EXACT AND APPROXIMATE TRAVELING WAVES OF REACTION-DIFFUSION SYSTEMS VIA A VARIATIONAL APPROACH

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Reaction-diffusion systems arise in many different areas of the physical and biological sciences, and traveling wave solutions play special roles in some of these applications. In this paper, we develop a variational formulation of the existence problem for the traveling wave solution. Our main objective is to use this variational formulation to obtain exact and approximate traveling wave solutions with error estimates. As examples, we look at the Fisher equation, the Nagumo equation, and an equation with a fourth-degree nonlinearity. Also, we apply the method to the multi-component Lotka–Volterra competition-diffusion system.

Keywords: Reaction-diffusion equations; traveling waves; variational formulation of existence problem; exact and approximate solutions.

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1. Introduction

Reaction-diffusion systems arise in the modeling of diverse phenomena encountered in different areas of the physical and biological sciences (see [2, 5, 8], for example). These systems of parabolic partial differential equations are typically of the form

\frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + f_1(u_1, \ldots, u_n),

\vdots

\frac{\partial u_n}{\partial t} = d_n \frac{\partial^2 u_n}{\partial x^2} + f_n(u_1, \ldots, u_n).
Here, $x \in \mathbb{R}$, $t > 0$, and for each $j = 1, \ldots, n$, $f_j$ is a nonlinear function of its arguments. We have $d_j \geq 0$ and $u_j = u_j(x, t)$. The above system can be recast in matrix form as

$$u_t = Du_{xx} + f(u)$$  \hspace{1cm} (1.1)

where $u = (u_1, \ldots, u_n)$, $f = (f_1, \ldots, f_n)$, $D = \text{diag}(d_1, \ldots, d_n)$, and the subscripts denote partial derivatives with respect to the corresponding variables. It is usual to assume that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(p) = f(q) = 0$ for some $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$.

In many applications, one looks for a special type of solution of (1.1) called a traveling wave solution, i.e. a solution of (1.1) of the form $u(x, t) = U(z)$ where $U = (U_1, \ldots, U_n)$, $z = kx - \omega t$, and $k, \omega \in \mathbb{R}$. Without loss of generality, we can assume that $k > 0$ since (1.1) is invariant with respect to the coordinate transformation $x \to -x$. The parameter $k$ controls the behavior of the solution at infinity while the constant $-\omega/k$ is called the wave speed. Rewriting (1.1) in $z$ alone yields the system of ordinary differential equations

$$k^2 D\ddot{U} + \omega \dot{U} + f(U) = 0$$  \hspace{1cm} (1.2)

where $\dot{}$ denotes differentiation with respect to $z$. Here we are interested in a solution of (1.2) that satisfies the boundary conditions

$$U(-\infty) = p, \quad U(\infty) = q.$$  \hspace{1cm} (1.3)

We call $U$ a traveling pulse when $p = q$ and a traveling front when $p \neq q$.

One of the most important results in the theory of reaction-diffusion systems is the existence of a traveling wave solution of (1.1), i.e. the existence of a solution of (1.2) and (1.3) under hypotheses on the diffusion matrix $D$ and the nonlinearity of $f$. Some of the approaches that have been employed in the literature include methods based on the Conley index, Leray–Schauder degree, maximum principles, and singular perturbations, to name a few; we refer the reader to [3, 4, 7, 11, 12] for comprehensive lists of references.

In this paper, we take a variational approach to partially address the existence question. It is not our intention to give precise sufficient conditions on $D$ and $f$ that will guarantee the existence of a traveling wave solution, which is quite difficult considering the generality of the framework that we are adopting. Rather, our aim here is to develop a variational formulation of the existence problem and utilize it to obtain exact and approximate traveling wave solutions.

In order to derive our variational approach, we convert the system of autonomous second-order ordinary differential equations given by (1.2) to a formally equivalent system of first-order partial differential equations. (Note that the “equivalence” of the two systems is not true in general. Since formulating the conditions for such an equivalence is difficult in the general case, the following method is valid only for those cases for which the two systems are equivalent.) The change
of variables involves first-order partial derivatives, and for all \( j = 1, \ldots, n \), we set
\[
\xi_j = U_j(z), \quad v_j(\xi) = v_j(\xi_1, \ldots, \xi_n) = \dot{U}_j(z) = \dot{\xi}_j(z).
\tag{1.4}
\]
Then
\[
\ddot{U}_j = \frac{\partial v_j}{\partial \xi_1} \dot{\xi}_1 + \cdots + \frac{\partial v_j}{\partial \xi_n} \dot{\xi}_n = \nabla v_j \cdot \dot{\xi} = v \cdot \nabla v_j,
\]
and (1.2) becomes
\[
k^2 \nabla_D v + \omega v + f = 0,
\tag{1.5}
\]
where we define \( \nabla_D v = (d_1 v \cdot \nabla v_1, \ldots, d_n v \cdot \nabla v_n) \). In other words, instead of analyzing (1.2), we develop a variational formulation for (1.5), and then go back to (1.2) using the change of variables in (1.4). The boundary conditions in (1.3) come into play when we define the so-called admissible curves in Sec. 2.

