \quad \text{at} \quad x = \ell(t) \quad (2.7)
\]

where \( \dot{\ell}(t) = u(\ell(t), t) \) and \( \ell(0) = 1 \). In Section 5, we will describe more general situations with different boundary conditions. The momentum equation can be integrated to obtain

\[
\mu s u_x = 1 \quad (2.8)
\]

throughout the entire length of the thread. Initially, we will assume the temperature is fixed and the cross-sectional area is uniform,

\[
\theta = 0 \quad \text{and} \quad s = 1 \quad \text{at} \quad t = 0. \quad (2.9)
\]

The basic equations (2.2), (2.3), and (2.8) can be solved conveniently using the Lagrangian coordinates \( (\xi, \tau) \). The relationship between the Eulerian and Lagrangian variables is given by \( \tau = t \) and \( x = u \) where \( x(\xi, \tau) \) is the location of the
material point that was initially located at \( x = \xi \). The Lagrangian equations are given by

\[
\begin{align*}
    s_\tau &= -e^\theta, \\
    s^{1/2} \theta_\tau &= \mathcal{H}(x), \\
    sx_\xi &= 1,
\end{align*}
\]

subject to the initial and boundary conditions

\[
\begin{align*}
    s(\xi, 0) &= 1, \\
    \theta(\xi, 0) &= 0, \\
    x(0, \tau) &= 0.
\end{align*}
\]

3. **Asymptotic solution for large \( \mathcal{H} \).** In many applications, the dimensionless heating parameter \( \mathcal{H} \) is typically large. For example, in the case of pulling glass microelectrodes, \( \mathcal{H} \sim 10^2 \), which is based on the temperature of the heater and other physical and geometrical parameters of the puller and the glass tubes [5]. For large \( \mathcal{H} \), we will show that the dynamics can be decomposed into two distinct phases. First, the thread heats up sufficiently rapidly that very little stretching occurs. Eventually, when the thread temperature is high enough, the viscosity decreases and the thread begins to stretch. Once the thread has started to extend, the stretching proceeds quickly, and there is relatively little temperature change.

In this section, to understand and quantify these two stages in the thread evolution, we perform an asymptotic analysis of our equations in the limit \( \mathcal{H} \to \infty \).

3.1. **The heating phase.** When \( \mathcal{H} \) is large, (2.10b) implies that the temperature increases rapidly over a timescale \( \tau = O(\mathcal{H}^{-1}) \). To describe this phase, we rescale the time variable

\[
\tau = \mathcal{H}^{-1} T,
\]

so that (3.2) becomes

\[
\begin{align*}
    s_T &= -\mathcal{H}^{-1} e^\theta, \\
    s^{1/2} \theta_T &= H(x), \\
    sx_\xi &= 1.
\end{align*}
\]

We assume the dependent variables have asymptotic expansions in powers of the small parameter \( \mathcal{H}^{-1} \), e.g.,

\[
s = s_0 + \mathcal{H}^{-1} s_1 + \cdots
\]

and similarly for \( \theta \) and \( x \). Solving (3.2) to leading order and applying the initial conditions (2.11), we obtain

\[
\begin{align*}
    s_0 &= 1, \\
    \theta_0 &= TH(x), \\
    x_0 &= \xi.
\end{align*}
\]

Thus, the thread heats up with no leading-order flow.

3.2. **The transition phase.** The asymptotic solution (3.4) breaks down when \( \theta_0 \) becomes so large that the right-hand side of (3.2a) is no longer negligible, i.e., becomes order one. This happens when \( e^T = O(\mathcal{H}) \), and to facilitate the analysis, we define

\[
T = \log(H) + T, \quad \theta = \log(H) + \phi.
\]

Evidently, the thread first achieves the temperature required for significant flow to occur in a neighborhood of the maximum \( x_\ast \) of the heating function \( H(x) \). If we expand the solution (3.4) in such a neighborhood, we find that

\[
\theta_0 \sim \frac{1}{\delta^2} + T - \frac{\kappa}{2\delta^2} (\xi - x_\ast)^2 \quad \text{as} \quad \xi \to x_\ast,
\]

so that

\[
e^{\theta_0} \sim \mathcal{H} \exp \left( T - \frac{\kappa}{2\delta^2} (\xi - x_\ast)^2 \right) \quad \text{as} \quad \xi \to x_\ast
\]

where, for convenience, we denote \( \delta = 1/\sqrt{\log \mathcal{H}} \ll 1 \).
This suggests that the appropriate rescalings for the other variables are
\[ \xi = x_* + \delta \eta, \quad s = S, \quad x = x_* + \delta y, \]
and the leading-order matching conditions are
\[ S \to 1 \quad \phi \sim T - \frac{\kappa \eta^2}{2}, \quad \text{as} \quad T \to -\infty, \]
\[ y \sim \eta \quad \text{as} \quad \eta \to -\infty. \]

