Ferromagnetism at nanoscale

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Joint work with

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- Theresa Simon, Institut für Angewandte Mathematik, University of Bonn, Germany
- Valeriy Slastikov, School of Mathematics, University of Bristol, UK

Further details: MS19

MS19: NANOMAGNETISM: AN...

Session 27K: MS19-1: Nanomagnetism: Analysis, Modeling, Simulation, Experiments

Wed. May 26, 2021 1:00 PM - 3:00 PM Room K

MS19: NANOMAGNETISM: AN...

Session 29L: MS19-2: Nanomagnetism: Analysis, Modeling, Simulation, Experiments

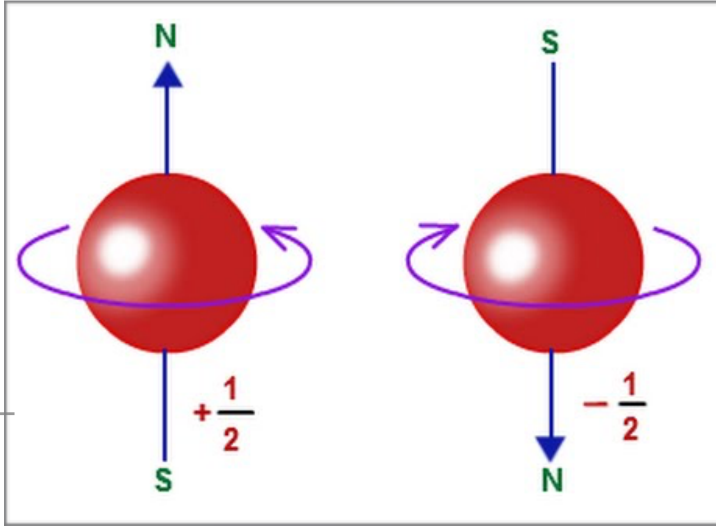
Wed. May 26, 2021 4:30 PM - 6:30 PM Room L

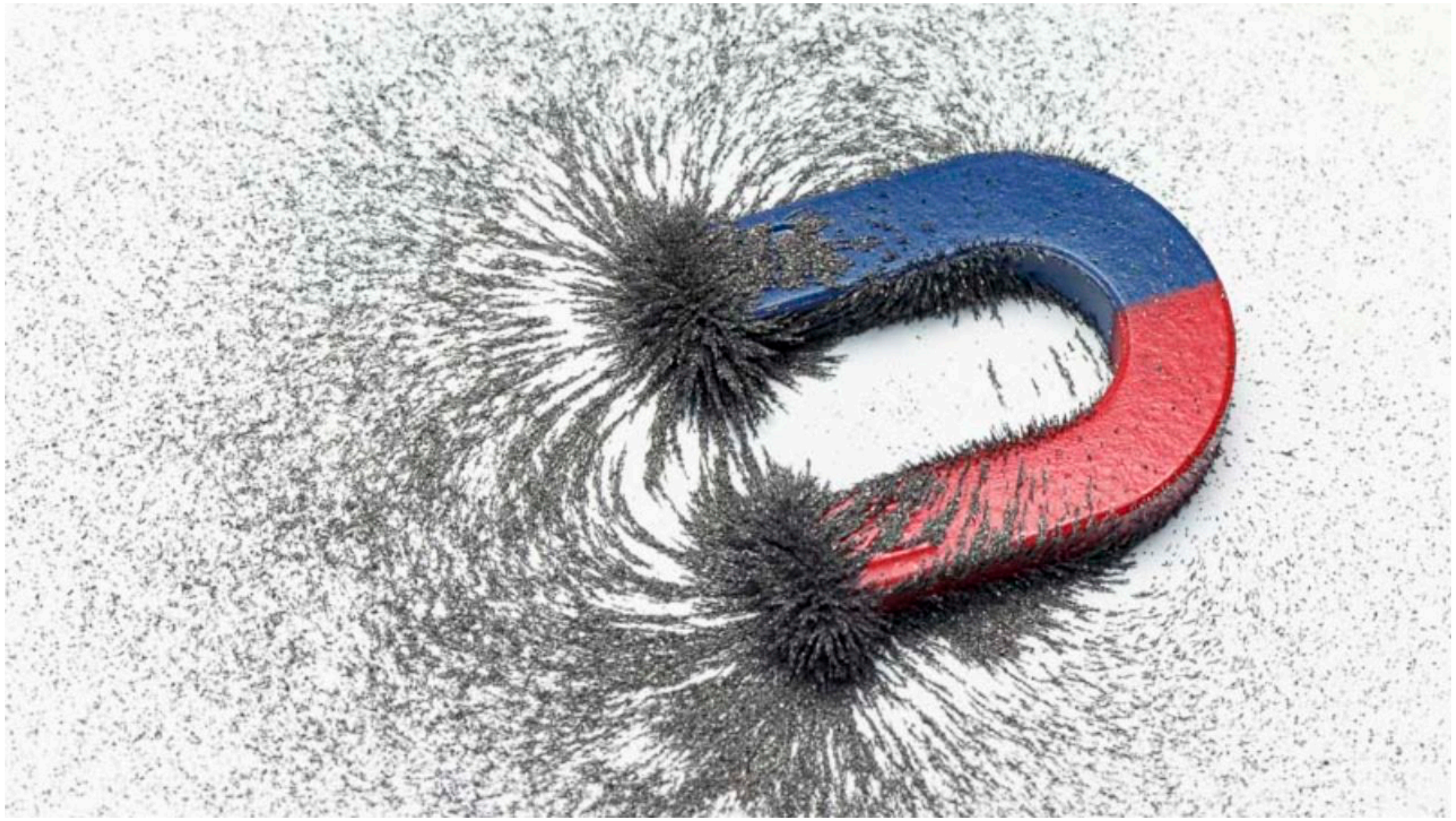
MS19: NANOMAGNETISM: AN...

Session 32J: MS19-3: Nanomagnetism: Analysis, Modeling, Simulation, Experiments

Thu. May 27, 2021 4:30 PM - 8:30 PM Room J

Magnetism and magnets





- one of the oldest known natural phenomena showing ***action at a distance***
- however, deeply rooted in quantum mechanics => understanding only begins to emerge in the 20th century: *electron spin*

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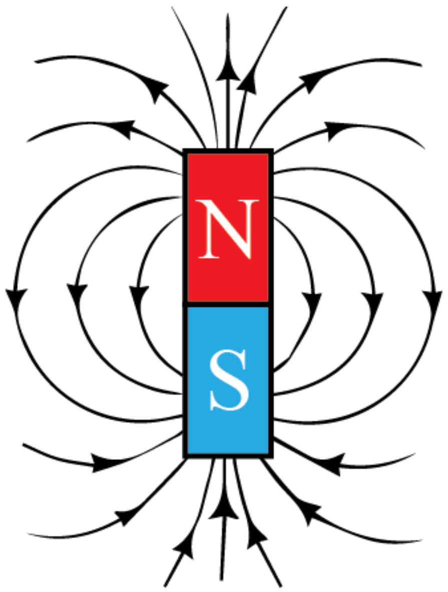
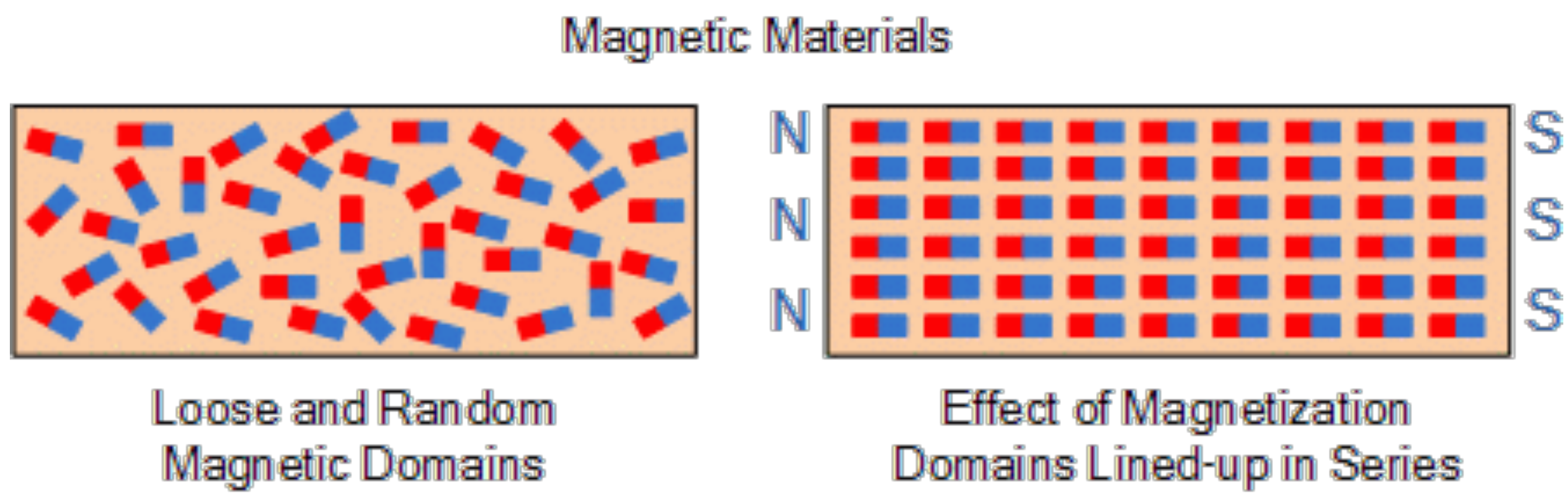
W. Pauli, P. Dirac, W. Heisenberg, 1920s

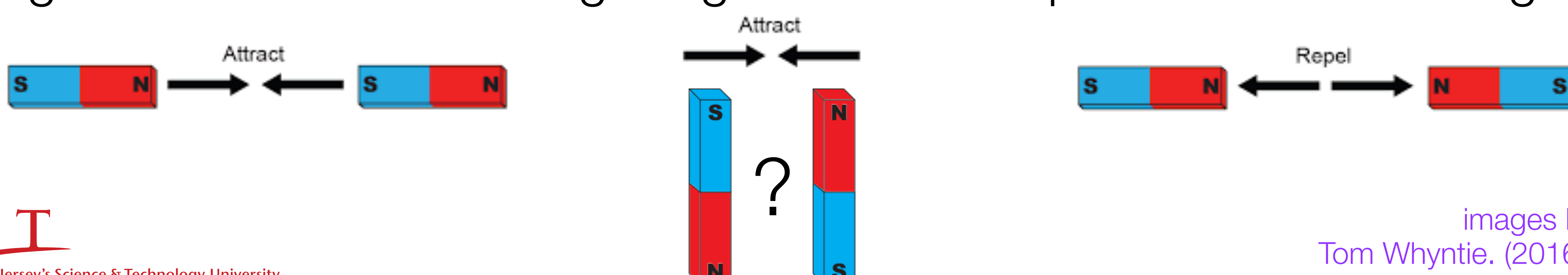
images borrowed from:
learning-center.homesciencetools.com
starecat.com

Magnetism and magnets





- spins act as tiny magnetic dipoles
- quantum-mechanical interaction between spins: <u>*exchange*</u>
- in transition metals below the *critical temperature*, exchange results in local spin alignment into the <u>*ferromagnetic state*</u>
- magnetic field mediates long-range attraction/repulsion between magnets