The outline of the remainder of this paper is as follows. In Sec. 2, we introduce our notation and define an appropriate space of functions together with a functional acting on this space of functions. Some useful properties of the functional are also proved. In Sec. 3, we define what we call a \( \Gamma \)-variational solution and then prove our main result, which is the equivalence of the existence of a \( \Gamma \)-variational solution and the maximization of the functional. Then we relate the maximizing function with a solution of (1.2) and (1.3). In Sec. 4, we show how to apply our result to find exact and approximate traveling wave solutions with some examples. Finally, we give concluding remarks in Sec. 5.

2. Preliminary Definitions and Results

Let us first introduce our notation. Denote by \( X \) the set of all smooth vector fields \( v : \mathbb{R}^n \to \mathbb{R}^n \). We will assume throughout that \( f \in X \).

A smooth curve \( \Gamma \subseteq \mathbb{R}^n \) is said to be admissible if the endpoints of \( \Gamma \) are \( p \) and \( q \), and there exists a bijective parametrization \( \xi(s) : [0, 1] \to \Gamma \) of \( \Gamma \) such that \( \xi(0) = p \) and \( \xi(1) = q \). Informally, \( \Gamma \) is the trajectory in phase space representing a solution of (1.2) and (1.3). Note that we can also replace \([0, 1]\) above by any other compact interval. We shall assume that \( \Gamma \) is admissible.

For every admissible curve \( \Gamma \) with a parametrization \( \xi(s) : [0, 1] \to \Gamma \), define
\[
\langle v, w \rangle_\Gamma = \int_0^1 v(\xi(s)) \cdot w(\xi(s)) \, ds \quad \text{for all } v, w \in X.
\]
We see that \( \langle \cdot, \cdot \rangle_\Gamma \) defines an inner product on the vector space \( X \). Consequently, we can also define a norm on this space by
\[
\|v\|_\Gamma = (\langle v, v \rangle_\Gamma)^{1/2} = \left( \int_0^1 v(\xi(s)) \cdot v(\xi(s)) \, ds \right)^{1/2} \quad \text{for all } v \in X.
\]
Let \( X_\Gamma \) be the set of all vector fields \( v \in X \) such that
\[
\langle f, v \rangle_\Gamma - \langle f, \nabla D v \rangle_\Gamma \langle v, \nabla D v \rangle_\Gamma > 0.
\] (2.1)

This positivity condition will be essential later on to ensure that \( k^2 > 0 \), so \( k \) is real valued. In our variational formulation, we will also need the functional defined formally by
\[
J_\Gamma(v) = \langle f, v \rangle_\Gamma^2 \frac{\|\nabla D v\|^2_\Gamma + \langle f, \nabla D v \rangle_\Gamma^2 - 2 \langle f, v \rangle_\Gamma \langle f, \nabla D v \rangle_\Gamma}{\|v\|^2_\Gamma - \langle v, \nabla D v \rangle_\Gamma^2}.
\]

The motivation behind this definition is that it allows us to express the norm of the left-hand side of (1.5) in terms of the functional \( J_\Gamma \) and the norm of \( f \) (see (2.6) below).

Some properties of \( X_\Gamma \) and \( J_\Gamma \) are contained in the following lemma:

**Lemma.** Suppose that \( v \in X_\Gamma \). Then
\[
\|v\|_\Gamma^2 \langle v, \nabla D v \rangle_\Gamma^2 - \langle v, \nabla D v \rangle_\Gamma^2 > 0
\] (2.2)

and
\[
J_\Gamma(v) \leq \|f\|^2_\Gamma.
\] (2.3)

Furthermore, if \( k^2 \) and \( \omega \) are given by
\[
k^2 = \frac{\langle f, v \rangle_\Gamma \langle v, \nabla D v \rangle_\Gamma - \langle f, \nabla D v \rangle_\Gamma \|v\|^2_\Gamma}{\|v\|^2_\Gamma - \langle v, \nabla D v \rangle_\Gamma^2},
\] (2.4)

and
\[
\omega = \frac{\langle f, \nabla D v \rangle_\Gamma^2 - \langle f, v \rangle_\Gamma \|\nabla D v\|^2_\Gamma}{\|v\|^2_\Gamma - \langle v, \nabla D v \rangle_\Gamma^2},
\] (2.5)

then we have
\[
\|k^2 \nabla D v + \omega v + f\|^2_\Gamma = \|f\|^2_\Gamma - J_\Gamma(v).
\] (2.6)