After the rescalings (3.5) and (3.8), the governing equations (3.2) become
\[ S_T = -e^\phi, \quad S^{1/2} \phi_T = H (x_* + \delta y), \quad S y_\eta = 1. \]

Now we expand the variables in powers of the asymptotically small parameter \( \delta \), e.g.,
\[ S \sim S_0 + \delta^2 S_1 + \cdots. \]

At leading-order, the right-hand side of (3.11b) is equal to 1, and the equations
\[ S_0 T = -e^{\phi_0}, \quad S_0^{1/2} \phi_0 T = 1 \]
may be solved simultaneously, subject to the leading-order matching conditions (3.9), to obtain
\[ S_0 = \left\{ 1 + W \left( -\frac{1}{2} e^{T - \kappa \eta^2/2} \right) \right\}^2, \quad \phi_0 = T - \frac{\kappa \eta^2}{2} - W \left( -\frac{1}{2} e^{T - \kappa \eta^2/2} \right) \]
where \( W \) is the Lambert \( W \)-function that solves \( W(x)e^{W(x)} = x \). We plot these solutions in Figure 1. As \( T \) increases, \( S_0 \) decreases, and reaches its minimum value of zero at \( \eta = 0 \) when \( T = \log 2 - 1 \). At the same time, \( \phi_0 \) reaches a critical value of \( \log 2 \) at \( \eta = 0 \), where it ceases to be smooth.

Finally, we solve
\[ y_0 \eta = S_0^{-1} = \left\{ 1 + W \left( -\frac{1}{2} e^{T - \kappa \eta^2/2} \right) \right\}^{-2} \]
for \( y_0 \) and apply the matching condition (3.10) to obtain
\[ y_0 = \eta + \int_{-\infty}^{\eta} \left\{ 1 + W \left( -\frac{1}{2} e^{T - \kappa \eta^2/2} \right) \right\}^{-2} - 1 \right\} \, d\tilde{\eta}. \]

In particular, we find that
\[ y \sim \eta + y_\infty (T) \quad \text{as} \quad \eta \to +\infty \]
where
\[ y_\infty (T) = 2y(0, T) = 2 \int_0^\infty \left\{ 1 + W \left( -\frac{1}{2} e^{T - \kappa \eta^2/2} \right) \right\}^{-2} - 1 \right\} \, d\eta \]
\[ = \frac{1}{\sqrt{\kappa}} \int_{-\infty}^{T-W(-e^{T/2})} \frac{4 - e^\theta}{(2 - e^\theta) \sqrt{2T - 2\theta + e^\theta}}. \]

We see in Figure 2 that \( y_\infty \) tends to infinity at the critical time \( T = \log 2 - 1 \) when the cross-sectional area \( S_0 \) tends to zero. It is straightforward to analyze (3.18) in this limit to obtain
\[ y_\infty (T) \sim \frac{\pi}{\sqrt{2\kappa (\log 2 - 1 - T)}} \quad \text{as} \quad T \to \log 2 - 1. \]
3.3. The stretching phase. The “transition” phase, analyzed above, describes the initial flow of the thread. The evolutions of the temperature $\phi$ and the cross-sectional areas $S$ are coupled, but the rescalings (3.8) imply that the flow is confined to a neighborhood of $x = x_\ast$. This is not true when $\delta y$ is no longer small, so the right-hand side of (3.11b) cannot be approximated as unity. If we examine the solutions (3.14) as $T$ approaches the critical value $\log 2 - 1$, we find that

$$S_0 \sim 2 (\log 2 - 1 - T) + \kappa \eta^2, \quad \phi_0 \sim \log 2 - \sqrt{2 (\log 2 - 1 - T)} + \kappa \eta^2. \quad (3.20)$$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Scaled cross-sectional area $S_0$ and temperature $\phi_0$ versus scaled Lagrangian coordinate $\eta$ for times $T = -2, -1.5, -1, -0.5, \log 2 - 1$ (here $\kappa = 1$).}
\end{figure}
The function $y_\infty (T)$ defined by (3.18).