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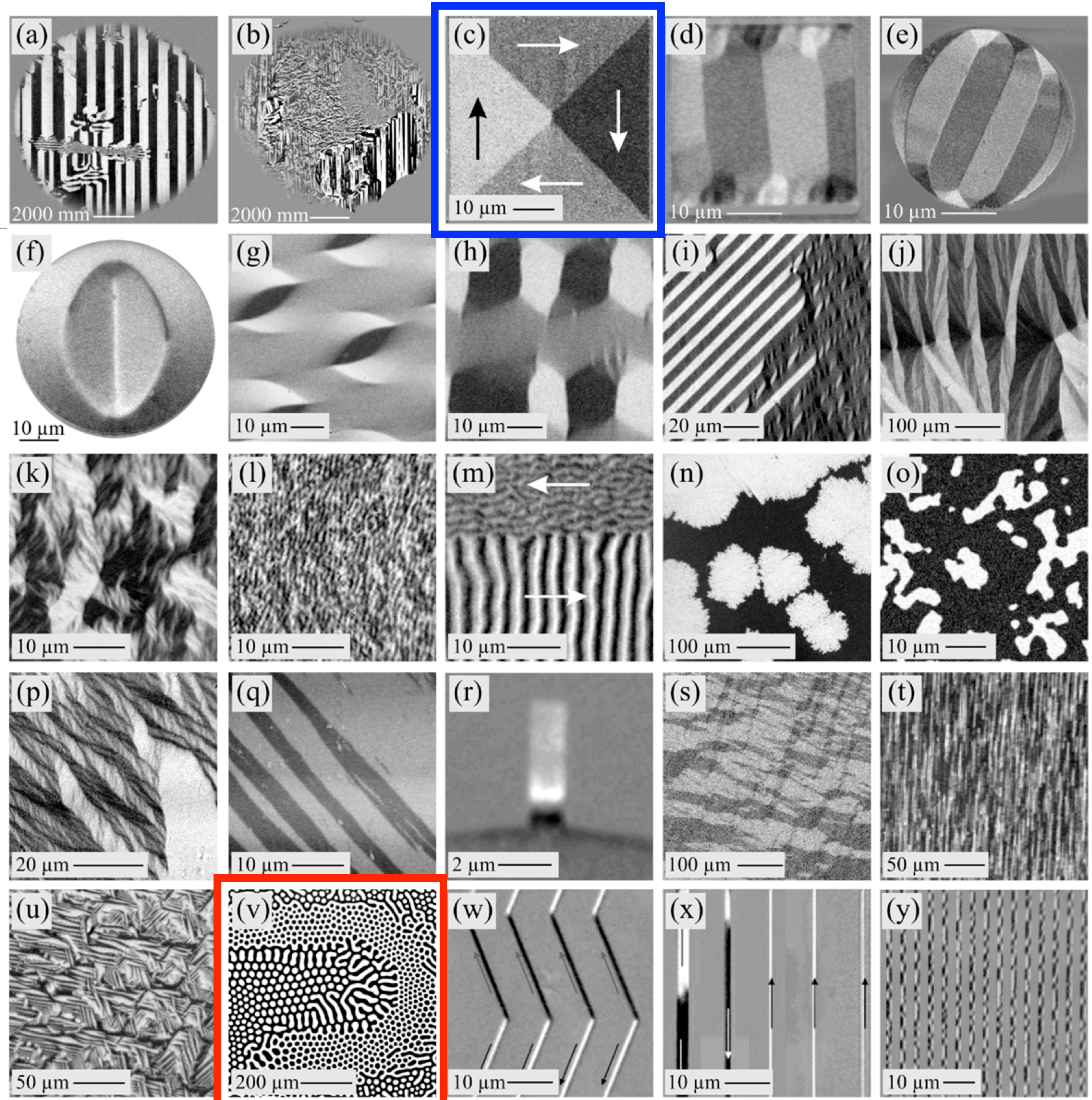
images borrowed from:
Tom Whyntie. (2016), zenodo.com
mammothmemory.net

Magnetic domains

- stray field frustrates the ferromagnetic order

- gives rise to a great variety of spin textures

- principle of pole avoidance



J. McCord, *J. Phys. D: Appl. Phys.* **48**, 333001 (2015)

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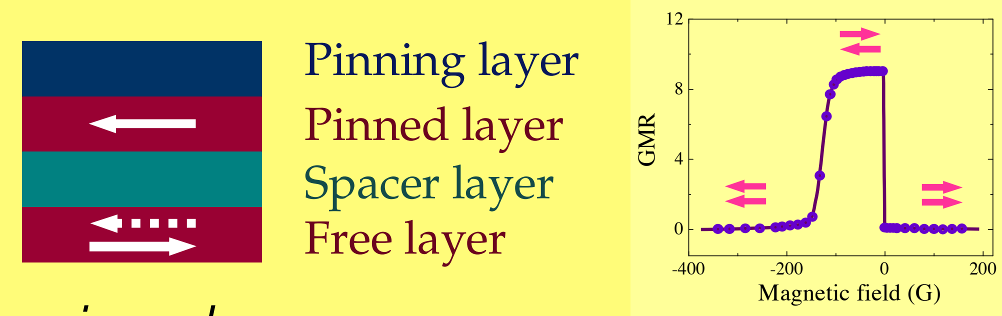
Birth of nanomagnetism



spin valve

Tsymbal and Pettifor, 1999

The Nobel Prize in Physics 2007

Photo: U. Montan
Albert Fert
Prize share: 1/2

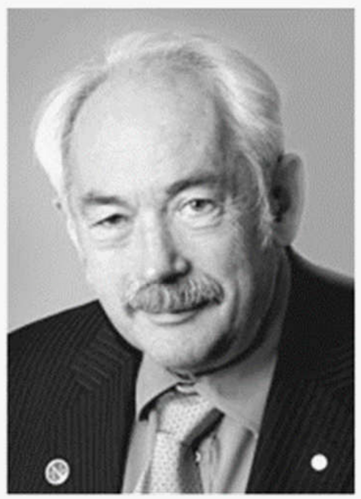
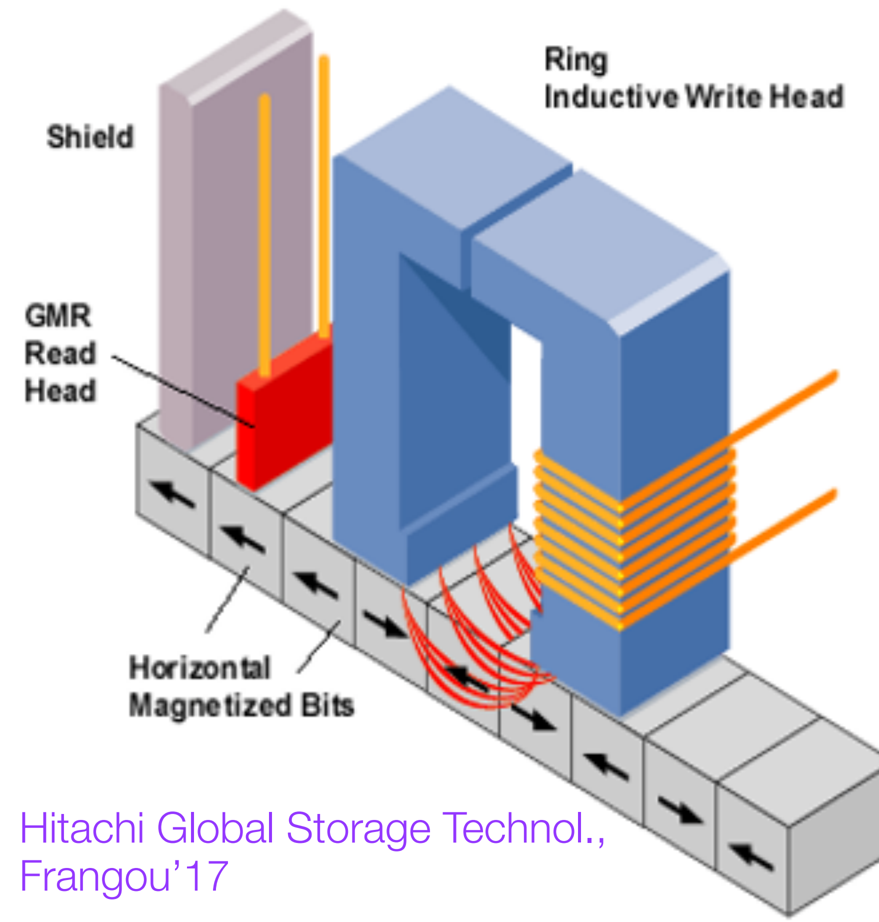
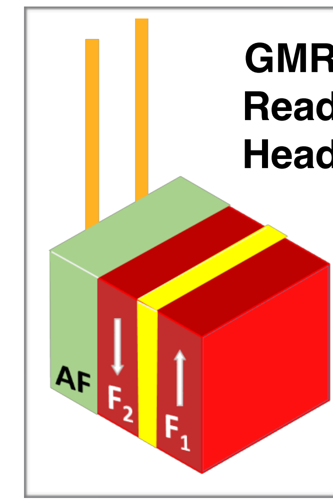


Photo: U. Montan
Peter Grünberg
Prize share: 1/2

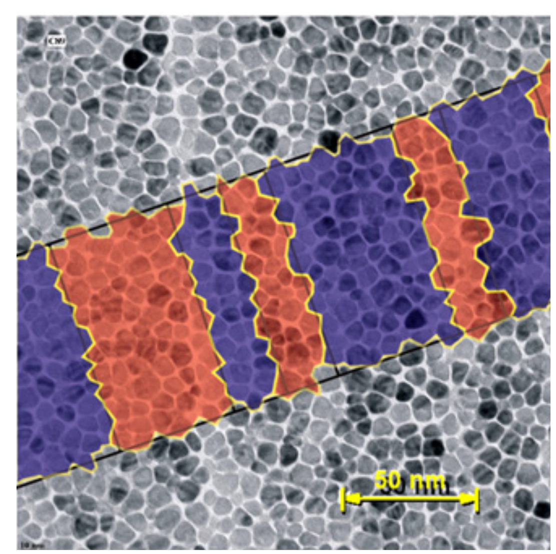
The Nobel Prize in Physics 2007 was awarded jointly to Albert Fert and Peter Grünberg *"for the discovery of Giant Magnetoresistance"*