**Proof.** Let \( v \in X_\Gamma \). Then \( v \) and \( \nabla D v \) are linearly independent, since if they were not, then \( \nabla D v = cv \) for some \( c \in \mathbb{R} \), and we would have the contradiction
\[
\langle f, v \rangle_\Gamma \langle v, \nabla D v \rangle_\Gamma - \langle f, \nabla D v \rangle_\Gamma \|v\|^2_\Gamma = c \langle f, v \rangle_\Gamma \|v\|^2_\Gamma - c \langle f, v \rangle_\Gamma \|v\|^2_\Gamma = 0.
\]

Hence, (2.2) follows from the Cauchy–Schwarz inequality and ensures that \( J_\Gamma \) is well defined on \( X_\Gamma \).

To prove (2.3), let \( c_1, c_2 \in \mathbb{R} \) and define
\[
\phi_1 = \frac{v}{\|v\|_\Gamma}, \quad \phi_2 = \frac{\nabla D v}{\|\nabla D v\|_\Gamma},
\] (2.7)
Note that $\|v\|_1 > 0$ and $\|\nabla_D v\|_1 > 0$ from (2.1). Then $\|\phi_1\|_\Gamma = \|\phi_2\|_\Gamma = 1$ and $\langle \phi_1, \phi_2 \rangle^2_\Gamma < 1$ from (2.2). Moreover,
\[
\|f - c_1 \phi_1 - c_2 \phi_2\|^2_1 = (f - c_1 \phi_1 - c_2 \phi_2, f - c_1 \phi_1 - c_2 \phi_2)_\Gamma \\
= \|f\|^2_1 - 2c_1 (f, \phi_1)_\Gamma - 2c_2 (f, \phi_2)_\Gamma \\
+ c_1^2 + 2c_1c_2 (\phi_1, \phi_2)_\Gamma + c_2^2.
\]

Let $c_1$ and $c_2$ be the solution of the linear system
\[
c_1 + \langle \phi_1, \phi_2 \rangle c_2 = (f, \phi_1)_\Gamma, \quad (\phi_1, \phi_2)_\Gamma c_1 + c_2 = (f, \phi_2)_\Gamma, \tag{2.8}
\]
which is consistent since $\langle \phi_1, \phi_2 \rangle^2_\Gamma < 1$. Multiplying the first and second equations in (2.8) by $c_1$ and $c_2$, respectively, and adding the resulting equations, we obtain
\[
c_1^2 + 2c_1c_2 (\phi_1, \phi_2)_\Gamma + c_2^2 = c_1 (f, \phi_1)_\Gamma + c_2 (f, \phi_2)_\Gamma.
\]
Thus,
\[
0 \leq \|f - c_1 \phi_1 - c_2 \phi_2\|^2_1 = \|f\|^2_1 - c_1 (f, \phi_1)_\Gamma - c_2 (f, \phi_2)_\Gamma. \tag{2.9}
\]
We are done if we can show that $J_\Gamma(v) = c_1 (f, \phi_1)_\Gamma + c_2 (f, \phi_2)_\Gamma$. If we factor out $\|v\|^2_1 \|\nabla_D v\|^2_1$ in the numerator and denominator of $J_\Gamma$, then we can rewrite it as
\[
J_\Gamma(v) = \frac{(f, \phi_1)^2_1 + (f, \phi_2)^2_1 - 2(f, \phi_1)_\Gamma (f, \phi_2)_\Gamma (\phi_1, \phi_2)_\Gamma}{1 - \langle \phi_1, \phi_2 \rangle^2_\Gamma}. 
\]
On the other hand, the solution of (2.8) is
\[
c_1 = \frac{(f, \phi_1)_\Gamma - (f, \phi_2)_\Gamma (\phi_1, \phi_2)_\Gamma}{1 - \langle \phi_1, \phi_2 \rangle^2_\Gamma}, \quad c_2 = \frac{(f, \phi_2)_\Gamma - (f, \phi_1)_\Gamma (\phi_1, \phi_2)_\Gamma}{1 - \langle \phi_1, \phi_2 \rangle^2_\Gamma}. \tag{2.10}
\]
Therefore, $J_\Gamma(v) = c_1 (f, \phi_1)_\Gamma + c_2 (f, \phi_2)_\Gamma$ and (2.3) follows.

