Abstract Interpretation (2/2)

Martin Kellogg

Reading quiz: abstract interpretation

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- today's quiz is on paper, and also covers the topics of last week's class
- you have 15 minutes to complete it. When you're finished, bring it to Kazi in the back.
- you may use any **hand-written** notes that you took during last class
    - this includes notes on a tablet or similar, if you wrote them with a stylus
        - but I will be looking over your shoulder if you do :)

Agenda: abstract interpretation, part 2

- review and clarifications from last week
- more on soundness
- refinement and branching
- widening
- Stein’s algorithm example
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  - usually represented as tables

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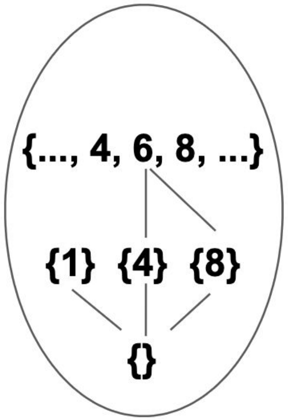

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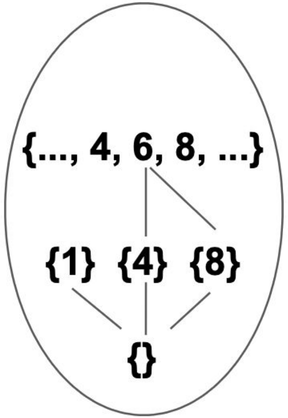

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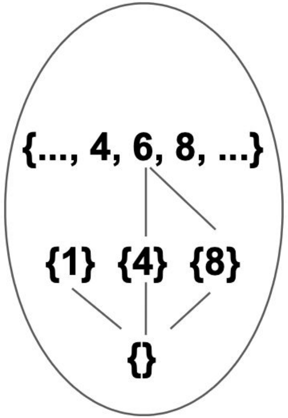

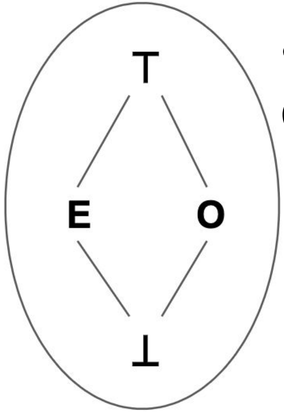

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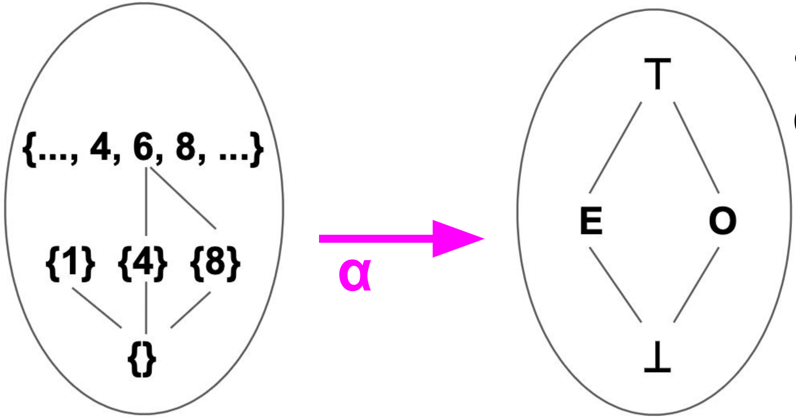

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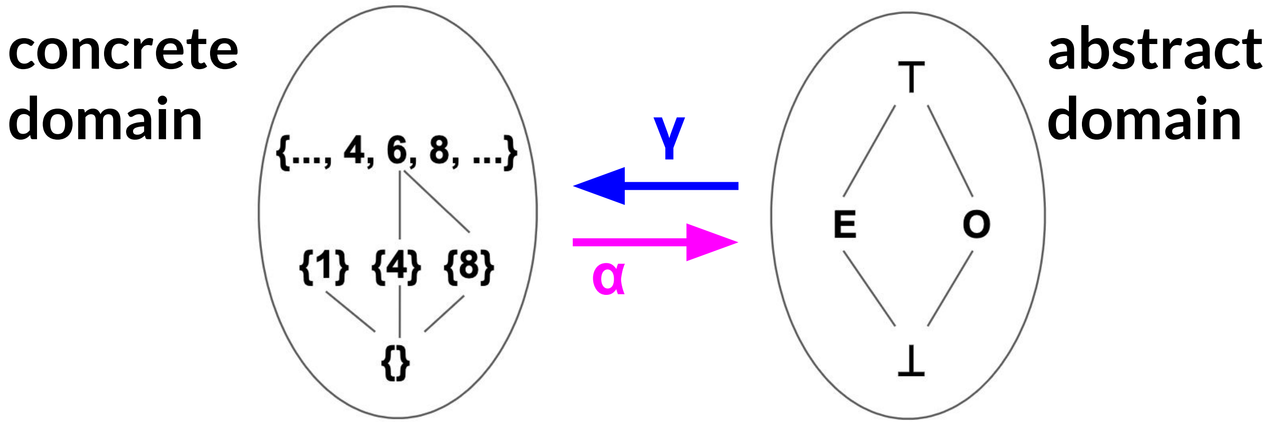