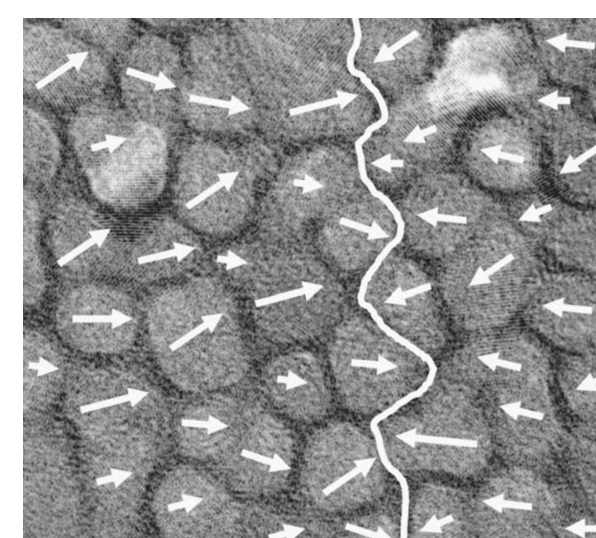


Hitachi Global Storage Technol., Frangou'17

paved the way to a new discipline in applied physics: ***spintronics***

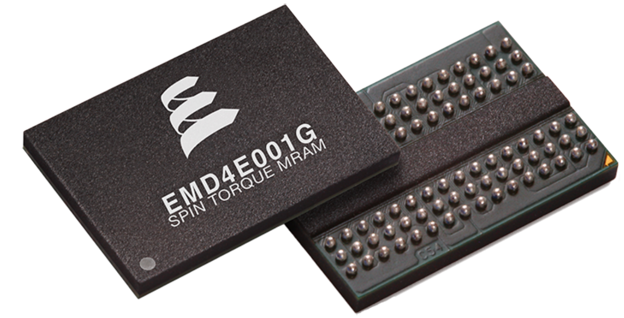


Terris'09

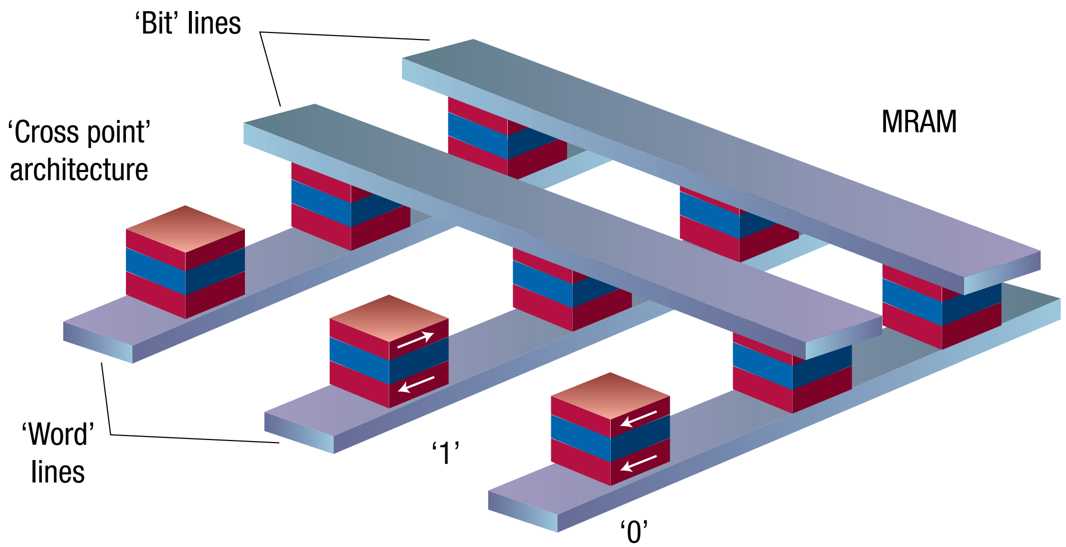


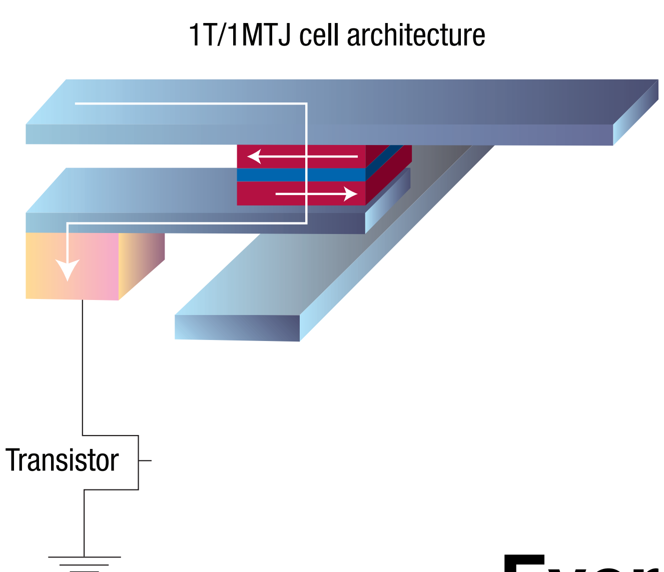
Piramanayagam'07

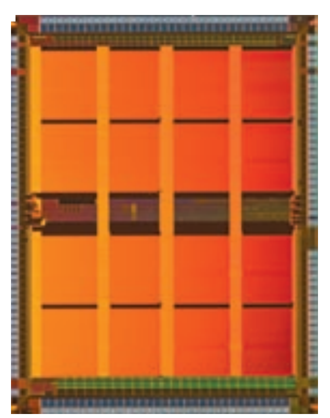
Non-volatile computer memories



magnetoresistive random access memory (MRAM):

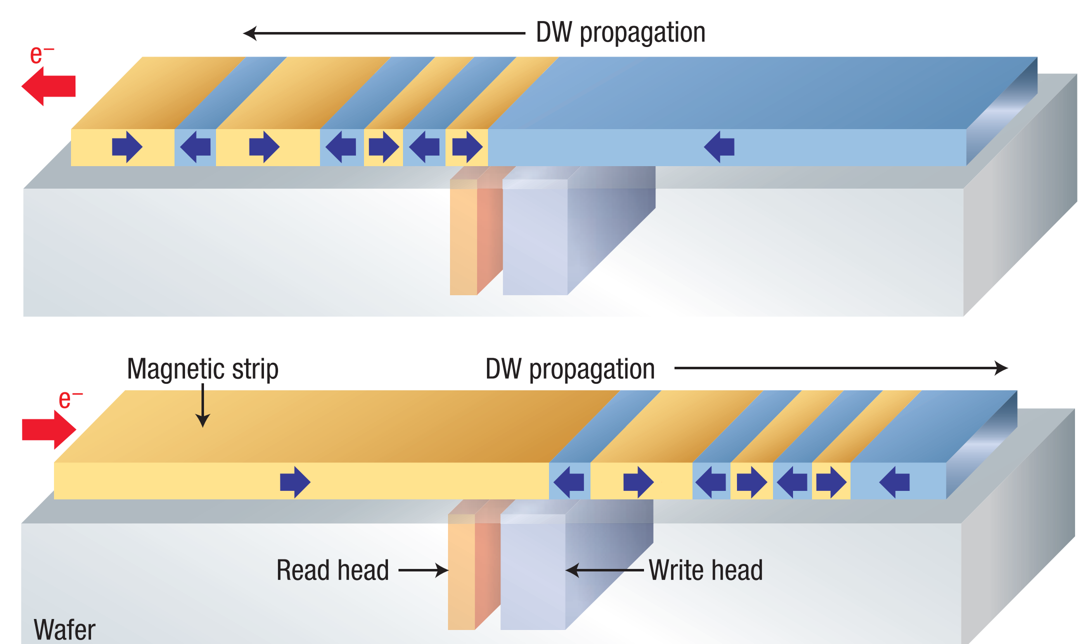






Everspin now ships 1Gb, 667 MHz chips

racetrack memory:



"hard disk drive" without moving parts

C. Chappert et al., *Nature Mat.* **6**, 813 (2007)

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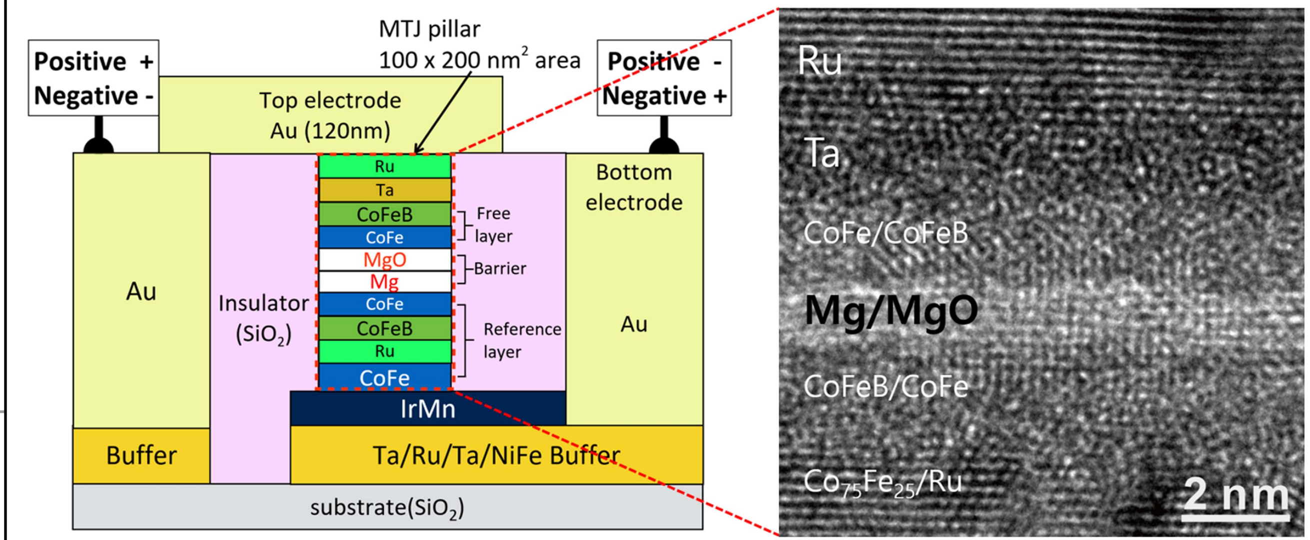
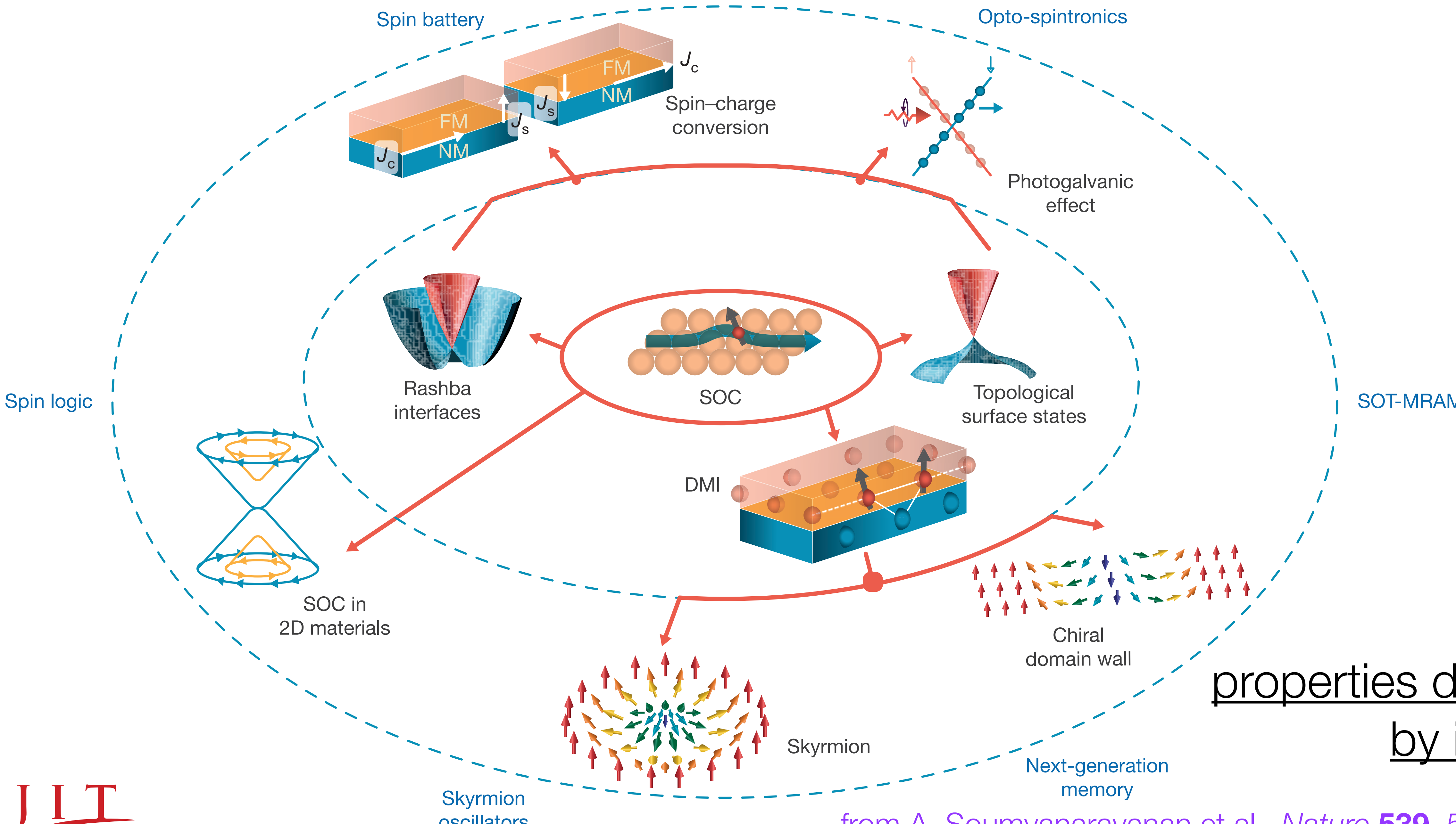
Next generation materials

atomically thin multilayers with strong spin-orbit coupling (SOC):



