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One-dimensional Néel walls under applied external fields

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Abstract
We present a detailed analysis of one-dimensional Néel walls in thin uniaxial ferromagnetic films in the presence of an in-plane applied external field in the direction normal to the easy axis. Within the reduced one-dimensional thin film model, we formulate a non-local variational problem whose minimizers are given by one-dimensional Néel wall profiles. We prove existence, uniqueness (up to translations and reflections), regularity, strict monotonicity and the precise asymptotics of the decay of the minimizers in the considered variational problem.

Mathematics Subject Classification: 78A30, 35Q60, 82D40

1. Introduction

Ferromagnetic materials are at the heart of modern information storage technology, whose need to keep up with the ever-growing amount of digital data, currently in excess of 10^{21} bytes worldwide, is readily apparent [1]. This is why these materials have attracted a huge degree of attention since the early days of the digital age. The basic principle of magnetic storage relies on the tendency of the electron spins in ferromagnetic materials to align along certain preferred directions, giving rise to *magnetic domains* [2]. Registering and manipulating the magnetization orientation in a given domain is then used to read and write discrete data encoded by the magnetization orientation in each domain.

One common magnetic storage solution relies on the use of thin uniaxial ferromagnetic films in which the magnetization vector prefers to align along either direction of the easy

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magnetocrystalline axis in the film plane [1–5]. When the film thickness becomes sufficiently small (under a few tens of nanometres), the magnetization vector is constrained to lie almost entirely in the film plane. In this situation magnetic domains in epitaxial (monocrystalline) films usually consist of relatively large regions, in which the magnetization vector is nearly constant and oriented in the direction of one of the two possible directions along the easy axis. These regions are separated by narrow transition regions, called *domain walls*, in which the magnetization vector rapidly rotates between the two orientations [2, 6–9]. One of the most common wall types in such materials is the *Néel wall*, which separates two regions of opposite magnetization by an in-plane rotation and is oriented along the easy axis to ensure zero net magnetic charge. In real materials these walls are often pinned to the material imperfections, and their motion determines magnetization reversal under the action of applied magnetic fields [2].

Studies of Néel walls have a long and somewhat controversial history (see the discussions in [2, 10]), but at present the structure of the Néel wall in very thin films appears to be rather well understood on the basis of micromagnetic arguments [2, 9, 11–17]. The basic features of the predicted one-dimensional Néel wall profiles had been verified experimentally in [18] (see also [19, 20]). Rigorous mathematical studies of the Néel walls are more recent and go back to the work of García-Cervera [14, 16], who undertook some analysis of the associated one-dimensional variational problems and performed extensive numerical studies of the energy functional obtained by Aharoni from the full micromagnetic energy after restricting the admissible configurations to profiles which depend only on one spatial variable [21]. Melcher further studied the minimizers of the same functional in the class of magnetization configurations constrained to the film plane and established symmetry and monotonicity of the energy minimizing profiles connecting the two opposite directions of the easy axis [22]. Using a further one-dimensional thin film reduction of the micromagnetic energy introduced in [17], Capella, Melcher and Otto outlined the proof of uniqueness of the Néel wall profile and its linearized stability with respect to one-dimensional perturbations [23]. Stability of geometrically constrained one-dimensional Néel walls with respect to large two-dimensional perturbations in soft materials was demonstrated asymptotically in [24]. More recently, Γ-convergence studies of the one-dimensional wall energy in the limit of very soft films and in the presence of an applied in-plane field normal to the easy axis were undertaken in [25, 26], and a rigorous derivation of the effective magnetization dynamics driven by the reduced thin film energy introduced in [23] from the full three-dimensional Landau–Lifshitz–Gilbert equation was presented in [27].

In this paper, we perform a detailed variational study of the Néel walls, understood as one-dimensional minimizers of the reduced thin film micromagnetic energy, in uniaxial materials in the presence of an applied in-plane magnetic field in the direction perpendicular to the easy axis, extending previous results for Néel walls in the absence of the applied field. We prove existence, uniqueness (up to translations and reflections), regularity, strict monotonicity and the precise decay of the energy minimizing wall profiles. Our variational setting is slightly different from that adopted in the earlier works and relies on the angle variable rather than the two-dimensional unit vector representation of the magnetization. For this reason our proofs differ in a few technical aspects from those of [22]. In fact, one of the purposes of our work was to clarify some of the arguments in the analyses of [16, 22, 23]. In particular, we spell out the details of the proof of uniqueness of minimizers within our setting and fill in the missing argument for proving strict monotonicity of the angle variable as the function of coordinate, which is needed to establish stability of the Néel wall profile in [23]. We also establish the precise asymptotic behaviour of the Néel wall profiles at infinity, which is new even in the case of zero applied field. Let us note that while in this paper we are not concerned with the logarithmic tail of the Néel walls in very soft materials, which was one of the main focuses of [14, 16, 22], our decay estimates could be made quantitative in this regime to yield the intermediate asymptotics of the Néel wall profile away from the core.