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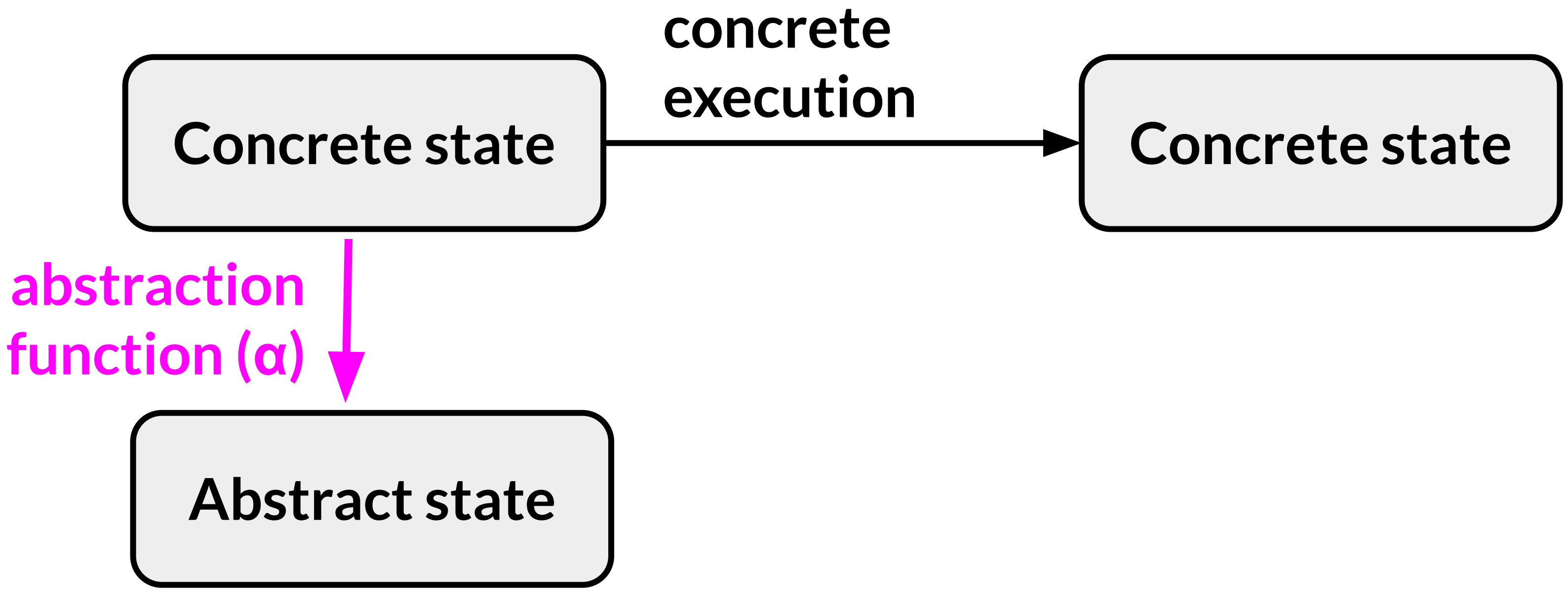