C.-M. Choi et al., *Semicond. Sci. Technol.* 32, 105007 (2017)



<u>properties dominated by interfaces</u>

from A. Soumyanarayanan et al., *Nature* **539**, 509-517 (2016)

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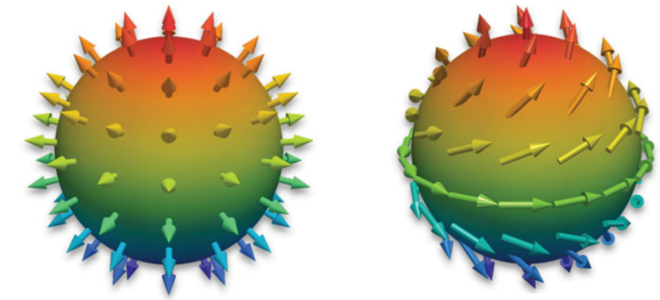
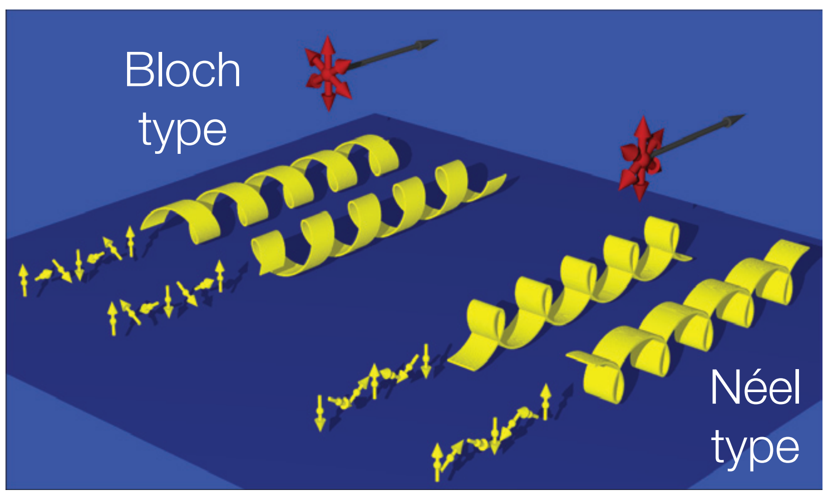
Topological spin textures

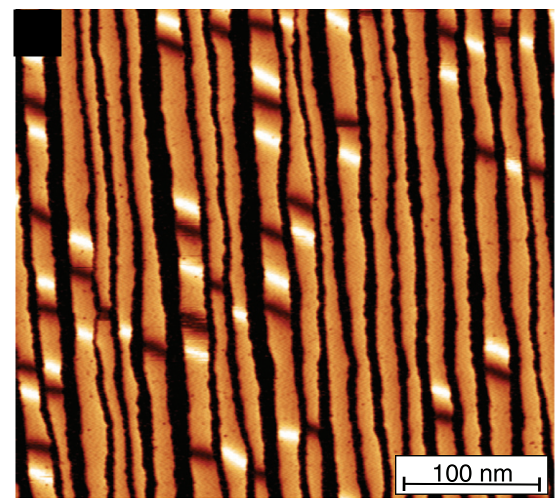


T. Lancaster, *Contemp. Phys.* **60**, 246-261 (2019)

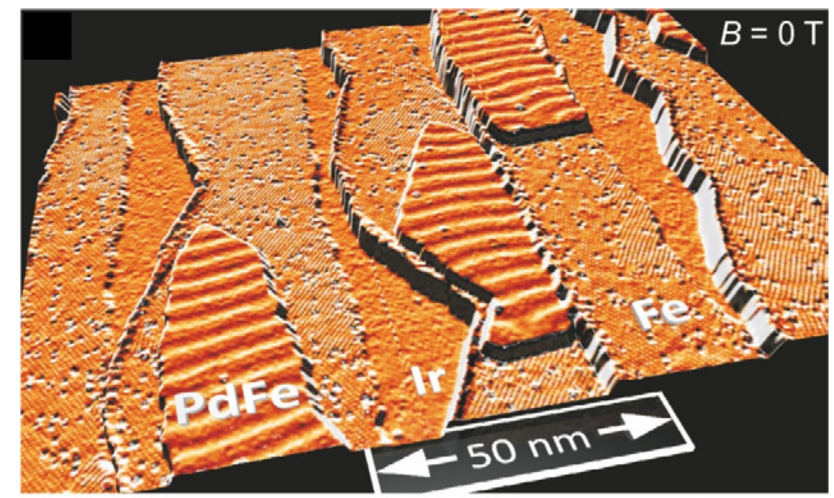
spin spirals and chiral domain walls from **Dzyaloshinskii-Moriya interaction** (DMI):



2ML Fe on W(110)

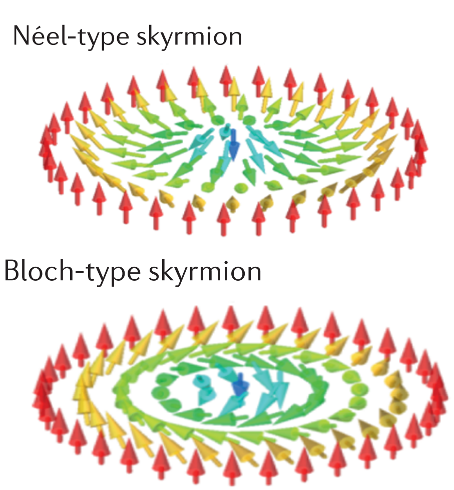


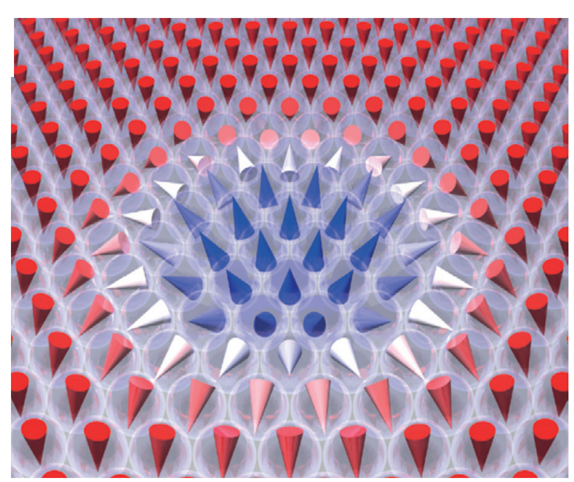
Pd/Fe bilayer on Ir(111)

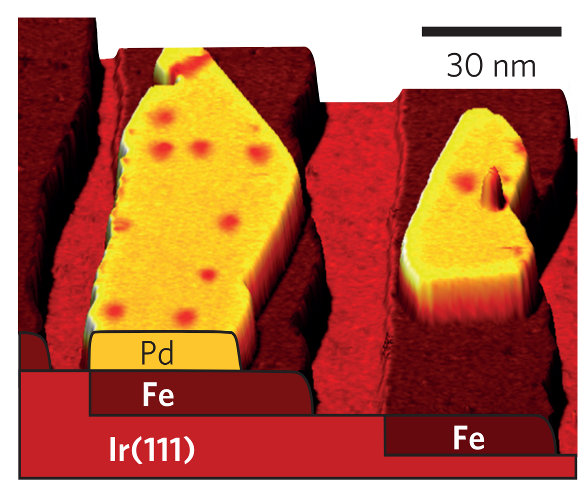


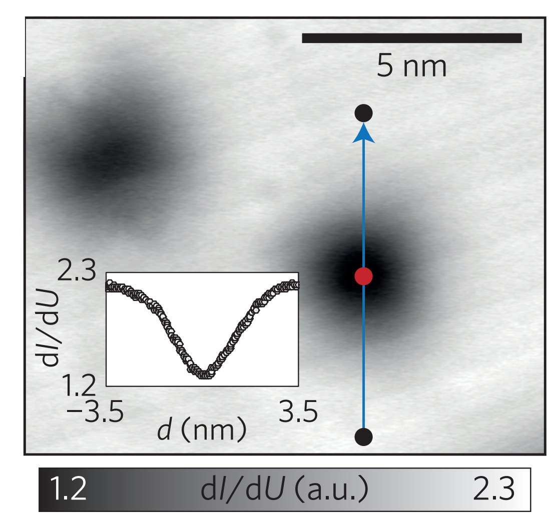
K. von Bergmann et al., J. Phys.: Condens. Matter **26**, 394002 (2014)

magnetic skyrmions:









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C. Hanneken et al., *Nature Nanotechnol.* **10**, 1039–1042 (2015)

Micromagnetic modeling framework

continuum mesoscopic theory $D \subset \mathbb{R}^3 \qquad \mathbf{M} : D \to \mathbb{R}^3 \qquad |\mathbf{M}| = M_s$

<u>example</u>: uniaxial crystal

$$\boxed{\begin{aligned} E(\mathbf{M}) = \frac{A}{M_s^2} \int_D |\nabla \mathbf{M}|^2 \, d^3 r + \frac{K}{M_s^2} \int_D \left(M_1^2 + M_3^2 \right) d^3 r \\ - \mu_0 \int_D \mathbf{M} \cdot \mathbf{H} \, d^3 r + \mu_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{M}(\mathbf{r}) \, \nabla \cdot \mathbf{M}(\mathbf{r}')}{8\pi |\mathbf{r} - \mathbf{r}'|} \, d^3 r \, d^3 r' \end{aligned}}$$

Landau and Lifshitz, 1935; Néel, 1944; Kittel, 1949; Brown, 1963; Hubert and Schaefer, 1998

<u>statics</u>:

the observed magnetization patterns are local or global energy minimizers

<u>dynamics</u>:

the Landau-Lifshitz-Gilbert equation (in the Landau-Lifshitz form)

$$(1 + \alpha^2) \frac{\partial \mathbf{M}}{\partial t} = -\gamma_0 \left(\mathbf{M} \times \mathbf{H}_{\text{eff}} + \frac{\alpha}{M_s} \mathbf{M} \times \mathbf{M} \times \mathbf{H}_{\text{eff}} \right), \qquad \mathbf{H}_{\text{eff}} = -\frac{1}{\mu_0} \frac{\delta E}{\delta \mathbf{M}}$$

<u>stochasticity can be added</u>

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Stray field energy

definition:

$$\boxed{E_{\mathrm{s}}(\mathbf{M}) = \frac{\mu_0}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\operatorname{div} \mathbf{M}(\mathbf{r}) \operatorname{div} \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3 r \, d^3 r'}$$

equivalent to:

$$E_{\mathrm{s}}(\mathbf{M}) = \frac{\mu_0}{2} \int_{\Omega} \mathbf{M} \cdot \nabla U_{\mathrm{d}} \, d^3 r = \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla U_{\mathrm{d}}|^2 \, d^3 r \qquad \Delta U_{\mathrm{d}} = \operatorname{div} \mathbf{M}$$

leading to the static Maxwell's equations for $\mathbf{H}_{\mathrm{d}} = -\nabla U_{\mathrm{d}}$

$$\operatorname{div} \mathbf{B}_{\mathrm{d}} = 0, \qquad \operatorname{curl} \mathbf{H}_{\mathrm{d}} = \mathbf{0},$$

coupled through the magnetization of the material:

$$\mathbf{B}_{\mathrm{d}} = \mu_0 (\mathbf{H}_{\mathrm{d}} + \mathbf{M}).$$

Remark: can also use the vector potential: $\mathbf{B} = \operatorname{curl} \mathbf{A}$, in the Coulomb gauge:

$$\operatorname{curl} (\operatorname{curl} \mathbf{A}_{\mathrm{d}}) = -\Delta \mathbf{A}_{\mathrm{d}} = \mu_0 \operatorname{curl} \mathbf{M} \qquad \operatorname{div} \mathbf{A}_{\mathrm{d}} = 0$$