Now suppose that $k^2$ and $\omega$ are given by (2.4) and (2.5), respectively. Then
\[
\omega = \frac{(f, \phi_2)_\Gamma (\phi_1, \phi_2)_\Gamma}{\|\phi_2\|_1 - (f, \phi_1)_\Gamma} = \frac{(f, \phi_2)_\Gamma}{\|\phi_1\|_1} - \frac{(f, \phi_1)_\Gamma (\phi_1, \phi_2)_\Gamma}{\|\phi_1\|_1 - (\phi_1, \phi_2)_\Gamma} \\
= \omega \left( \frac{1}{\|\phi_1\|_1} - \frac{(f, \phi_1)_\Gamma (\phi_1, \phi_2)_\Gamma}{\|\phi_1\|_1 - (\phi_1, \phi_2)_\Gamma} \right) \\
= - \frac{c_1}{\|\phi_1\|_1} \\
\]
and
\[
k^2 = \frac{(f, \phi_1)_\Gamma (\phi_1, \phi_2)_\Gamma}{\|\phi_2\|_1 - (f, \phi_2)_\Gamma} = \frac{1}{\|\phi_2\|_1 - (f, \phi_2)_\Gamma} - \frac{(f, \phi_1)_\Gamma (\phi_1, \phi_2)_\Gamma}{\|\phi_1\|_1 - (\phi_1, \phi_2)_\Gamma} \\
= - \frac{c_2}{\|\nabla_D v\|_\Gamma}. 
\]
In this case, we have
\[ f - c_1 \phi_1 - c_2 \phi_2 = f - \frac{c_1}{\|v\|_\Gamma} v - \frac{c_2}{\|\nabla_D v\|_\Gamma} \nabla_D v = f + \omega v + k^2 \nabla_D v. \]

But from (2.9), we see that
\[ \|f - c_1 \phi_1 - c_2 \phi_2\|^2_\Gamma = \|f\|^2_\Gamma - J_\Gamma(v); \]
hence, (2.6) holds.

3. Variational Formulation of the Existence Problem

We now return to the equivalent system that we derived in Sec. 1. A solution of (1.5) is a triple \((v,k,\omega)\) where \(k > 0\), \(\omega \in \mathbb{R}\), and \(v \in X\) such that
\[ k^2 \nabla_D v + \omega v + f = 0 \text{ on } \mathbb{R}^n. \]

However, looking for a solution with \(v \in X\) is quite general; hence, we need to restrict \(X\) to an appropriate subset. For a given admissible curve \(\Gamma\), we say that \((v,k,\omega)\) is a \(\Gamma\)-variational solution of (1.5) if \(k > 0\), \(\omega \in \mathbb{R}\), and \(v \in X_\Gamma\) such that
\[ k^2 \nabla_D v + \omega v + f = 0 \text{ on } \Gamma, \]
i.e.
\[ k^2 \nabla_D v(\xi(s)) + \omega v(\xi(s)) + f(\xi(s)) = 0 \text{ for all } s \in [0,1], \]
where \(\xi(s) : [0,1] \to \Gamma\) is a bijective parametrization of \(\Gamma\). We use this nomenclature because a \(\Gamma\)-variational solution arises when we maximize the functional \(J_\Gamma\) that was defined in Sec. 2. This is proved in the following theorem:

**Theorem.** The triple \((v_*, k_*, \omega_*)\) is a \(\Gamma\)-variational solution of (1.5) if and only if
\[ k_*^2 = \frac{\langle f, v_* \rangle_\Gamma \langle v_*, \nabla_D v_* \rangle_\Gamma - \langle f, \nabla_D v_* \rangle_\Gamma \|v_*\|^2_\Gamma}{\|v_*\|^2_\Gamma \|\nabla_D v_*\|^2_\Gamma - \langle v_*, \nabla_D v_* \rangle^2_\Gamma}, \]
\[ \omega_* = \frac{\langle f, \nabla_D v_* \rangle_\Gamma \langle v_*, \nabla_D v_* \rangle_\Gamma - \langle f, v_* \rangle_\Gamma \|\nabla_D v_*\|^2_\Gamma}{\|v_*\|^2_\Gamma \|\nabla_D v_*\|^2_\Gamma - \langle v_*, \nabla_D v_* \rangle^2_\Gamma}, \]
and
\[ J_\Gamma(v_*) = \sup_{v \in X_\Gamma} J_\Gamma(v) = \|f\|^2_\Gamma \text{ for some } v_* \in X_\Gamma. \]