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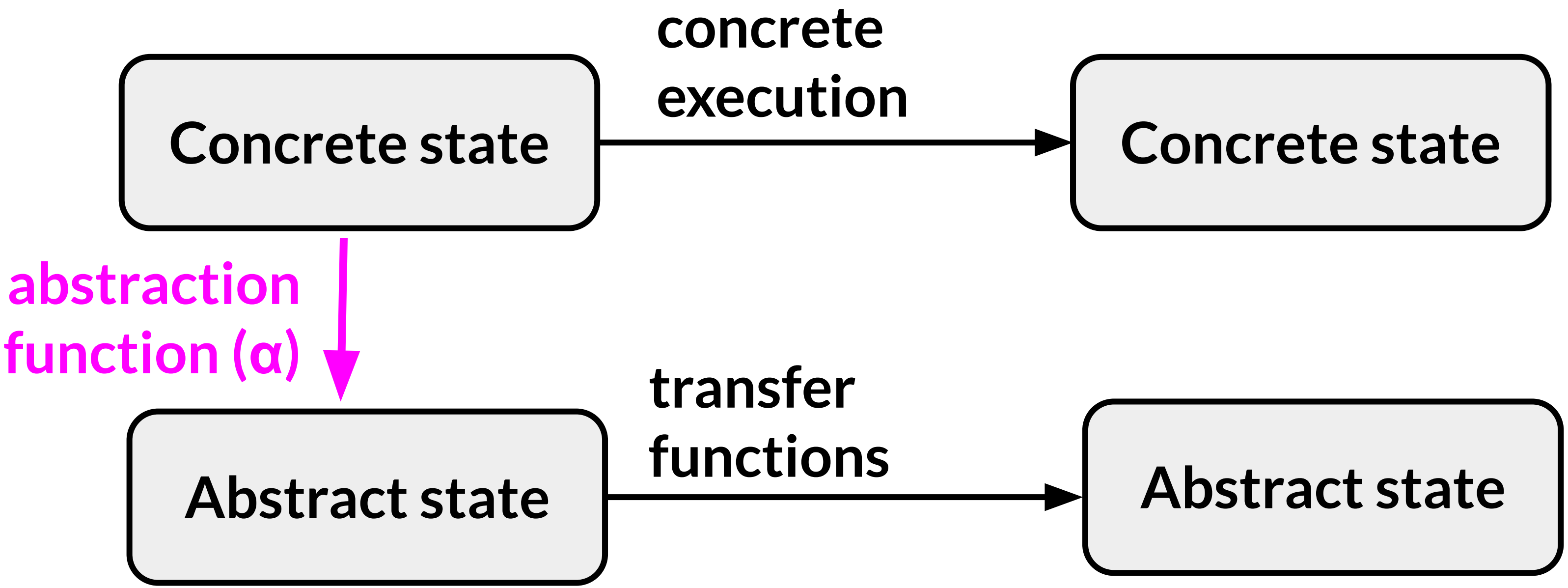

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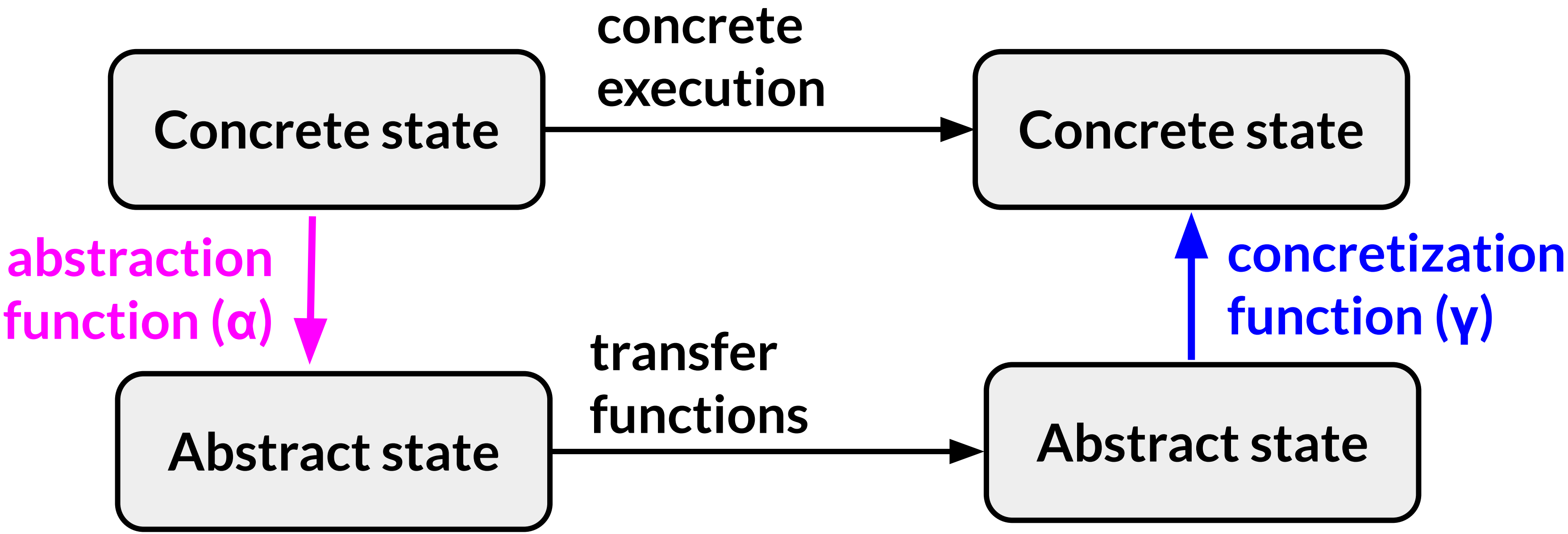