The rest of our paper is organized as follows. In section 2 we discuss the basic micromagnetic energy and derive the reduced one-dimensional energy that describes the Néel walls in the applied in-plane field oriented normally to the easy axis. Then in section 3 we present the variational setting for our analysis and state our main result. Section 4 contains a few auxiliary results and section 5 contains the proof of the main theorem. We also discuss some open problems at the very end of section 5.

2. Model

In this paper we are interested in the analysis of the energy minimizing magnetization configurations in thin uniaxial ferromagnetic films of large extent with the easy axis in the film plane. We also wish to include the effect of an applied in-plane field in the direction normal to the easy axis. The starting point in the studies of such systems is the energy functional, introduced by Landau and Lifshitz, which leads to a non-convex, non-local variational problem. The functional, written in the CGS units, is [2, 10, 28, 29]

$$E(\boldsymbol{M}) = \frac{A}{2|M_s|^2}\int_\Omega |\nabla \boldsymbol{M}|^2\,\mathrm{d}^3r + \frac{K}{2|M_s|^2}\int_\Omega \Phi(\boldsymbol{M})\,\mathrm{d}^3r - \int_\Omega \boldsymbol{H}_{\mathrm{ext}}\cdot\boldsymbol{M}\,\mathrm{d}^3r + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nabla\cdot\boldsymbol{M}(\boldsymbol{r})\nabla\cdot\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3r\,\mathrm{d}^3r' + \frac{M_s^2}{2K}\int_\Omega |\boldsymbol{H}_{\mathrm{ext}}|^2\,\mathrm{d}^3r. \tag{1}$$

Here $\Omega \subset \mathbb{R}^3$ is the domain occupied by the ferromagnetic material, $\boldsymbol{M} : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetization vector that satisfies $|\boldsymbol{M}| = M_s$ in Ω and $\boldsymbol{M} = 0$ in $\mathbb{R}^3 \setminus \Omega$, the positive constants M_s, A and K are the material parameters referred to as the saturation magnetization, exchange constant and the anisotropy constant, respectively, $\boldsymbol{H}_{\mathrm{ext}}$ is an applied external field, and $\Phi : \mathbb{R}^3 \to \mathbb{R}$ is a non-negative potential that has several minima at which Φ vanishes. Note that $\nabla\cdot\boldsymbol{M}$ in the double integral is understood in the distributional sense.

The micromagnetic energy in (1) is composed of five terms: the exchange energy term which penalizes the spatial variations of the magnetization $\boldsymbol{M}$, the anisotropy term reflecting the magnetocrystalline properties of the material, the Zeeman energy favouring the alignment of $\boldsymbol{M}$ with the applied external field, the stray-field energy, which is non-local and favours vanishing distributional divergence, i.e. $\nabla\cdot\boldsymbol{M} = 0$ both in Ω and on $\partial\Omega$ (the so-called pole-avoidance principle), and an inessential constant term added for convenience. In the case of a uniaxial material of interest to us, there exists a distinguished axis identified through a unit vector $\boldsymbol{e}$, and Φ is given by $\Phi(\boldsymbol{M}) = M_s^2 - (\boldsymbol{M}\cdot\boldsymbol{e})^2$, so that the minima of Φ are $\{\pm\boldsymbol{e}M_s\}$ [2, 10, 29].

In the case of extended monocrystalline thin films with the in-plane easy axis we have $\Omega = \mathbb{R}^2 \times (0, d)$, and without loss of generality we may assume that $\boldsymbol{e} = \boldsymbol{e}_2$, where $\boldsymbol{e}_i$ is the unit vector in the ith coordinate direction. For thin films (moderately soft, ultra-thin) of practical interest to magnetic device applications such as MRAMs (magnetoresistive random access memories) [4–6], a significant reduction of the energy in (1) is possible, giving rise to the *reduced thin film energy* [9, 17]. To better explain the relevant parameter regime, let us introduce the following quantities

$$\ell = \left(\frac{A}{4\pi M_s^2}\right)^{1/2}, \qquad L = \left(\frac{A}{K}\right)^{1/2}, \qquad Q = \left(\frac{\ell}{L}\right)^2, \tag{2}$$

called the exchange length, the Bloch wall thickness, and the material quality factor, respectively. When the film is *ultra-thin* and *soft*, we have $d \lesssim \ell \lesssim L$, but at the same time one also has a balance $Ld/\ell^2 \sim 1$ for many materials [30]. In this situation the dimensionless parameter

$$\nu = \frac{4\pi M_s^2 d}{KL} = \frac{Ld}{\ell^2} = \frac{d}{\ell\sqrt{Q}}, \tag{3}$$

which is referred to as the *thin film parameter* [17], becomes a single measure of the strength of the magnetostatic interaction relative to both anisotropy and exchange.