**Proof (Necessity).** Suppose that \((v_*, k_*, \omega_*)\) is a \(\Gamma\)-variational solution of (1.5). Then \(v_* \in X_\Gamma\) and
\[ k_*^2 \nabla_D v_* + \omega_* v_* + f = 0 \text{ on } \Gamma. \]
Taking the inner product of \( v_* \) and \( \nabla_D v_* \), respectively, with (3.4) yields the system of linear equations
\[
\langle v_*, \nabla_D v_* \rangle \Gamma k_*^2 + \| v_* \|_\Gamma^2 \omega_* + \langle f, v_* \rangle_\Gamma = 0,
\]
\[
\| \nabla_D v_* \|_\Gamma^2 k_*^2 + \langle v_*, \nabla_D v_* \rangle_\Gamma \omega_* + \langle f, \nabla_D v_* \rangle_\Gamma = 0,
\]
with solutions \( k_*^2 \) and \( \omega_* \) expressed by (3.1) and (3.2), respectively. Note that \( k_*^2 > 0 \) from (2.1). Consequently, since (3.1) and (3.2) are satisfied, we can use (2.6) together with (3.4) to get
\[
0 = \| k_*^2 \nabla_D v_* + \omega_* v_* + f \|_\Gamma^2 = \| f \|_\Gamma^2 + J_\Gamma(v_*) \quad \text{or} \quad J_\Gamma(v_*) = \| f \|_\Gamma^2.
\]
To prove that \( v_* \) maximizes \( J \), take any \( v \in X_\Gamma \). Then (2.3) implies that \( J_\Gamma(v) \leq \| f \|_\Gamma^2 = J_\Gamma(v_*) \) and so \( \sup_{v \in X_\Gamma} J_\Gamma(v) \leq J_\Gamma(v_*) \). Furthermore, since \( v_* \in X_\Gamma \), we also have \( J_\Gamma(v_*) \leq \sup_{v \in X_\Gamma} J_\Gamma(v) \). Combining the above results proves (3.3).

**(Sufficiency).** Assume that (3.1)–(3.3) hold. Then \( k_*^2 \) in (3.1) is well defined since \( v_* \in X_\Gamma \), and we can take the positive root for \( k_* \). It remains for us to prove that \( v_* \) satisfies (3.4). Since (2.4) and (2.5) are satisfied for \( k = k_* \) and \( \omega = \omega_* \), (2.6) and (3.3) together imply that
\[
\| k_*^2 \nabla_D v_* + \omega_* v_* + f \|_\Gamma^2 = \| f \|_\Gamma^2 - J_\Gamma(v_*) = 0;
\]
thus, (3.4) is satisfied.

We now identify a \( \Gamma \)-variational solution of (1.5) with a solution of the boundary-value problem (1.2) and (1.3). Suppose that there exist an admissible curve \( \Gamma \) and a \( \Gamma \)-variational solution \((v, k, \omega)\) of (1.5). Let \( \xi(s) : [0, 1] \to \Gamma \) be a bijective parametrization of \( \Gamma \) such that \( \xi(0) = p \) and \( \xi(1) = q \).

For a fixed \( s_0 \in (0, 1) \), consider the initial-value problem
\[
\dot{U} = v(U), \quad U(0) = \xi(s_0).
\] (3.5)

Note that the initial condition in (3.5) eliminates the translation invariance of the traveling wave solution. Since \( v \) is a smooth vector field, there exists a unique solution \( U = U(z; \xi(s_0)) \) of (3.5). From the equivalence of (1.2) and (1.5) via (1.4), we know that \( U \) also satisfies (1.2), but not necessarily (1.3). Define an auxiliary function \( F : (0, 1) \times \mathbb{R} \to \mathbb{R}^n \) by
\[
F(s, z) = U(z; \xi(s_0)) - \xi(s).
\] (3.6)

Then \( U \) satisfies the boundary conditions in (1.3) if in the limit, the following consistency relations hold:
\[
F(0, -\infty) = F(1, \infty) = 0.
\] (3.7)

In summary, finding a solution to (1.2) and (1.3) is equivalent to the following:

(i) Fix an admissible curve \( \Gamma \) and find a \( \Gamma \)-variational solution \((v, k, \omega)\) of (1.5) by maximizing the functional \( J_\Gamma \).
(ii) For a fixed \( s_0 \in (0, 1) \), solve the initial-value problem (3.5) for the unique solution \( U = U(z; \xi(s_0)) \).

(iii) Define the auxiliary function \( F \) as in (3.6) and verify the consistency conditions in (3.7).

4. Exact and Approximate Traveling Wave Solutions

Here we show how we can apply our result to find exact and approximate traveling wave solutions. Before we consider actual examples, let us give an outline of the procedure to follow.

First, we propose an admissible curve \( \Gamma \) which, as mentioned previously, represents the unknown trajectory in phase space of the traveling wave solution. However, if we do not expect the trajectory to be too complicated, we may assume that \( \Gamma \) is a line or a simple curve in \( \mathbb{R}^n \). Then we consider the functional \( J_\Gamma \) on \( X_\Gamma \) and take a trial function \( v \in X_\Gamma \), which for simplicity we assume to be a polynomial or algebraic vector-valued function of \( \xi_1, \ldots, \xi_n \).