Now considering (3.11c), we deduce that variations in $y$ of order $\delta^{-1}$ occur on a time scale $O(\delta^2)$, and hence we introduce the rescaling

$$ T = \log 2 - 1 + \delta^2 t, \quad \eta = \delta \zeta. \quad (3.21) $$

We rescale the dependent variables $S$ and $\phi$ and return to the variable $x$ using

$$ S = \delta^2 s, \quad \phi = \log 2 + \delta \psi, \quad y = \delta^{-1} (x - x_*). \quad (3.22) $$

After the application of (3.21) and (3.22), the governing equations (3.11) become

$$ s_t = -2e^{\delta \psi}, \quad s^{1/2} \psi_t = H (x), \quad sx_\zeta = 1. \quad (3.23) $$

Using asymptotic expansions of the form

$$ s \sim s_0 + \delta s_1 + \cdots, \quad (3.24) $$

integrating (3.23a) to leading order, and matching with (3.20), we obtain

$$ s_0 = \kappa \zeta^2 - 2t. \quad (3.25) $$

Next we substitute this into (3.23c), integrate, and match with (3.16) to obtain

$$ x_0 = x_* + \frac{1}{\sqrt{-2\kappa t}} \left\{ \frac{\pi}{2} + \tan^{-1} \left( \zeta \sqrt{\frac{\kappa}{-2t}} \right) \right\}. \quad (3.26) $$

Finally, in principle, $\psi_0$ is determined by integration of

$$ \psi_0 t = \frac{1}{\sqrt{\kappa \zeta^2 - 2t}} H \left( x_* + \frac{1}{\sqrt{-2\kappa t}} \left\{ \frac{\pi}{2} + \tan^{-1} \left[ \zeta \sqrt{\frac{\kappa}{-2t}} \right] \right\} \right), \quad (3.27) $$

subject to

$$ \psi_0 \sim -\sqrt{\kappa \zeta^2 - 2t} \quad \text{as} \quad t \to -\infty. \quad (3.28) $$

It is through (3.27) that the details of the heating function $H(x)$, rather than just its local behavior near $x = x_*$, enter the solution. However, since the $\psi$ part of (3.22b) is only an $O(\delta)$ correction to the temperature, we do not solve (3.27).

The minimum of $s$ is always at $\zeta = 0$, which corresponds to the Eulerian position

$$ x = x_* + \frac{\pi}{2\sqrt{-2t}}. \quad (3.29) $$
By eliminating \( \zeta \) between (3.25) and (3.26), we find that the canonical behavior near this minimum is

\[
 s \sim -2t \csc^2 \left( (x - x_*) \sqrt{-2\kappa t} \right).
\] (3.30)

As shown in Figure 3, the value of the local minimum decreases and the profile spreads as \( t \) increases. The area \( s \) tends to zero as \( t \) tends to zero and the thread length tends to infinity.

### 3.4. Uniform approximation

By combining the solutions in the heating, transition, and stretching phases, we obtain the following approximations for the solution in Lagrangian variables that are valid to leading order throughout the evolution of the thread:

\[
 s \sim \left[ 1 + W \left( -\frac{1}{2\mathcal{H}} e^{\mathcal{H} \tau} \mathcal{H}^{-\kappa(x-x_*)^2/2} \right) \right]^2,
\] (3.31)

\[
 \theta \sim \mathcal{H} \tau H(\xi) - W \left( -\frac{1}{2\mathcal{H}} e^{\mathcal{H} \tau} \mathcal{H}^{-\kappa(x-x_*)^2/2} \right),
\] (3.32)

\[
 x \sim \xi + \frac{1}{\sqrt{2\kappa \log \mathcal{H}}} \left[ \frac{\pi}{2} + \tan^{-1} \left( \frac{(\xi - x_*) \sqrt{\kappa \log \mathcal{H}}}{\sqrt{2} \log \mathcal{H}} \right) \right].
\] (3.33)