Variational principles for magnetostatic energy

a minimax principle:

Brown, 1963
James and Kinderlehrer, 1990

$$E_\mathrm{s}(\mathbf{M}) = \max_{U \in \mathring{H}^1(\mathbb{R}^3)} \mu_0 \int_{\mathbb{R}^3} \left(\mathbf{M} \cdot \nabla U - \frac{1}{2} |\nabla U|^2 \right) d^3 r$$

maximize in U at fixed $\mathbf{M}$, then minimize in $\mathbf{M}$

another representation:

$$E_\mathrm{s}(\mathbf{M}) = \frac{1}{2} \int_\Omega \left(\mu_0 |\mathbf{M}|^2 - \mathbf{M} \cdot \mathrm{curl}\, \mathbf{A}_\mathrm{d} \right) d^3 r = \frac{1}{2\mu_0} \int_{\mathbb{R}^3} |\mathrm{curl}\, \mathbf{A}_\mathrm{d} - \mu_0 \mathbf{M}|^2 \, d^3 r$$

leads to

Asselin and Thiele, 1986

$$E_\mathrm{s}(\mathbf{M}) = \min_{\substack{\mathbf{A} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3) \\ \mathrm{div}\, \mathbf{A} = 0}} \frac{1}{2\mu_0} \int_{\mathbb{R}^3} |\mathrm{curl}\, \mathbf{A} - \mu_0 \mathbf{M}|^2 \, d^3 r$$

new minimization principle:

see also Coulomb, 1981

$$E_\mathrm{s}(\mathbf{M}) = \frac{1}{2} \mu_0 M_\mathrm{s}^2 V + \min_{\mathbf{A} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(\frac{1}{2\mu_0} |\nabla \mathbf{A}|^2 - \mathbf{M} \cdot \mathrm{curl}\, \mathbf{A} \right) d^3 r.$$

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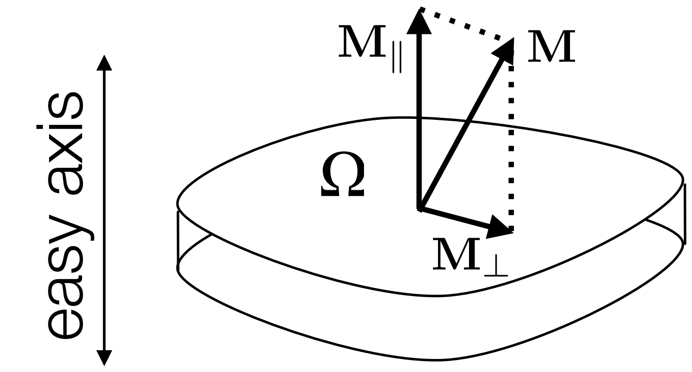
Di Fratta, M, Rybakov and Slastikov, 2020

Micromagnetics of thin films



statics:

$\Omega \subseteq \mathbb{R}^2$ - film shape

$$\boxed{\begin{aligned} E(\mathbf{M}) = \frac{A}{M_s^2}\int_{\Omega\times(0,d)} |\nabla \mathbf{M}|^2\, d^3r + \frac{Kd}{M_s^2}\int_\Omega |\overline{\mathbf{M}}_\perp|^2 d^2r - \mu_0 \int_{\Omega\times(0,d)} \mathbf{M}\cdot\mathbf{H}\, d^3r \\ + \mu_0 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla\cdot\mathbf{M}(\mathbf{r})\,\nabla\cdot\mathbf{M}(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|}\, d^3r\, d^3r' + \frac{Dd}{M_s^2}\int_\Omega \left(\overline{M}_\parallel \nabla\cdot\overline{\mathbf{M}}_\perp - \overline{\mathbf{M}}_\perp\cdot\nabla\overline{M}_\parallel\right) d^2r \end{aligned}}$$

M, Slastikov, 2016

Here $\mathbf{M} = (\mathbf{M}_\perp, M_\parallel)$, $\mathbf{M}_\perp \in \mathbb{R}^2$ $M_\parallel \in \mathbb{R}$ $|\mathbf{M}| = M_s$ in $\Omega \times (0,d) \subset \mathbb{R}^3$

Parameters and their representative values:

- exchange constant $A = 10^{-11}\,\mathrm{J/m}$
- anisotropy constant $K = 1.25 \times 10^6\ \mathrm{J/m^3}$
- saturation magnetization $M_s = 1.09 \times 10^6\ \mathrm{A/m}$
- DMI strength $D = 1\ \mathrm{mJ/m^2}$

film thickness $d = 0.5 - 5\,\mathrm{nm}$
lateral dimension:
$L = 50 - 500\,\mathrm{nm}$

applied field strength $\mu_0 H = 100\ \mathrm{mT}$

exchange length $\ell_{ex} = 3.66\ \mathrm{nm}$

Need reduced micromagnetic models

mathematically, the full 3D problem poses a **formidable challenge**:

- *vectorial*
- *nonlinear*
- *nonlocal*
- *multiscale*
- *topological constraints*

physically, the full 3D model **breaks down** on atomic scales

need a simplified model which is valid for the relevant parameter range
and still captures quantitatively the physical features of the system

Solution: introduce reduced thin film models that are amenable to analysis

Use the tools from *rigorous asymptotic analysis* of calculus of variations

A. De Simone, R. V. Kohn, S. Müller, F. Otto,
Recent Analytical Developments in Micromagnetics, 2006

Dimension reduction

$$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$$

<u>assume</u> the magnetization $\mathbf{m} = \mathbf{M}/M_s$ does not vary significantly across the film thickness, measure lengths in the units of ℓ_{ex}, scale energy by Ad

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa\, \mathbf{m}_\perp \cdot \nabla m_\parallel \right\} d^2 r$$

$$+ \frac{1}{2\pi\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\sqrt{|\mathbf{r}-\mathbf{r}'|^2 + \delta^2}} - 2\pi\delta^{(2)}(\mathbf{r}-\mathbf{r}')\delta \right) m_\parallel(\mathbf{r}) m_\parallel(\mathbf{r}')\, d^2 r\, d^2 r'$$

$$+ \delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_\delta(|\mathbf{r}-\mathbf{r}'|) \nabla \cdot \mathbf{m}_\perp(\mathbf{r})\, \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')\, d^2 r\, d^2 r'$$

Here:

$$\boxed{Q = \frac{2K}{\mu_0 M_s^2}, \qquad \kappa = D\sqrt{\frac{2}{\mu_0 M_s^2 A}}, \qquad \ell_{ex} = \sqrt{\frac{2A}{\mu_0 M_s^2}}, \qquad \delta = \frac{d}{\ell_{ex}}}$$

$$K_\delta(r) = \frac{1}{2\pi\delta} \left\{ \ln\left(\frac{\delta + \sqrt{\delta^2 + r^2}}{r} \right) - \sqrt{1 + \frac{r^2}{\delta^2}} + \frac{r}{\delta} \right\} \simeq \frac{1}{4\pi r} \qquad \delta \ll 1$$

C. Garcia-Cervera, Ph.D. thesis (1999)

Reduced thin film energy

$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$

regime $\delta \ll 1$: Taylor-expand in Fourier space

$$\boxed{\begin{aligned} E(\mathbf{m}) \simeq & \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel \right\} d^2 r \\ & + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r\, d^2 r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r\, d^2 r' \end{aligned}}$$

M, Slastikov, 2016

the expression for the stray field energy is rigorously justified via Γ-expansion

Knüpfer, M, Nolte, 2019

for bounded 2D samples, extra boundary terms appear

Di Fratta, M, Slastikov, 2021

proper definition of the non-local terms via Fourier:

$$\frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{k}| \, |\widehat{m}_\parallel(\mathbf{k})|^2 \frac{d^2 k}{(2\pi)^2} = \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r\, d^2 r', \quad \Big\} \textit{ surface charges}$$

$$\frac{1}{2} \int_{\mathbb{R}^2} \frac{|\mathbf{k} \cdot \widehat{\mathbf{m}}_\perp(\mathbf{k})|^2}{|\mathbf{k}|} \frac{d^2 k}{(2\pi)^2} = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \, \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r\, d^2 r'. \quad \Big\} \textit{ volume charges}$$

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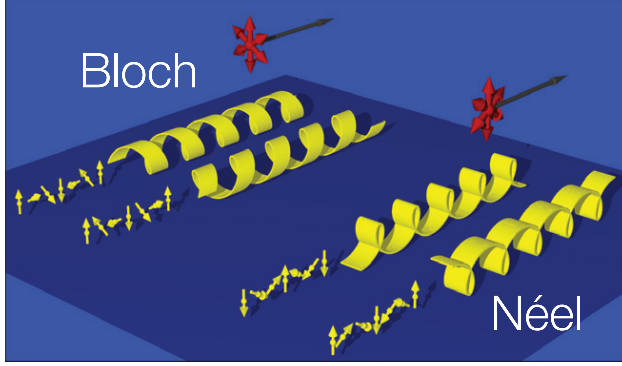
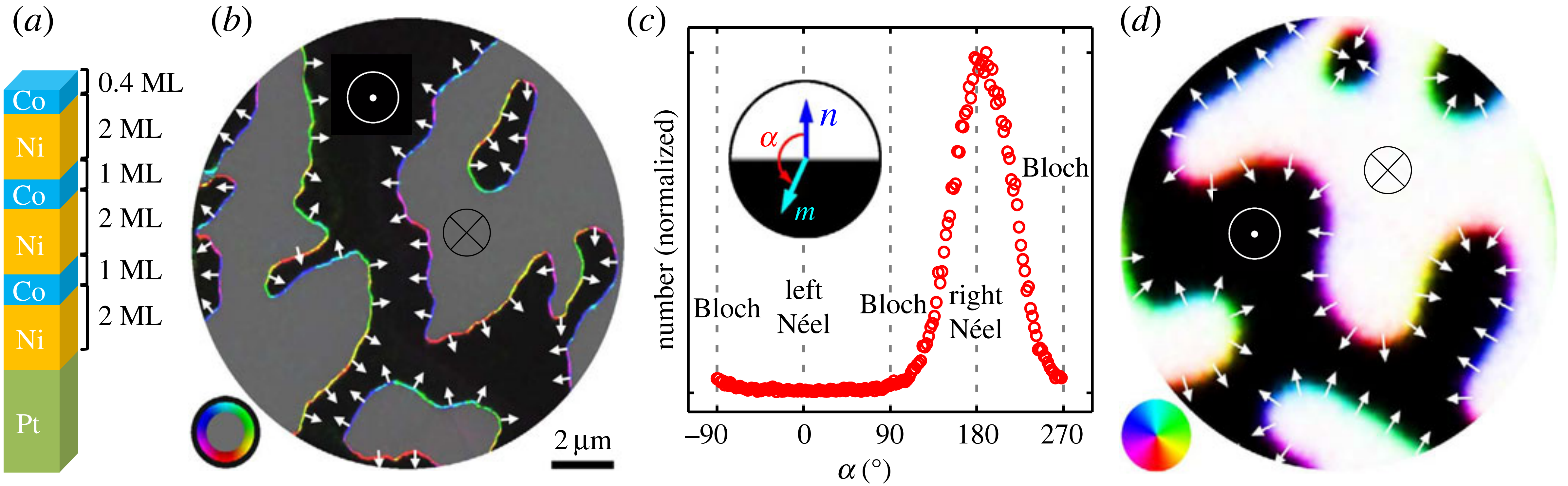
Chiral domain walls



ultrathin films + perpendicular magnetic anisotropy (PMA) + interfacial Dzyaloshinskii-Moriya interaction (DMI):