Define the error functional

\[
E_\Gamma(v) = \|k^2 \nabla_D v + \omega v + f\|_\Gamma \quad \text{for all } v \in X_\Gamma,
\]

where \( k^2 \) and \( \omega \) are defined as in (2.4) and (2.5), respectively. Then from (2.6), we observe that \( E_\Gamma(v)^2 = \|f\|_\Gamma^2 - J_\Gamma(v) \). If the trial function \( v \) that we choose satisfies \( J_\Gamma(v) = \|f\|_\Gamma^2 \), then \( (v, k, \omega) \) is an exact \( \Gamma \)-variational solution of (1.5). In general, of course, we have \( J_\Gamma(v) \leq \|f\|_\Gamma^2 \), so we may need to modify \( v \) to obtain a better approximation, i.e. a smaller value for \( E_\Gamma(v) \). We remark that the computation of \( E_\Gamma(v) \) can be easily carried out with any symbolic computation software; here we use the program Maxima. Then we proceed with Steps (ii) and (iii) as explained in the last part of Sec. 3.

We now give some examples.

4.1. Nonlinear diffusion equation

We begin by considering the scalar case when \( n = 1 \). We may assume without loss of generality that \( d_1 = 1 \) since we can always rescale \( z \). We also assume that \( p_1 = 0 \) and \( q_1 = 1 \). Thus, following [3], we consider

\[
k^2 \ddot{U}_1 + \omega \dot{U}_1 + f_1(U_1) = 0
\]

with the boundary conditions

\[
U_1(-\infty) = 0, \quad U_1(\infty) = 1.
\]

The change of variables in (1.4) is now \( \xi_1 = U_1(z) \) and \( v_1(\xi_1) = \dot{U}_1(z) \), so that (1.5) simplifies to the Abel equation of the second kind

\[
k^2 v_1 \frac{dv_1}{d\xi_1} + \omega v_1 + f_1 = 0.
\]
For an admissible curve $\Gamma \subseteq \mathbb{R}$, we assume for simplicity that $\xi_1(s) = s$ for all $s \in [0, 1]$. Other possible admissible curves could be $\xi_1(s) = s^2$ or $\xi_1(s) = s^{1/2}$.

A trial function $v_1 \in X_\Gamma$ can be proposed by looking at $v_1(\xi_1) = \dot{U}_1(z)$. Since we expect that $\dot{U}_1(\pm\infty) = 0$ and $\dot{U}_1(z) > 0$ for all $z \in \mathbb{R}$ [3], we have to choose $v_1$ such that $v_1(0) = v_1(1) = 0$ and $v_1(\xi_1) > 0$ for all $\xi_1 \in (0, 1)$. The simplest possible functional form is $v_1(\xi_1) = \xi_1 - \xi_1^2$. This could be generalized to $v_1(\xi_1) = \xi_1 - \xi_1^m$ where $m > 1$. In order to verify that (2.4) also holds, we need to specify a functional form for $f_1$. Furthermore, $k^2$ and $\omega$ can be calculated from (2.4) and (2.5), respectively.

To determine $U_1$, we solve the initial-value problem

$$\dot{U}_1 = v_1(U_1), \quad U_1(0) = \xi_1(s_0).$$

In the general case when $v_1(\xi_1) = \xi_1 - \xi_1^m$ with $m > 1$, we have a Bernoulli equation, which has a nontrivial solution that always satisfies $U_1(-\infty) = 0$ and $U_1(\infty) = 1$ for any $s_0 \in (0, 1)$. In other words, the consistency conditions in (3.7) are always satisfied in this case.

Our first example is Fisher’s equation where $f_1(\xi_1) = \xi_1(1 - \xi_1)$. Here we take $\xi_1(s) = s$ for all $s \in [0, 1]$, but we cannot choose $v_1(\xi_1) = \xi_1 - \xi_1^2$ since (2.4) gives $k = 0$. Instead we take $v_1(\xi_1) = \xi_1 - \xi_1^{3/2}$. Then $E_\Gamma(v_1) = 0$, and we obtain an exact solution $(v_1, k, \omega)$ where

$$k = \frac{\sqrt{6}}{3}, \quad \omega = -\frac{5}{3}.$$  

To find $U_1$, we solve

$$\dot{U}_1 = v_1(U_1) = U_1 - U_1^{3/2}, \quad U_1(0) = \xi_1(s_0) = s_0,$$

giving

$$U_1(z) = \frac{s_0}{(s_0^{1/2} + (1 - s_0^{1/2})e^{-z/2})^2}.$$  

We see that $U_1(-\infty) = 0$ and $U_1(\infty) = 1$. This recovers the exact solution obtained in [1].