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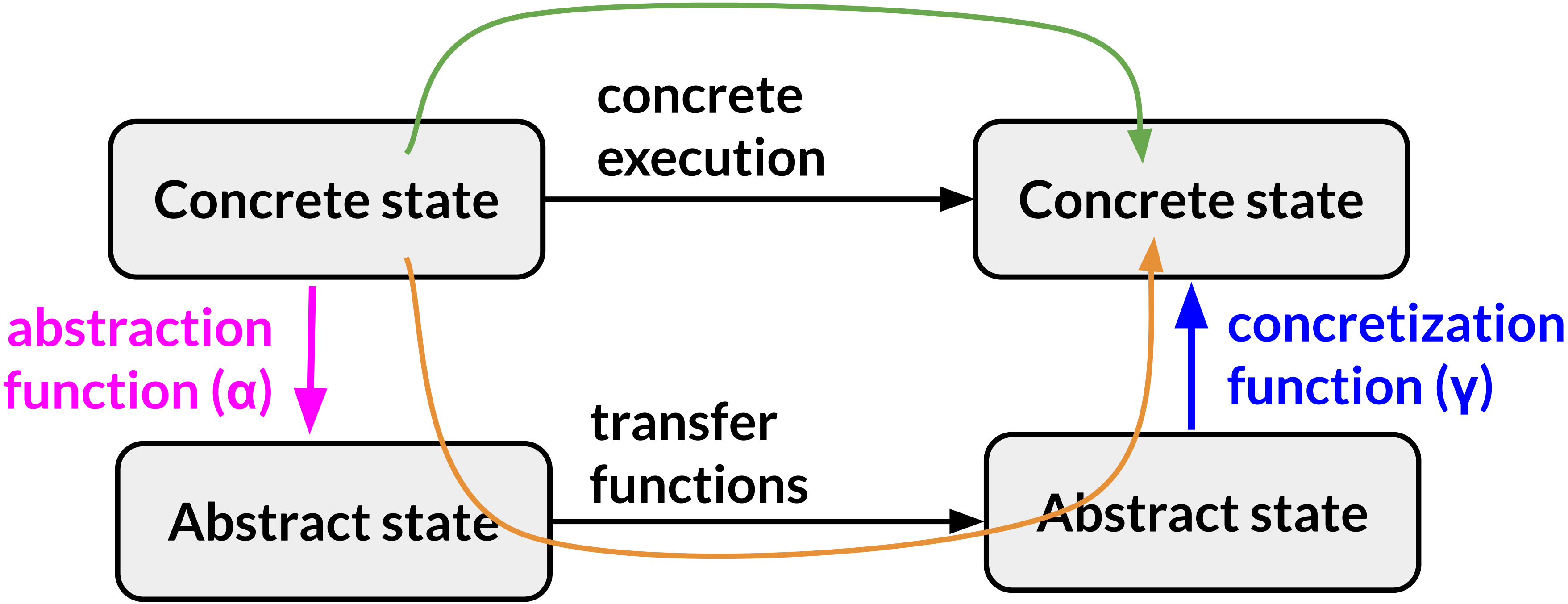

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- last week, I went through an extended example of how to get a parity analysis to work on **one program**
  - however, that was just an example!
    - an abstract interpretation is applicable to **any program**
  - one of the **key challenges** in abstract interpretation design is figuring out the **right set of examples** to handle precisely
    - when you're implementing your divide-by-zero analysis, I **strongly recommend** that you write out some examples as test cases!
      - you can just add them to the existing test