The asymptotic solution (3.31)-(3.33) is plotted in Figure 4 for the case

\[
 H(x) = \frac{1}{1 + \kappa(x-x_*)^2/2},
\] (3.34)

with \( \kappa = 1, x_* = 1/2, \) and \( \delta = 0.1. \) We use the unrealistically large value of \( \mathcal{H} = e^{100} \) for illustrative purposes. In the figure, we can clearly distinguish the three phases described above. Initially, in (a), the cross-sectional area \( s \) stays virtually constant, while the temperature \( \theta \) increases uniformly with \( T. \) In (b), we see the transition phase, in which both \( \theta \) and \( s \) evolve, and the thread starts to stretch, although mainly near the heating maximum \( x = x_* \). Finally, in (c), the thread starts to stretch significantly, while \( \theta \) approaches a constant value close to \( \log \mathcal{H}. \) The area tends to zero at a critical time, \( T = \log(2\mathcal{H}) - 1 \approx 99.693. \)
4. Breaking and numerical solutions. In the stretching phase, the thread thins dramatically. This leads to the development of extremely large local stresses. In typical applications, the materials break when a critical stress is exceeded. Cohen [1] has shown that the breaking stress of glass varies with temperature. The dimensionless breaking criterion is given in [5]

\[ s < \frac{\sqrt{\theta + R}}{C} \]  

(4.1)

where \( R \) is the ratio of the initial temperature of the glass to the temperature change required to significantly change the viscosity and \( C \) is the breaking stress.
at the initial temperature divided by the initial stress in the thread. For further details, we refer the reader to Huang et al. (2007) [5].

In order to determine the asymptotic solution at breaking, one can substitute the uniformly valid asymptotic approximations for $s$ and $\theta$ given by (3.31) and (3.32) into the breaking criterion (4.1). The location of the narrowest part of the tube for the asymptotic solution is at $\xi = x_*$, which also corresponds to the location of the maximal temperature. Therefore, the breaking criterion first must be satisfied at $\xi = x_*$.

The breaking criterion then reduces to an algebraic equation in the breaking time $\tau_b$ that can be solved numerically. It should be noted, however, that the temperature at breaking can be accurately approximated by the temperature at which the tube would pinch off, that is when $s \to 0$. These two temperatures will be extremely close since the thread extends very rapidly at breaking, and therefore, there will be very little heating between the breaking time and the pinching time. At pinching, $\theta = \log(2H)$. Substituting this into the breaking criterion, we obtain an expression for the breaking area

$$s_b = \frac{\sqrt{R + \log(2H)}}{C}.$$  (4.2)

We then can use (3.31) to determine the breaking time

$$\tau_b = \frac{\log(2H) + \log(1 - \sqrt{s_b}) - (1 - \sqrt{s_b})}{H}.$$  (4.3)

The gradient of the cross-sectional area at the breaking point is zero to leading order. Hence, the slope at the breaking point cannot be used to characterize the degree of taper of the thread tip. Therefore, it is natural to use the curvature at the breaking point as a measure of the degree of taper. The uniform asymptotic solution (3.31) can be used to compute the curvature of the cross-sectional area with respect to $x$

$$\frac{s_{xx}}{(1 + s_2^2)^{3/2}}|_{\xi = x_*} = s^2 s_{\xi\xi}|_{\xi = x_*} = 2s_b^2 \kappa (1 - \sqrt{s_b}) \log(H).$$  (4.4)

Similarly, the half-length of thread at breaking is given by

$$x_b = x_* + \left[-2\kappa \log(H) \log(1 - \sqrt{s_b}) + \sqrt{s_b}\right]^{-1/2}.$$  (4.5)

With these expressions, we can easily see how the cross-sectional area of the tip, the curvature of the cross-sectional area in the axial direction, and the length of the thread vary as functions of $H$ and $\kappa$.

We used the heating profile (3.34) and integrated the (Lagrangian) equations using a straightforward Euler scheme in time and Simpson’s rule in $\xi$. At each time step, we checked that the breaking criterion is not violated and terminated the computation when it was violated. This provided us with the final shape at breaking.

In Figures 5 and 6, we plot the numerical and asymptotic solutions for the shape and temperature profiles at breaking for two different values of $H$. For the larger value of $H$, the shape is very well approximated near the breaking point, but less well away from the breaking point. This is because the leading-order asymptotic solution neglects any stretching that is not in the vicinity of the breaking point. For the smaller value of $H$, the agreement is quite poor away from the breaking point, but the shape is still well approximated near the breaking point. The agreement in
the shape profile is good near the breaking point even though the agreement in the temperature profile is quite poor.