The next example is Nagumo’s equation where $f_1(\xi_1) = \xi_1(1 - \xi_1)(\xi_1 - a)$ and $0 < a < 1/2$. Take $\xi_1(s) = s$ for all $s \in [0, 1]$ and $v_1(\xi_1) = \xi_1 - \xi_1^2$. Then we obtain $E_\Gamma(v_1) = 0$, i.e. we have an exact solution $(v_1, k, \omega)$ of (1.5) where

$$k = \frac{\sqrt{2}}{2}, \quad \omega = a - \frac{1}{2}.$$  

To find $U_1$, we solve

$$\dot{U}_1 = v(U_1) = U_1 - U_1^2, \quad U_1(0) = \xi_1(s_0) = s_0,$$

giving

$$U_1(z) = \frac{s_0}{s_0 + (1 - s_0)e^{-z}}.$$  

It is clear that $U_1(-\infty) = 0$ and $U_1(\infty) = 1$. This recovers the well-known solution obtained in [6].
For our final example, we consider \( f_1(\xi_1) = \xi_1(1 - \xi_1)(\xi_1 + a)(\xi_1 - a) \) and \( 0 < a < \sqrt[4]{\frac{2}{7}} \). As far as we know, there is no exact traveling wave solution for this case. Take \( \xi_1(s) = s \) for all \( s \in [0, 1] \) and \( v_1(\xi_1) = \xi_1 - \xi_1^2 \). Then we get \( E_1(v_1) = 1/17640 \approx 0.00006 \), i.e. we almost have an exact solution. Note that the error does not depend on \( a \). In addition, we have
\[
k = \frac{\sqrt{2}}{2}, \quad \omega = a^2 - \frac{2}{7}.
\]
To find \( U_1 \), we solve the the same problem as (4.1) and obtain the Nagumo solution (4.2).

### 4.2. Lotka–Volterra competition-diffusion system

For an example of a multi-component case, we shall consider the Lotka–Volterra competition-diffusion system [9]
\[
k^2 \frac{\partial^2 U_1}{\partial \xi_1^2} + \omega \frac{\partial U_1}{\partial \xi_1} + U_1(1 - U_1 - cU_2) = 0,
\]
\[
k^2 d \frac{\partial^2 U_2}{\partial \xi_2^2} + \omega \frac{\partial U_2}{\partial \xi_2} + U_2(a - bU_1 - U_2) = 0,
\]
together with the boundary conditions
\[
(U_1, U_2)(-\infty) = (0, a), \quad (U_1, U_2)(\infty) = (1, 0).
\]

We assume that \( a, b, c, d > 0 \) and look for a traveling wave solution such that \( \dot{U}_1(z) > 0 \) and \( \dot{U}_2(z) < 0 \) for all \( z \in \mathbb{R} \).

In this case, \( d_1 = 1, d_2 = d, p = (p_1, p_2) = (0, a) \), and \( q = (q_1, q_2) = (1, 0) \). The change of variables in (1.4) is \( \xi = (\xi_1, \xi_2) = (U_1, U_2)(z) \) and \( v = (v_1, v_2)(\xi_1, \xi_2) = (\dot{U}_1, \dot{U}_2)(z) \). Thus, if we denote \( f_1(\xi_1, \xi_2) = \xi_1(1 - \xi_1 - c\xi_2) \) and \( f_2(\xi_1, \xi_2) = \xi_2(\xi_1 - b\xi_1 - \xi_2) \), then (1.5) becomes
\[
k^2 \left( v_1 \frac{\partial v_1}{\partial \xi_1} + v_2 \frac{\partial v_1}{\partial \xi_2} \right) + \omega v_1 + f_1 = 0,
\]
\[
k^2 d \left( v_1 \frac{\partial v_2}{\partial \xi_1} + v_2 \frac{\partial v_2}{\partial \xi_2} \right) + \omega v_2 + f_2 = 0.
\]

For an admissible curve \( \Gamma \subseteq \mathbb{R}^2 \), we take a polynomial or algebraic plane curve. Assume that \( v_1 \), say, depends only on \( \xi_1 \) and has the form \( v_1(\xi_1, \xi_2) = \xi_1 - \xi_1^m \) for some \( m > 1 \). This implies that the ordinary differential equation for \( \dot{U}_1 \) can be solved independently of \( U_2 \) (in fact, it is a Bernoulli equation). However, note that once we have chosen \( \Gamma \) and \( U_1 \) is known, then \( U_2 \) is fixed and, consequently, \( v_2 \) is determined. These considerations also guarantee that the consistency relations in (3.7) are satisfied.