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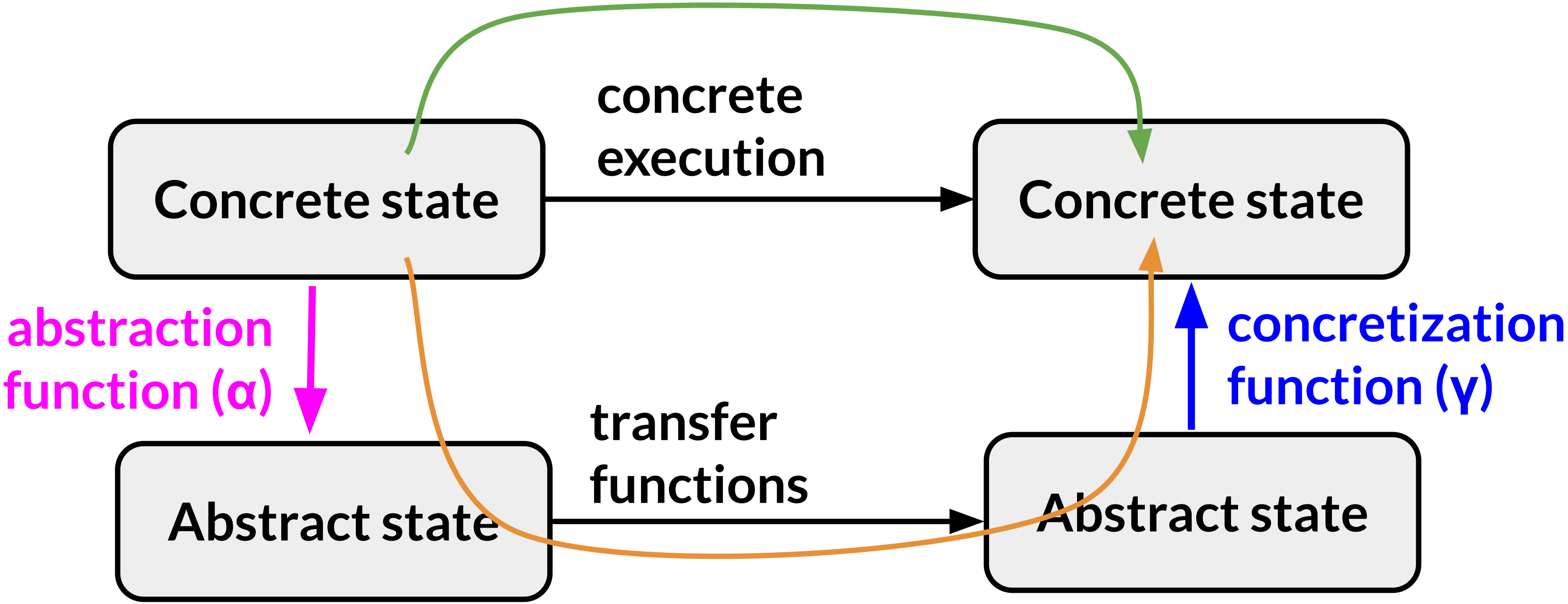
More on soundness: using a Galois connection



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possible results of concrete execution (**green line**)

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concretization

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More on soundness: example proof

- let’s carry out an example proof using this technique ourselves on the **plus transfer function** from our simple parity analysis

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{ even, odd } = top
      /           \
  {even}       {odd}
      \           /
      {} = bottom
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+	T	even	odd	⊥
T	T	T	T	⊥
even	T	even	odd	⊥
odd	T	odd	even	⊥
⊥	⊥	⊥	⊥	⊥

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even	$\mp$	even	odd	⊥
odd	$\mp$	odd	even	⊥
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    - $\{\} \subseteq \{\}$

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    - {} $\subseteq$ {}
    - QED

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⊥	±	±	±	±

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  - we could do them one-by-one…

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- Now we need to handle the more **complex cases** in the middle
  - we could do them one-by-one…
  - but we can skip some because addition is commutative
    - so we don’t need to worry about order

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More on soundness: example proof

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  - we co…
  - but w… is commutative
    - so… rder

+	$\top$	even	odd	$\bot$
$\top$	$\cancel{\top}$	$\cancel{\top}$	$\cancel{\top}$	$\cancel{\bot}$
even	$\cancel{\top}$	even	**odd**	$\cancel{\bot}$
odd	$\cancel{\top}$	**odd**	even	$\cancel{\bot}$
$\bot$	$\cancel{\bot}$	$\cancel{\bot}$	$\cancel{\bot}$	$\cancel{\bot}$

in other words, the two **orange** cases are the same!