In Figure 7, we plot the numerical and asymptotic solutions at breaking for the length of the thread and the curvature at the breaking point of the cross-sectional area in the axial direction as functions of $H$ for fixed $\kappa$. As $H$ decreases, the total extension of the thread increases. This is because for weak heating, the axial variation of the viscosity is small and so the thread extends significantly throughout its entire length, and therefore, the stretching is less localized. The curvature of the cross-sectional area decreases as $H$ increases for the same reason. The agreement between the numerical and asymptotic solutions is reasonable for all values of $H$. 

Figure 5. Profiles of the cross-sectional area $s$ and the temperature $\theta$ at breaking for $H = 2.2 \times 10^{10}$, $R = 6$, $C = 255$, and $\kappa = 10$. The solid and dashed curves represent asymptotic and numerical solutions, respectively.
In Figure 8, we plot the numerical and asymptotic solutions at breaking for the length and the curvature at the breaking point of the cross-sectional area as functions of $\kappa$ for fixed $H$. As $\kappa$ increases, the total extension of the thread decreases because the heating, and hence the stretching, becomes more localized. The curvature of the cross-sectional area increases as $\kappa$ increases for the same reason. Therefore, increasing $\kappa$ and increasing $H$ both have the same qualitative effect. However, both the length and the curvature of the cross-sectional area are much more sensitive to changes in $\kappa$ than to changes in $H$. 

Figure 6. Profiles of the cross-sectional area $s$ and the temperature $\theta$ at breaking for $H = 2.2 \times 10^2$, $R = 6$, $C = 255$, and $\kappa = 10$. The solid and dashed curves represent asymptotic and numerical solutions, respectively.
The total extension at breaking and the curvature of the cross-sectional area at the breaking point are plotted against the dimensionless heating rate $\mathcal{H}$ for $R = 6$, $C = 255$, and $\kappa = 10$. The solid and dashed curves represent asymptotic and numerical solutions, respectively.

In principle, the asymptotic approximation is only valid when the relevant small parameter $\delta = 1/\sqrt{\log(\mathcal{H})} \ll 1$. However, as in many other cases involving asymptotic expansions, we found that the approximation is reasonable even for a moderately large value of $\mathcal{H} \sim 10^2$, which is relevant in applications. For illustrative purposes, we have used values of $\mathcal{H}$ which are unrealistically large ($10^{10}$ or $e^{100}$).

5. Generalizations. The analysis carried out above is easily extended to situations where the end $\xi = 0$, rather than being held fixed, is moved to $x = X(t)$ at time $t$. For example, $X(t)$ might be determined by requiring that the center-of-mass
Figure 8. The total extension at breaking and the curvature of the cross-sectional area at the breaking point are plotted against the curvature in the heating rate $\kappa$ for $H = 2.2 \times 10^2$ and $2.2 \times 10^{10}$, $R = 6$, and $C = 255$. The solid and dashed curves represent asymptotic and numerical solutions, respectively.

of the thread remain fixed, that is,
\[
\int_0^1 x(\xi, \tau) \, d\xi = \frac{1}{2}. \tag{5.1}
\]

At the same time, we can generalize the analysis to cases where the applied force varies with time. The dimensionless equations and initial conditions become, in Lagrangian form,
\[
s_T = -\mathcal{H}^{-1} F(T)e^\theta, \quad s^{1/2}\theta_T = H(x), \quad s x_\xi = 1. \tag{5.2}
\]
\[
s(\xi, 0) = 1, \quad \theta(\xi, 0) = 0, \quad x(0, T) = X(T). \tag{5.3}
\]
Here, $F(T)$ is the dimensionless applied force, and $X(T)$ is assumed for the moment to be a known function; it will be determined \textit{a posteriori} using a condition like (5.1).

The asymptotic procedure demonstrated above can be followed. During the initial heating phase, we have

$$s_0 = 1, \quad \theta_0 = TH(x),$$

as before. The center-of-mass condition, (5.1), (or, presumably, any other sensible criterion to determine $X$) gives $X_0 = 0$ and hence

$$x_0 = \xi.$$  \hspace{1cm} (5.5)

The transition phase is characterized by the same rescalings, (3.5) and (3.8), as before. We assume that the terms containing the Lambert $W$ function are effectively constant throughout this phase; the idea is that the Lambert $W$ function only will vary significantly during the stretching phase when the end $\xi = 1$ starts to accelerate rapidly. Therefore, the solutions (3.14) for $S$ and $\phi$ are unchanged, while (3.16) is simply modified to

$$y_0(\eta, T) = \eta + Y_0(T) + \int_{-\infty}^{\eta} \left\{ \left( 1 + W \left( -\frac{1}{2} e^{T - \kappa \tilde{\eta}^2 / 2} \right) \right)^{-2} - 1 \right\} \, d\tilde{\eta} \hspace{1cm} (5.6)$$

where

$$X(T) = \delta Y(T).$$ \hspace{1cm} (5.7)