Néel-type walls

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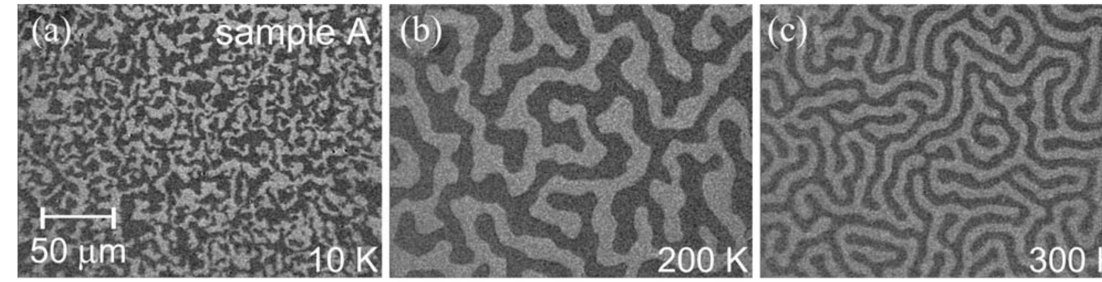
G. Chen et al., *Nat. Commun.* **4**, 1–6 (2013)

Contribution of the stray field



M. Yamanouchi et al., IEEE Magn. Lett. 2, 3000304 (2011)

regime $\delta \ll 1$: $\mathbf{m} \in H^1_{loc}(\mathbb{R}^2; \mathbb{S}^2), \quad m_\parallel = -1 \quad \text{outside} \quad B_L(0)$

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel \right\} d^2 r$$
$$+ \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r'$$

<u>Fact</u>:   for any $0 < r < L$

$$\boxed{\frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r' \leq \frac{1}{\pi} \ln\left(\frac{L}{r}\right) \int_{\mathbb{R}^2} |\nabla m_\parallel| \, d^2 r + r \int_{\mathbb{R}^2} |\nabla m_\parallel|^2 d^2 r + \pi L}$$

DeSimone, Knüpfer, Otto, 2006; M, 2019; Knüpfer, M, Nolte, 2019

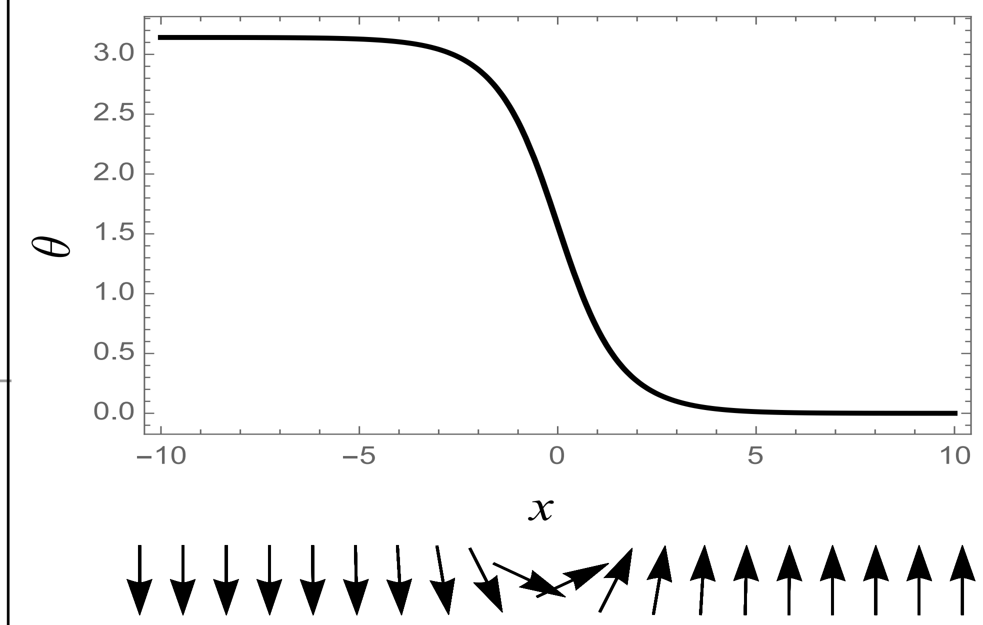
Tight lower bound, hence:

$$E(\mathbf{m}) \simeq \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel \right\} d^2 r - \underbrace{\cancel{\frac{\delta}{2\pi} \ln L \int_{\mathbb{R}^2} |\nabla m_\parallel| \, d^2 r}}_{\text{net contribution to the wall energy}}$$

negligible for $\delta \ln L \ll 1$, otherwise renormalizes the wall energy or causes an instability!

Knüpfer, Muratov, Nolte, 2017; M and Simon, 2019

One-dimensional chiral wall



Heide, Bihlmayer and Blügel, 2008
Thiaville et. al., 2012

profile:

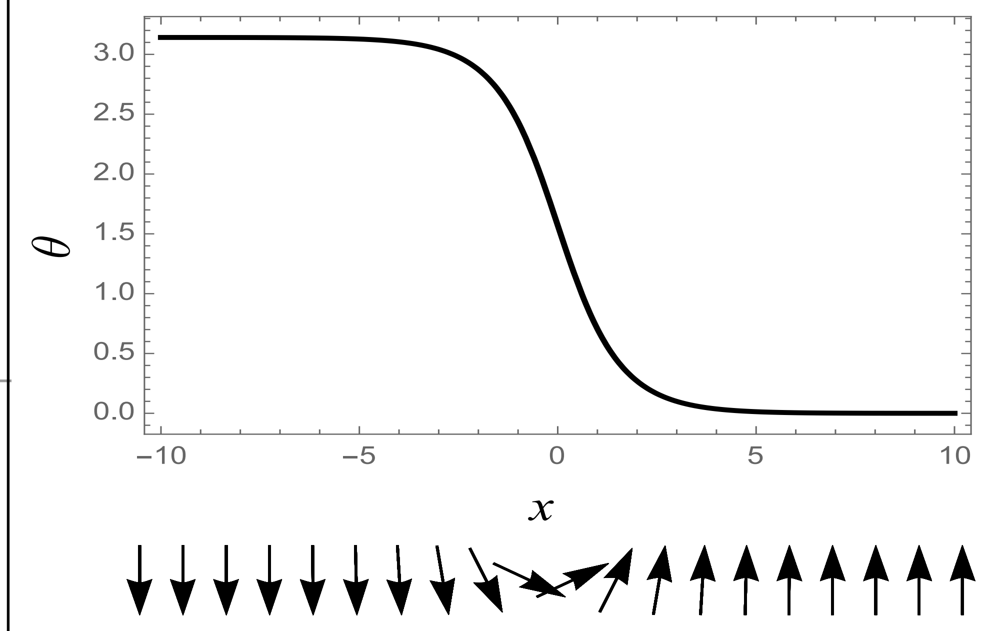
$$\boxed{\mathbf{m} = (\sin\theta, 0, \cos\theta), \quad \theta(x) = 2\arctan e^{-x\sqrt{Q-1}}}$$

<u>*Sharp lower bound:*</u>

$$\boxed{\sigma_{wall} = 4\sqrt{Q-1} - \pi\kappa > 0.}$$

$$E(\mathbf{m}) = \int_{-\infty}^{\infty} \left(|\mathbf{m}'|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa(\hat{x}\cdot\mathbf{m}_\perp)m'_{||} \right) dx$$

One-dimensional chiral wall



for any $R > 0$:

$$E(\mathbf{m}) \geq 2\int_{-R}^{R} \left(\sqrt{Q-1} - \kappa\sqrt{1-m_{\|}^2}\right) |m_{\|}'|\, dx$$

$$\geq 2\int_{-R}^{R} \left(\sqrt{Q-1} - \kappa\sqrt{1-m_{\|}^2}\right) m_{\|}'\, dx$$ assuming $\kappa \leq \sqrt{Q-1}$

$$= \left\{2m_{\|}(x)\sqrt{Q-1} - \kappa\left(m_{\|}(x)\sqrt{1-m_{\|}^2(x)} + \arcsin(m_{\|}(x))\right)\right\}\Big|_{-R}^{R}$$

hence:

$$E(\mathbf{m}) \geq \sigma_{wall}, \qquad \boxed{\sigma_{wall} = 4\sqrt{Q-1} - \pi\kappa > 0.}$$

Equality holds if and only if the rotation is Néel and

$$m_{\|}' = \sqrt{Q-1}(1-m_{\|}^2) \qquad \lim_{x\to\pm\infty} m_{\|}(x) = \pm 1$$

the unique function that satisfies this property is

$$m_{\|}(x) = \tanh(x\sqrt{Q-1}\,)$$

which is *precisely* the one obtained with the help of the ansatz!

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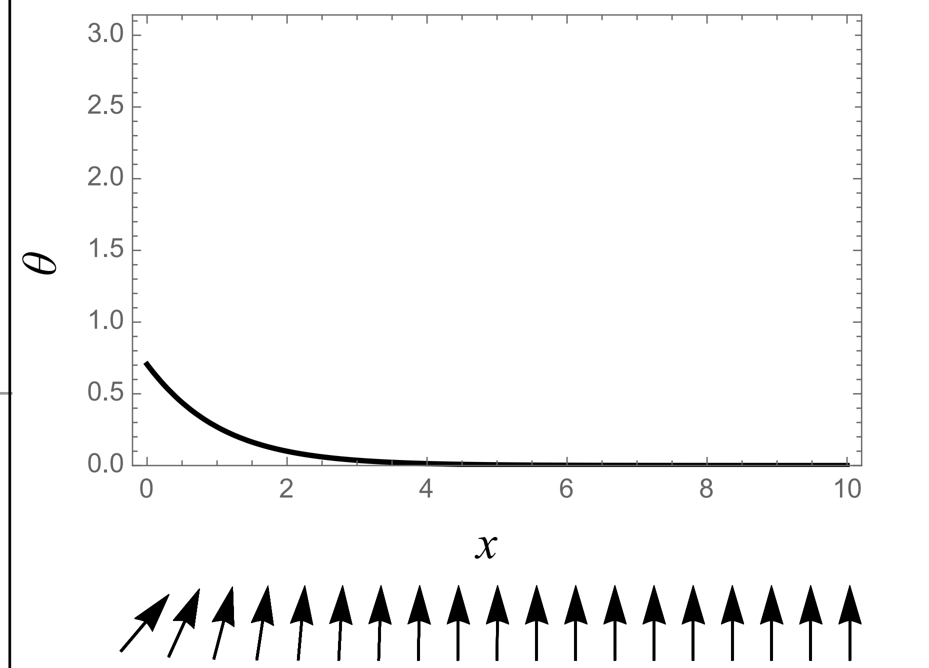
M and Slastikov, 2016

Edge domain wall



DMI has an effect near the sample boundary

<u>*Ansatz*</u>: $\theta_0 \in (0, \pi)$

$$\theta(x) = 2 \arctan e^{(x_0 - x)\sqrt{Q-1}}, \qquad x_0 = \frac{\ln \tan\left(\frac{\theta_0}{2}\right)}{\sqrt{Q-1}}.$$

Optimal? $0 < \kappa < \sqrt{Q-1}.$

$$E(\mathbf{m}) = 2\sqrt{Q-1}\int_0^\infty |\sin\theta|\,|\theta'|\,dx + \int_0^\infty \left(|\theta'| - \sqrt{Q-1}\,|\sin\theta|\right)^2 dx - \kappa\theta_0$$

$$\geq -\int_0^\infty \left(2\sqrt{Q-1}\,|\sin\theta| - \kappa\right)\theta' dx = \int_0^{\theta_0} \left(2\sqrt{Q-1}\,|\sin\theta| - \kappa\right) d\theta.$$

Yes, provided θ_0 minimizes $F(\theta_0) = 2\sqrt{Q-1}\,(1 - \cos\theta_0) - \kappa\theta_0$. Hence

$$\boxed{\sigma_{edge} = 2\sqrt{Q-1}\left(1 - \sqrt{1 - \frac{\kappa^2}{4(Q-1)}}\right) - \kappa \arcsin\left(\frac{\kappa}{2\sqrt{Q-1}}\right) < 0.}$$