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- we dispatch these three by considering each case individually
  - they're all basically the same, so we're only going to do one

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More on soundness: example proof

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More on soundness: example proof

- *c* is some addition statement $x + y$
  - we know that concretely x is odd and y is even (why?)
    - formally, we would state this as $x \% 2 = 1$ and $y \% 2 = 0$

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  - $\alpha(x)$ is **odd** (the abstract value), and $\alpha(y)$ is **even** (the AV)

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- what's $\alpha(c)$?
  - $\alpha(x)$ is **odd** (the abstract value), and $\alpha(y)$ is **even** (the AV)
- $T_+(\alpha(c))$ is just applying our transfer function: result is the **odd** AV
- $\gamma($**odd**$)$ is the set of all odd integers, which does contain itself

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  - $2a + 1 + 2b = 2(a+b) + 1$, which we can easily prove is the set of all odd integers
- what's $\alpha(c)$?
  - $\alpha(x)$ is **odd** (the abstract value), and $\alpha(y)$ is **even** (the AV)
- $T_+(\alpha(c))$ is just applying our transfer function: result is the **odd** AV
- γ(**odd**) is the set of all odd integers, which does contain itself $\square$

Agenda: abstract interpretation, part 2

- review and clarifications from last week
- more on soundness
- **refinement and branching**
- widening
- Stein’s algorithm example
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Refinement

Consider the following program:

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Insight: *anything* you can figure out by reasoning through the program by hand, an abstract interpretation can do too!

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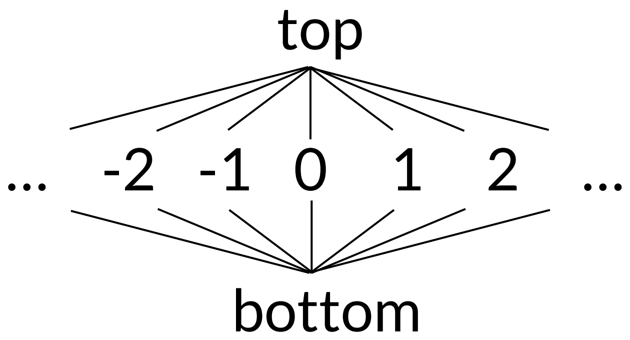

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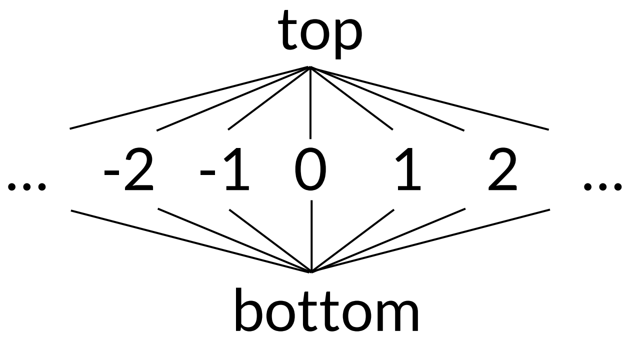

not enough! need sets

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draw in the correct lattice here:

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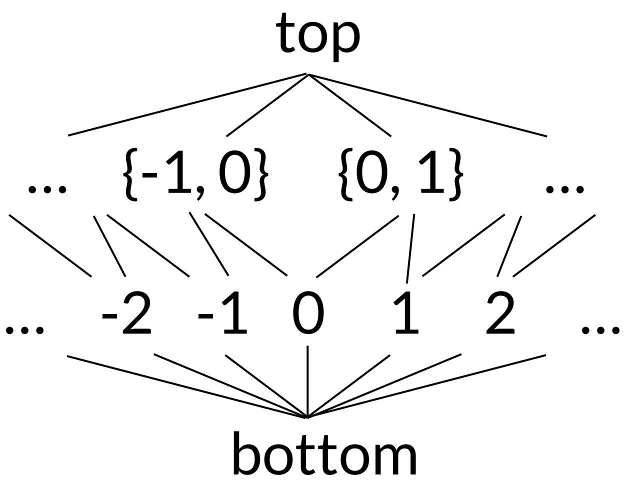