Finally, we apply the rescalings, (3.21) and (3.22), for the stretching phase, which transforms the governing equations to

$$s_t = -2 e^{\delta} F(t), \quad s^{1/2}_t = H(x), \quad s_x \xi = 1. \hspace{1cm} (5.8)$$

It is convenient to introduce the notation

$$I(t) = - \int_0^t F(\bar{t}) \, d\bar{t},$$ \hspace{1cm} (5.9)

so the leading-order solution of (5.8a) that matches with the transition phase is

$$s_0 = \kappa \xi^2 + 2I(t).$$ \hspace{1cm} (5.10)

Then, solving (5.8c) for $x$ and matching yields

$$x_0 = X_0(t) + x_+ + \frac{1}{\sqrt{2 \kappa I(t)}} \left\{ \frac{\pi}{2} + \tan^{-1} \left( \frac{\xi}{\sqrt{2I(t)}} \right) \right\}. \hspace{1cm} (5.11)$$

To make further progress, we need to specify boundary conditions that allow us to determine the position $X$ of the end of the thread and the applied tension $F$. We consider two particular scenarios of practical interest. First, we suppose that the center-of-mass of the thread remains fixed while a known pulling force is applied. Second, we consider the case where the top is held fixed at $X = 0$, but the force $F$ varies because of the acceleration of an applied weight.
5.1. Center-of-mass preserving threads. We assume here that the applied force \( F \), and therefore, the integral \( I(t) \) defined by (5.9), is given. By using (5.2c), the center-of-mass condition (5.1) may be turned into equations for the position \( X(T) \) of the top of the thread and its length \( \ell(T) \), namely,

\[
X(T) = \frac{1}{2} - \int_0^1 \frac{1 - \xi}{s(\xi, T)} \, d\xi, \quad \ell(T) = \int_0^1 \frac{d\xi}{s(\xi, T)}. \tag{5.12}
\]

It is readily shown that \( X \sim 0 \) and \( \ell \sim 1 \) to leading order throughout the heating and transition phases. However, the ends do move significantly during the stretching phase. To determine their position during this phase, we must examine the solution outside the inner layer near \( \xi = x_\ast \), where we find that

\[
s_0 = 1, \quad x_0 = \begin{cases} X(t) + \xi & \xi < x_\ast, \\ X(t) + \ell(t) + \xi - 1 & \xi > x_\ast. \end{cases} \tag{5.13}
\]

To evaluate the integrals in (5.12), we must combine (5.13) with the corresponding solutions (5.10) and (5.11) in the “inner” region near \( \xi = x_\ast \). For example, we obtain an expression for \( \ell_0(t) \) by writing

\[
\ell(t) = \int_{x_\ast - \alpha}^{x_\ast} \frac{d\xi}{s(\xi, t)} + \int_{x_\ast + \alpha}^{1} \frac{d\xi}{s(\xi, t)} + \int_{-\alpha/\delta^2}^{\alpha/\delta^2} \frac{d\zeta}{s(\zeta, t)} \tag{5.14}
\]

where \( \alpha \) is any constant in the range \( \delta^2 \ll \alpha \ll 1 \). By substituting for \( s \) and \( s \) from (5.10) and (5.13), respectively, we find the leading-order expression

\[
\ell_0(t) = 1 + \frac{\pi}{\sqrt{2} \kappa I(t)}, \tag{5.15}
\]

and a similar procedure yields

\[
X_0(t) = -\frac{\pi(1 - x_\ast)}{\sqrt{2} \kappa I(t)}. \tag{5.16}
\]

We therefore find that the two ends \( x = X \) and \( x = X + \ell \) move at rates scaled according to the position \( x_\ast \) of the maximal heating, that is,

\[
\frac{dX}{dt} = -(1 - x_\ast) \frac{d\ell}{dt}, \quad \frac{d}{dt} (X + \ell) = x_\ast \frac{d\ell}{dt}, \tag{5.17}
\]

so, in particular, the ends move at the same speed if and only if \( x_\ast = 1/2 \).