Suppose now that we have selected an admissible curve \( \Gamma \) and a trial function \( v \) from \( X_\Gamma \). Substituting these into the error functional \( E_\Gamma \) yields a nonnegative multivariate function, say \( E = E(a, b, c, d) \). Our objective is to find positive \( a, b, c, \) and \( d \) such that \( E(a, b, c, d) = 0 \). Due to the complexity of \( E \), let us assume that two of
the parameters are fixed and view $E$ as a function of the other two parameters, e.g. suppose that $a$ and $c$ are fixed and consider $E = E(b, d)$. A candidate pair $(b^*, d^*)$ for which $E(b^*, d^*) = 0$ can be obtained by determining the critical points of $E$, i.e. by solving the pair of equations

$$\frac{\partial E}{\partial b}(b^*, d^*) = 0, \quad \frac{\partial E}{\partial d}(b^*, d^*) = 0$$

for $(b^*, d^*)$. We then substitute these back into $E = E(b, d)$ to see which ones actually satisfy the equation $E(b^*, d^*) = 0$ and therefore yield exact solutions. If these steps can be carried out, then we compute $k^2$ and $\omega$ from (2.4) and (2.5), respectively.

We programmed the procedure outlined above in Maxima and the results are summarized as follows:

(i)

$$\xi_1(s) = s, \quad \xi_2(s) = as^2 - 2as + a,$$

$$v_1(\xi_1, \xi_2) = \xi_1 - \xi_1^2, \quad v_2(\xi_1, \xi_2) = -2a\xi_1(1 - \xi_1)^2,$$

$$b^* = -ac + \frac{5a}{3} + 2, \quad d^* = \frac{1}{3c},$$

$$k^2 = \frac{ac}{2}, \quad \omega = \frac{ac}{2} - 1,$$

$$U_1(z) = \frac{s_0}{s_0 + (1 - s_0)e^{-z}}, \quad U_2(z) = aU_1(z)^2 - 2aU_1(z) + a;$$

(ii)

$$\xi_1 = s, \quad \xi_2 = as - 2as^{1/2} + a,$$

$$v_1(\xi_1, \xi_2) = \xi_1 - \xi_1^{3/2}, \quad v_2(\xi_1, \xi_2) = -a\xi_1^{1/2}(1 - \xi_1^{1/2}),$$

$$b^* = -ac + 5a + 5, \quad d^* = \frac{ac - 5}{3},$$

$$k^2 = \frac{2}{3}(ac + 1), \quad \omega = \frac{ac}{2} - 1,$$

$$U_1(z) = \frac{s_0}{(s_0^{1/2} + (1 - s_0^{1/2})e^{-z/2})^2}, \quad U_2(z) = aU_1(z) - 2aU_1(z)^{1/2} + a;$$

(iii)

$$\xi_1(s) = s, \quad \xi_2(s) = a - as^{1/2},$$

$$v_1(\xi_1, \xi_2) = \xi_1 - \xi_1^{3/2}, \quad v_2(\xi_1, \xi_2) = -\frac{a}{2}\xi_1^{1/2}(1 - \xi_1^{1/2}),$$

$$b^* = -ac + 2a + \frac{5}{3}, \quad d^* = -3ac + 6a + 5,$$
Note that $a$ and $c$ in all of the above cases have to be restricted in such a way that $b^* > 0$ and $d^* > 0$. These solutions recover the ones obtained in [9]. In fact, all of the exact solutions obtained in [10] also can be recovered by the method we propose here. Moreover, our approach leads to a method for obtaining approximate traveling wave solutions and estimating the error in the approximation.

5. Concluding Remarks

The ubiquity of systems of reaction-diffusion equations to describe phenomena in different areas of the physical and biological sciences makes them a popular subject of study. However, the nonlinear nature of the forcing terms in these systems essentially force studies to be restricted to perturbation methods and/or numerical methods. Here, we have focused on one aspect of these systems, namely their traveling wave solutions. We have developed a variational formulation of the existence problem for these traveling waves. Our use of this variational formulation, however, was to obtain exact and approximate traveling wave solutions, including their error estimates.

By carefully defining the set of admissible curves and their parametrizations, we were able to prove a lemma that explicitly gives the two parameters, $k^2$ and $\omega$, in terms of the nonlinear source term in the original reaction-diffusion system and of the first- and second-order derivatives of the original traveling wave variable. With the equivalent system of first-order differential equations, we proved a theorem that guarantees a variational solution if and only if (3.1)–(3.3) are satisfied. This variational solution is then identified as the solution of the boundary-value problem satisfying the boundary conditions through a set of consistency relations.

An error functional was defined and several examples were given to demonstrate the use of this variational formulation to obtain exact and approximate solutions. For nonlinear diffusion equations, we gave three examples: (1) Fisher’s equation with a quadratic nonlinear source term, (2) Nagumo’s equation with a cubic nonlinear source term, and (3) an equation with a quartic nonlinear source term. Finally, we gave an example of a multi-component case, namely the Lotka–Volterra competition-diffusion system.

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