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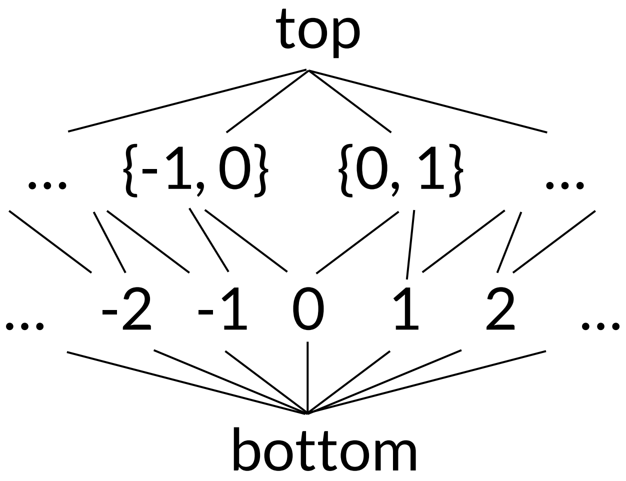

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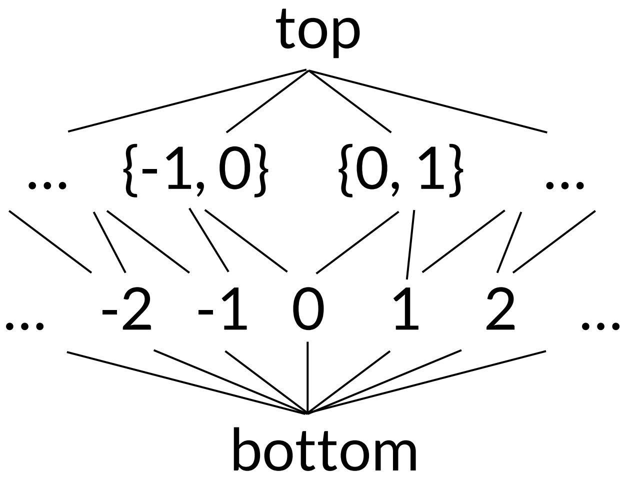

Does this permit us to prove the value of x at the end?
NO (need transfer function)

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    - loop guard is false, so we invert the operator

Refinement

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(on the whiteboard. Start by drawing a CFG, then execute the algorithm. Put the CFG to the side and don't erase it.)

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  - it would be nice if we could have sets of **arbitrary size**
    - and we shouldn't need to **reimplement** our analysis each time we need to reason about differently-sized sets
  - do you think that's possible?
    - We can use ***widening operators*** to allow this (sort of)

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- this is safe because the analysis isn't required to take **the least** upper bound so long as it chooses **an** upper bound
- example widening operator for constant propagation:
  - if an abstract value has changed at least five times, go to top

Widening

Let’s return to the previous example:

```
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    - but abstract interpretation was always imprecise, so that's okay
- A nice fact about implementing an abstract interpretation is that it is **always safe** to apply a widening operator
    - this means it's easy to support complex language features: just immediately widen any values that they impact
        - "go to top" is a sound policy in all situations

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- **Stein's algorithm example**
- analysis implementation demo

Another example: Stein's algorithm

```
def gcd(int a, int b):
  if a == 0 or b == 0:
    return 0
  int expt = 0
  while a is even and b is even:
    a = a / 2
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    expt = expt + 1
  while a != b:
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- "constant propagation" can prove no divisions by zero!

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- 1st: a is odd or b is odd
- 2nd: a eventually equals b

Another example: Stein's algorithm: parity

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Fortunately, we already know an analysis for parity. Let's use it (on the board; requires a CFG).

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Fortunately, we already know an analysis for parity. Let's use it (on the board; requires a CFG).

- we ran into a problem: we **can't prove** that a and b are eventually odd!
  - the transfer function for even / is2 returns T
- in this case, that's actually correct!
  - the program does not terminate on all inputs
  - -1, 1 is a counterexample

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Course announcements

- This week's homework is **individual** (you may not work with a partner)
  - this is a difference from previous homeworks!
- early next week I will send out a survey (via Discord) about what topic we should cover in the last week of class (April 25)
  - please give this some serious thought!
  - the survey will be open until next week's class, and I will announce the result during class