5.2. Acceleration effects. Now we consider the case where the top of the thread is fixed \( (X \equiv 0) \), but the acceleration term in the force balance equation (2.8) may not be neglected. The applied dimensionless force thus is related to the length of the thread by

\[
F(\tau) = 1 - A \frac{d^2 \ell}{d\tau^2}. \tag{5.18}
\]

The dimensionless parameter \( A = \ell_0 F_0^2 / 9 \mu_0^2 s_0^2 g \) is usually very small, but nevertheless may be significant when (5.18) is considered over the short-time scale associated with the stretching phase, that is,

\[
F(t) = 1 - a \frac{d^2 \ell}{dt^2}, \tag{5.19}
\]

where

\[
a = A \mathcal{H}^2 \log^2 \mathcal{H}. \tag{5.20}
\]
This is coupled to our expression (5.15) for the thread length, which we rearrange to give

$$I(t) = \frac{\pi^2}{2\kappa(\ell - 1)^2},$$

(5.21)

where we recall the definition (5.9) of $I$, that is,

$$I(t) = -\int_0^t F(\tilde{t}) \, d\tilde{t} = -t + a \frac{d\ell}{dt}.$$  (5.22)

We thus obtain a first-order nonlinear differential equation for the thread length, namely,

$$a \frac{d\ell}{dt} = t + \frac{\pi^2}{2\kappa(\ell - 1)^2},$$

(5.23)

which is to be solved subject to the matching condition

$$\ell \sim 1 + \frac{\pi}{\sqrt{-2\kappa t}} \quad \text{as} \quad t \to -\infty.$$  (5.24)

The solution of this problem may be written in the form

$$\ell(t) = 1 + \left[ \frac{\pi^4}{4\kappa^2a^2} \right]^{1/5} f \left( \left[ \frac{2\kappa}{\pi^2a^2} \right]^{1/5} t \right)$$

(5.25)

where the function $f$ is determined uniquely by the canonical problem

$$\frac{df}{dz} = z + \frac{1}{f^2}, \quad f \sim \frac{1}{\sqrt{-z}} + \frac{1}{4z^3} + \cdots \quad \text{as} \quad z \to -\infty.$$  (5.26)

As indicated in Figure 9, $f$ has the asymptotic behaviors

$$f \sim \begin{cases} 
\frac{1}{\sqrt{-z}} & \text{as} \quad z \to -\infty, \\
\frac{z^2}{2} & \text{as} \quad z \to +\infty,
\end{cases}$$  (5.27)

$$\ell \sim \begin{cases} 
1 + \frac{\pi}{\sqrt{-2\kappa t}} & \text{as} \quad t \to -\infty, \\
\frac{t^2}{2a} & \text{as} \quad t \to +\infty.
\end{cases}$$  (5.28)
The former of these agrees with (5.15), showing that acceleration effects lose importance as $t \to -\infty$. On the other hand, the behavior for large $t$ simply corresponds to the applied weight accelerating uniformly under gravity with the force exerted on it by the thread becoming negligible.

We plot $\ell$ versus $T$ in Figure 10 for various values of the inertia parameter $a$. With $a = 0$, the acceleration term is ignored and the solution is given by (5.15), that is,

$$\ell \sim 1 + \frac{\pi}{\sqrt{2\kappa \log \mathcal{H} (\log 2\mathcal{H} \mathcal{H} - 1 - T)}}. \quad (5.29)$$

The length tends to infinity after a finite time $\log(2\mathcal{H} - 1) \approx 99.693$ when $\mathcal{H} = e^{100}$. For larger values of $a$, we see that the dynamics is virtually unaffected by the inertia term until the stretching phase. A close-up of the behavior near the critical time shows that inertia slows down the stretching and prevents the length from tending to infinity in finite time.

6. **Application to glass microelectrodes.** Glass microelectrodes are widely used to measure membrane potentials and inject electric current and dyes into cells. This is done by inserting the electrode tips through cellular membranes or by ‘patching’ the electrode tip to the membrane. The data collected by these techniques provide crucial information about the electrical properties of the membrane, e.g., the voltage-gated and receptor-gated ion channels under various conditions, including during drug applications. For more descriptions of the medical applications of these electrodes, we refer interested readers to [6, 7] and references therein.

Electrophysiology laboratories generally produce these microelectrodes on a daily basis using commercially available glass tubes and mechanical microelectrode pullers. In practice, producing the desired microelectrode tip shape is difficult, and requires a time-consuming trial-and-error process. A quantitative understanding of the process of formation of the microelectrode tip region would facilitate a more efficient means of production.