Necessary ingredients for the <u>*sharp interface limit*</u>

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M and Slastikov, 2016

Ferromagnetism at the edge

model needs to be modified at the film edge — ***magnetic dead layers***

detailed *microscopic* physics matters: composition, roughness, fluctuations…

<u>Example</u>: mean-field treatment of the exchange $\rho \in L^1(\mathbb{R}^2 \times \mathbb{S}^2; [0,\infty])$

$$F(\rho) = -\frac{1}{2} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \int_\Omega \int_\Omega J_\delta(|\mathbf{r} - \mathbf{r}'|)(\mathbf{m} \cdot \mathbf{m}')\rho(\mathbf{r}, \mathbf{m})\rho(\mathbf{r}', \mathbf{m}')\, d^2r\, d^2r'\, d\mathcal{H}^2(\mathbf{m})\, d\mathcal{H}^2(\mathbf{m}')$$
$$+\beta^{-1} \int_{\mathbb{S}^2} \int_\Omega \rho(\mathbf{r}, \mathbf{m}) \ln \rho(\mathbf{r}, \mathbf{m})\, d^2r\, d\mathcal{H}^2(\mathbf{m}).$$

subject to

goes back to Onsager, 1949

$$\int_{\mathbb{S}^2} \rho(\mathbf{r}, \mathbf{m})\, d\mathcal{H}^2(\mathbf{m}) = 1 \text{ if } \mathbf{r} \in \Omega, \qquad \rho(\mathbf{r}, \mathbf{m}) = 0 \text{ if } \mathbf{m} \in \mathbb{S}^2 \text{ and } \mathbf{r} \in \mathbb{R}^2 \backslash \Omega$$

moments closure: $\overline{\mathbf{m}}(\mathbf{r}) := \int_{\mathbb{S}^2} \mathbf{m} \rho(\mathbf{r}, \mathbf{m})\, d\mathcal{H}^2(\mathbf{m}) \quad \Rightarrow$

$$\bar{\rho}(\mathbf{r}, \mathbf{m}) = \exp\Big(\beta(\mu(\mathbf{r}) + \boldsymbol{\lambda}(\mathbf{r}) \cdot \mathbf{m})\Big), \qquad \begin{cases} 1 = \dfrac{4\pi e^{\beta\mu} \sinh(\beta|\boldsymbol{\lambda}|)}{\beta|\boldsymbol{\lambda}|}, \\ \overline{\mathbf{m}} = \dfrac{4\pi e^{\beta\mu}(\beta|\boldsymbol{\lambda}| \cosh(\beta|\boldsymbol{\lambda}|) - \sinh(\beta|\boldsymbol{\lambda}|))}{\beta^2|\boldsymbol{\lambda}|^3} \boldsymbol{\lambda}. \end{cases}$$

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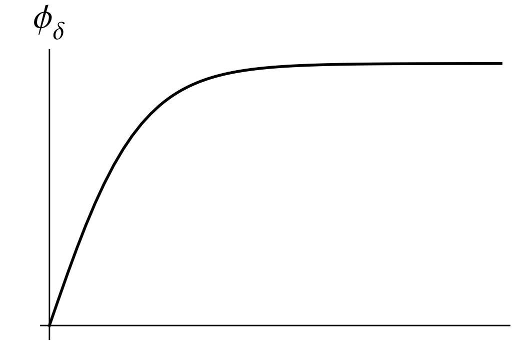
Thompson and Silver, 1973; Fatkullin and Slastikov, 2005; Di Fratta, M and Slastikov, 2021

Ferromagnetism at the edge



free energy as a function of average magnetization $\qquad \overline{\mathbf{m}} = \mathbf{0}$ in $\mathbb{R}^2 \backslash \Omega$

$$\overline{F}(\overline{\mathbf{m}}) = \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} J_\delta(|\mathbf{r} - \mathbf{r}'|) |\overline{\mathbf{m}}(\mathbf{r}) - \overline{\mathbf{m}}(\mathbf{r}')|^2 d^2 r \, d^2 r' + \int_{\mathbb{R}^2} U_\beta(|\overline{\mathbf{m}}|) \, d^2 r$$

compare with Alberti and Bellettini, 1998

gradient expansion $\qquad \overline{\mathbf{m}} \in H_0^1(\Omega; \mathbb{R}^3)$

$$\overline{F}_0(\overline{\mathbf{m}}) = \int_\Omega \left(g_\delta |\nabla \overline{\mathbf{m}}|^2 + U_\beta(|\overline{\mathbf{m}}|) \right) d^2 r, \qquad g_\delta := \tfrac{\pi}{4} \int_0^\infty r^3 J_\delta(r) \, dr = O(\delta^2).$$

Fatkullin and Slastikov, 2008

one-dimensional profile near the edge: $\quad |\overline{\mathbf{m}}(x)| = \phi_\delta(x)$

expansion of the exchange energy: $\qquad \overline{\mathbf{m}} = \phi_\delta \hat{\mathbf{m}}, \qquad |\hat{\mathbf{m}}| = 1$

$$\boxed{\overline{F}_0(\overline{\mathbf{m}}) - C_\delta \simeq \int_\Omega g_\delta \phi_\delta^2 |\nabla \hat{\mathbf{m}}|^2 d^2 r}$$

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compare with M. Tokas et al., AIP Advances **7**, 115022 (2017)

Di Fratta, M and Slastikov, 2021

Reduced thin film energy revisited

putting all the terms together: $\eta_\delta(\mathbf{r}) = \eta(\delta^{-1}\mathrm{dist}(\mathbf{r}, \mathbb{R}^2 \backslash \Omega))$

$$\boxed{\begin{aligned} E(\mathbf{m}) = & \int_{\mathbb{R}^2} \eta_\delta^2 \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 + \kappa(m_\parallel \nabla \cdot \mathbf{m}_\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel) \right\} d^2 r \\ & + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot (\eta_\delta \mathbf{m}_\perp)(\mathbf{r}) \nabla \cdot (\eta_\delta \mathbf{m}_\perp)(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' \\ & - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\eta_\delta(\mathbf{r}) m_\parallel(\mathbf{r}) - \eta_\delta(\mathbf{r}') m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r' \end{aligned}}$$

Chaves and M, 2013; M, Osipov and Vanden-Eijnden, 2015; Lund, M and Slastikov, 2018; Di Fratta, M and Slastikov, 2021

$\eta \in C^1([0, +\infty); [0, 1])$, $\eta'(t) \geq 0$ for all $t \geq 0$, $\eta(0) = 0$ and $\eta(t) = 1$ for all $t \geq L$

<u>in a suitable limit as</u> $\delta \to 0$ this energy Γ-converges to: $\mathbf{m} \in H^1(\Omega; \mathbb{S}^2)$

$$\begin{aligned} E_0(\mathbf{m}) := & \int_\Omega \left(|\nabla \mathbf{m}|^2 + \alpha |\mathbf{m}_\perp|^2 \right) d^2 r + \lambda \int_\Omega \left(m_\parallel \nabla \cdot \mathbf{m}_\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel \right) d^2 r \\ & + \gamma \int_{\partial \Omega} \left((\mathbf{m}_\perp \cdot \mathbf{n})^2 - m_\parallel^2 \right) d\mathcal{H}^1(\mathbf{r}) \end{aligned}$$

see also Kohn and Slastikov, 2005

Di Fratta, M and Slastikov, 2021

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Skyrmions

- topologically nontrivial configurations of nonlinear field theories

- introduced by Tony Skyrme in the early 1960s to empirically describe the low-energy properties of baryons

- received attention in the mathematical literature from the 1980s onward

Esteban, 1986; Esteban, 1992; Faddeev and Niemi, 1997; Esteban, 2004; Lin and Yang, 2004

- relevant example: $u : \mathbb{R}^2 \to \mathbb{S}^2$ *baby skyrmions*

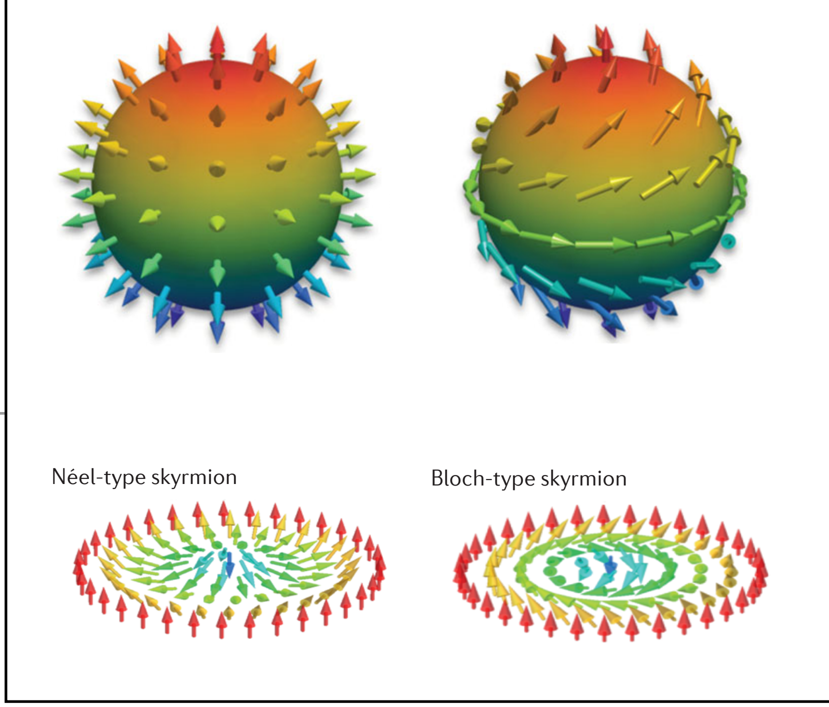
$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla u|^2 + \frac{\lambda}{2} |\partial_1 u \times \partial_2 u|^2 + \frac{\mu}{2} (1 - \mathbf{n} \cdot u)^2 \right\} \mathrm{d}x$$

- existence of minimizers of

$$E_k = \inf\{E(u) : E(u) < \infty, \deg(u) = k\} \qquad \deg(u) = \frac{1}{4\pi} \int_{\mathbb{R}^2} u \cdot (\partial_1 u \times \partial_2 u) \mathrm{d}x$$

Lin and Yang, 2004; Li and Zhu, 2011

Magnetic skyrmions



T. Lancaster, *Contemp. Phys.* **60**, 246-261 (2019)

maps $\mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^2$ with non-trivial topology

example: harmonic maps

$$\boxed{E(\mathbf{m}) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r}$$

all minimizers with prescribed degree are known

Belavin and Polyakov, 1975

after stereographic projection, reduces to harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$

they are *holomorphic* or *anti-holomorphic* maps

specifically, all degree 1 minimizing maps belong to:

Eells and Sampson, 1964
Lemaire, 1978
Wood, 1974
Brezis, Coron, 1985

$$\boxed{\mathcal{B} := \left\{ S\Phi(\rho^{-1}(\bullet - x)) : S \in \mathrm{SO}(3),\ \rho > 0,\ x \in \mathbb{R}^2 \right\}}$$

i.e., *dilations, rotations and translations* of:

$$\Phi(x) := \left(-\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right)$$