The basic method for production of these microelectrodes is to heat a uniform glass tube in the middle while applying an axial extensional force. There are a number of different electrode pulling devices available commercially, which only differ in the details of the heating process and applied force.
The applicability of a glass microelectrode depends on two critical characteristics of the tip region, namely electrode tip radius and the degree of taper. Tip radius relative to the size of a cell is a crucial parameter. A short steeply tapered tip does not penetrate tissue easily, whereas long gently tapered tips are easily broken. Additionally, the profile of the microelectrode tip region is significant because it determines its electrical resistance and capacitance.

Tubes that are initially uniform in the axial direction have constant ratio of the inner and outer radii, which we denote by \( \beta \), throughout the pulling process [5]. Therefore, it suffices to consider only the cross-sectional area, and the equations for the pulling of glass tubes are identical to those for pulled threads. We hence can use the solution developed in Section 3 to describe the formation of glass microelectrodes. In fact, the only difference appears in the definition of the heating strength, \( \mathcal{H} \), that is given by

\[
\mathcal{H} = \frac{6\mu_0 \sqrt{s_0} k_B \varepsilon_h \alpha \theta_h^4 \sqrt{\pi}}{\rho c_p \theta_s F_0 \sqrt{(1 - \beta^2)(1 - (1 - \alpha)(1 - \varepsilon_h))}},
\]

which differs from (2.4) only in the additional factor of \( \sqrt{(1 - \beta^2)} \) in the denominator.

The length of the electrode is given by the half-length at breaking (4.5). The effects of heating strength and curvature of the heating profile on the profiles of the cross-sectional area and temperature, the length at breaking, and the curvature of the cross-sectional area at the breaking point of the electrode are shown in Figures 5-8 with suitable interpretation of \( \mathcal{H} \) defined in (6.1).

7. Discussion. We have considered the stretching of thin cylindrical threads which are made of materials that have temperature-dependent viscosities. As each thread is pulled with a prescribed force, it is continuously heated from an external axially nonuniform heater. The model equations for this system are given, and we have studied the dynamics of the thread in the limit of large dimensionless heating rate. Uniformly-valid asymptotic solutions were derived for the shape and temperature profiles, and we showed that to leading order, these depend only on the local properties of the heating near its maximal heating value.

The threads are subject to a critical breaking criterion, and we use this criterion to determine when breaking will occur. Then we derive simple analytical expressions to leading order for the spatial distributions of the shape and temperature at this breaking time, and they clearly show the influences of heating strength and the degree of localization of the heating profile. Specifically, the details of the heating profile away from the maximal strength have negligible effect on the thread shape. The asymptotic shape profiles give good agreement with numerical simulations. As an example, these results are applied to the formation of glass microelectrodes, which are used in electrophysiology laboratories throughout the world.

We have not attempted to compare our approximate solutions directly to experimental observations. This is due to the fact that the detailed heating profile of a heater is unknown for a practical setup. Nonetheless, the results obtained here are useful from an application point of view. In particular, the heating profile is associated with the geometric dimensions of the heater and the glass tubes being stretched. Therefore, the shape of the electrode (when the glass tube breaks) is influenced by the geometrical parameters of the heater and the glass tube. One can, for example, use glass tubes with a variety of diameters, and our theory can be
used to predict the evolution of the electrode shape as the tube diameter is varied. Alternatively, our theory can be used as a guiding principle for modifying the design of the pullers (heater dimensions) so that electrodes with desirable shapes can be pulled.

Acknowledgments. The authors wish to express their gratitude to the Banff International Research Station and its staff for providing a wonderful environment during the initiation of this research. JJW acknowledges full support from the Research Council of the Hong Kong Special Administrative Region, China (CityU 1220/02P). HH acknowledges support from the Natural Sciences and Engineering Research Council of Canada and from the Mathematics of Information Technology and Complex Systems, Networks of Centres of Excellence. RMM acknowledges support from the Department of Mathematical Sciences, New Jersey Institute of Technology. We also would like to thank the referees very much for their valuable comments and suggestions.

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Received August 2006; revised January 2007.

E-mail address: howell@maths.ox.ac.uk
E-mail address: wylie@cpmaspc07.cityu.edu.hk
E-mail address: hhuang@yorku.ca
E-mail address: miura@njit.edu