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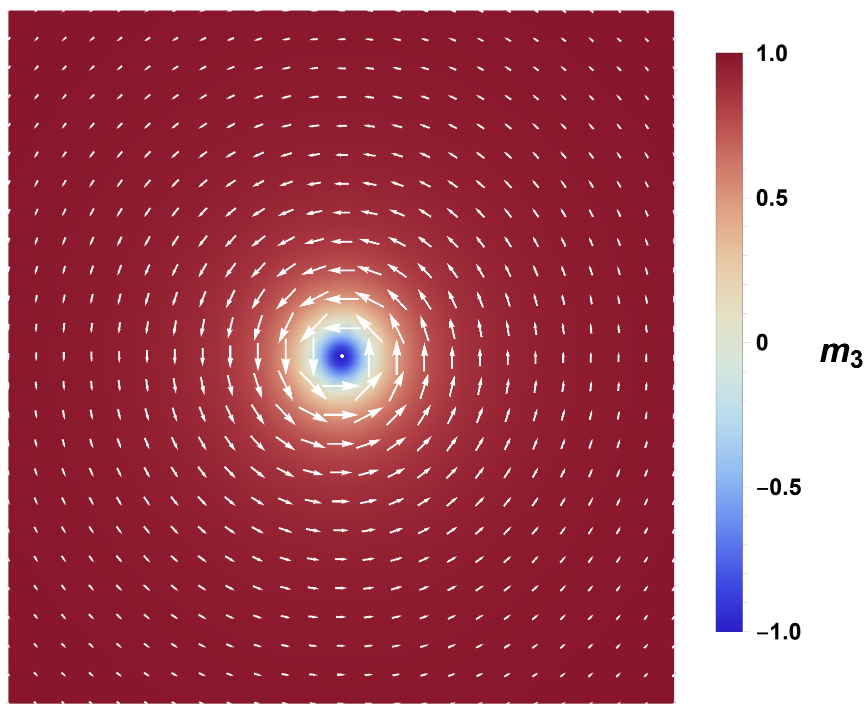
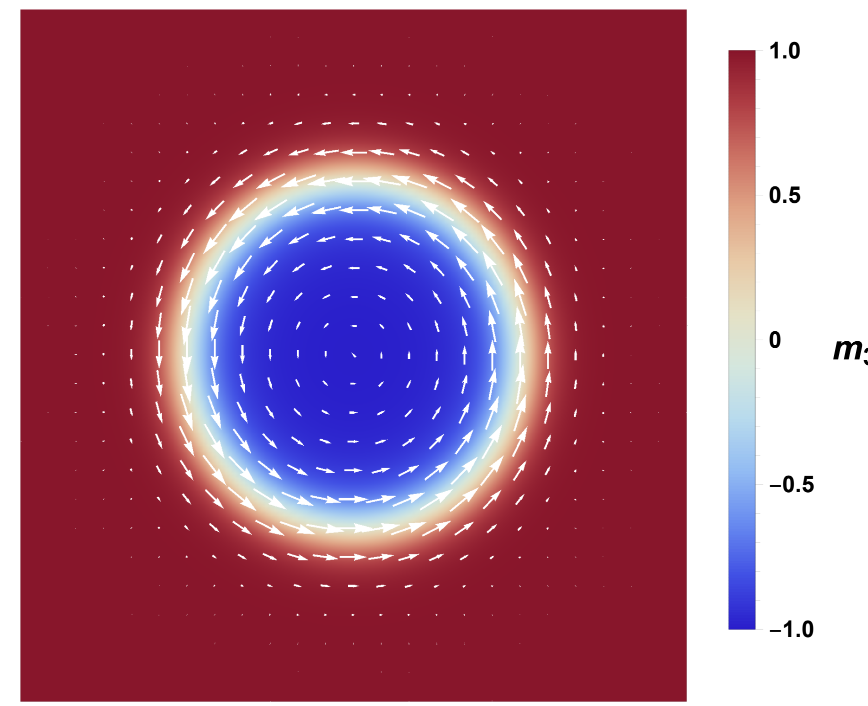
Admissible class?

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \{|\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel\}\, d^2r$$
$$+ \frac{\delta}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r\, d^2r' - \frac{\delta}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r'$$

compact skyrmion vs. skyrmionic bubble





for bubble skyrmion, the stray field energy *diverges* with radius:

$$E_s(\mathbf{m}_R) \sim -R \ln R$$

M, Simon, 2019

hence $\quad \mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^2,\ \nabla \mathbf{m} \in L^2,\ \mathbf{m}_\perp \in L^2 \quad \not\Rightarrow \quad E(\mathbf{m}) > -\infty$

no hope to construct solutions as absolute minimizers with prescribed degree

contrast with the local case

Bogdanov, Yablonskii, 1989; Ivanov, 1990
Bogdanov, Kudinov, Yablonskii, 1989
Melcher, 2014; Li, Melcher, 2018

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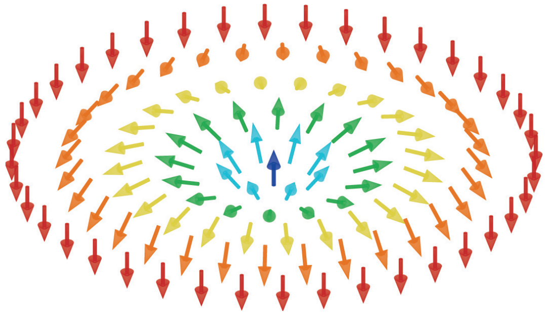
Compact skyrmions as local minimizers



introduce:

$$\boxed{\mathcal{N}(\mathbf{m}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \mathbf{m}\cdot(\partial_1\mathbf{m}\times\partial_2\mathbf{m})\, d^2r}$$

$$\mathcal{A} := \left\{ \mathbf{m} \in \mathring{H}^1(\mathbb{R}^2;\mathbb{S}^2) : \mathcal{N}(\mathbf{m}) = 1,\ \mathbf{m}+\mathbf{e}_3 \in L^2(\mathbb{R}^2),\ \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2r < 16\pi \right\}$$

why 16π? Topological lower bound: $\mathbf{m} \in \mathring{H}^1(\mathbb{R}^2;\mathbb{S}^2)$

$$\boxed{\int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2r \geq 8\pi\, |\mathcal{N}(\mathbf{m})|}$$

$$|\nabla\mathbf{m}|^2 \pm 2\mathbf{m}\cdot(\partial_1\mathbf{m}\times\partial_2\mathbf{m}) = |\partial_1\mathbf{m} \mp \mathbf{m}\times\partial_2\mathbf{m}|^2$$

allows to exclude splitting in the concentration compactness arguments

Theorem 1. *Let $Q>1$, $\delta>0$ and $\kappa\in\mathbb{R}$ be such that $(2|\kappa|+\delta)^2 < 2(Q-1)$. Then there exists $\mathbf{m}\in\mathcal{A}$ such that*

$$E(\mathbf{m}) = \inf_{\widetilde{\mathbf{m}}\in\mathcal{A}} E(\widetilde{\mathbf{m}}).$$

Bernand-Mantel, M and Simon, 2020

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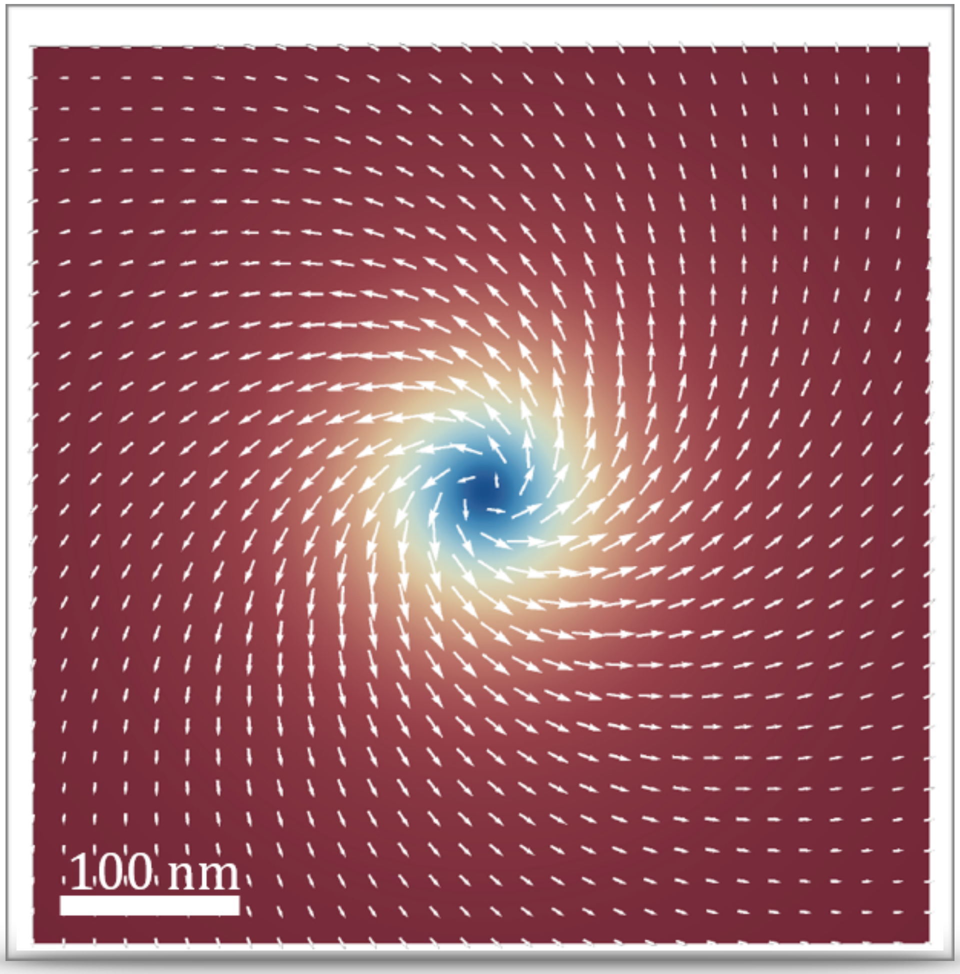
complete asymptotic description in the *conformal limit*

Mixed Bloch-Néel skyrmion

5 nm thick GdCo



$$A = 20\,\mathrm{pJ/m}$$
$$M_\mathrm{s} = 10^5\,\mathrm{A/m}$$
$$K = 6346\,\mathrm{J/m^3}$$
$$D = 0.018\,\mathrm{mJ/m^2}$$

MuMax3

$4096 \times 4096\ \mathrm{nm}^2$

$4 \times 4 \times 5\ \mathrm{nm}^3$ mesh

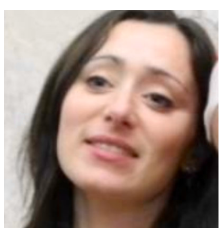
competition between DMI and dipolar effects

Bernand-Mantel, M and Simon, 2020

The team

Gabriel
Chaves

Valeriy
Slastikov



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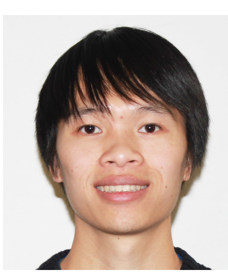

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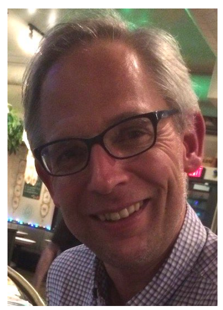

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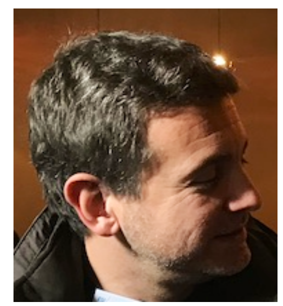
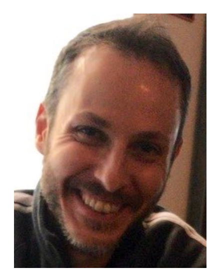


Massimiliano
Morini



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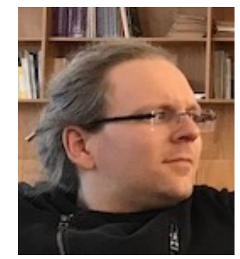

Theresa
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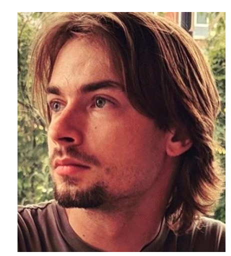

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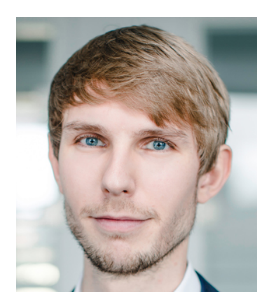

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