The reduced thin film energy is formally obtained from the full micromagnetic energy in (1) by assuming that $\boldsymbol{M}$ does not vary in the direction of $\boldsymbol{e}_3$ (the direction normal to the film), setting the component of $\boldsymbol{M}$ along $\boldsymbol{e}_3$ to zero and passing to the limit $Q \to 0$ and $d \to 0$ jointly, subject to $\nu = O(1)$ fixed, after rescaling lengths with L [17] (see also [9]). Assuming further that $\boldsymbol{H}_{\text{ext}} = \boldsymbol{e}_1 hK/M_s$, after a suitable rescaling we arrive at the following reduced energy functional:

$$E(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2 \,\mathrm{d}^2 r + \frac{1}{2}\int_{\mathbb{R}^2} (\boldsymbol{m}\cdot\boldsymbol{e}_1 - h)^2 \,\mathrm{d}^2 r \\ + \frac{\nu}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\cdot\boldsymbol{m}(\boldsymbol{r})\nabla\cdot\boldsymbol{m}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^2 r\,\mathrm{d}^2 r', \tag{4}$$

where now $\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{S}^1$ is the unit vector in the direction of the magnetization in the film plane. Note that the assumptions on $\boldsymbol{M}$ used in this derivation are justified by the strong penalization of the variations of $\boldsymbol{M}$ across the normal direction to the film by the exchange energy and by the strong penalization of the normal component of $\boldsymbol{M}$ by the shape anisotropy [2]. Also, up to a constant factor the last term in (4) is simply the square of the homogeneous $H^{-1/2}$-norm of $\nabla\cdot\boldsymbol{m}$ in $\mathbb{R}^2$ [9].

The reduced energy in (4) is the starting point of the analysis of the rest of our paper. Without loss of generality we may assume that $h \geqslant 0$. Also note that for $h \geqslant 1$ the energy in (4) admits a unique global minimum $\boldsymbol{m} = \boldsymbol{e}_1$ and no Néel walls are, therefore, possible in this situation. For $h \in [0, 1)$, on the other hand, there are two global minimizers $\boldsymbol{m}_\pm = (h, \pm\sqrt{1-h^2})$ corresponding to the two monodomain states. In the following, we will always assume that h is in this non-trivial range, in which Néel walls connecting the two states appear. Let us point out that at the same time we do not allow the external field to have a component in the direction of the easy axis, since in this case only one monodomain state exists as the global minimizer of the energy. Under an applied field in the direction of the easy axis Néel walls begin to move, invading the domain with higher energy density by the domain with the lower energy density [2, 23]. Similarly, the considered wall orientation along the easy axis is the only one that makes the stray-field energy of a one-dimensional profiles finite. When the wall makes a non-zero angle with the easy axis (compare with [31]), it carries a net magnetic charge, which makes the associated magnetostatic potential for the wall in the whole of $\mathbb{R}^2$ infinite.

3. Variational formulation and statement of the main result

We now turn to the study of one-dimensional Néel wall profiles. For that we assume that $\boldsymbol{m}$ varies only along $\boldsymbol{e}_1$ and compute the energy of such a configuration per unit length of the wall. It is convenient to introduce the new variable $\vartheta = \vartheta(x)$ which gives the angle that the vector $\boldsymbol{m}$ makes with $\boldsymbol{e}_2$ in the counter-clockwise direction as a function of the coordinate along $\boldsymbol{e}_1$. Thus, setting

$$\boldsymbol{m}(x) = (-\sin\vartheta(x), \cos\vartheta(x)) \in \mathbb{S}^1 \tag{5}$$

for every $x \in \mathbb{R}$, we can rewrite the one-dimensional version of the functional in (4) in terms of the angle variable ϑ to obtain the one-dimensional Néel wall energy (see [17]):

$$
\begin{aligned}
E(\vartheta;\mathbb{R}) &:= \frac{1}{2}\int_{\mathbb{R}}\left\{|\vartheta_x|^2 + (\sin\vartheta - h)^2 + \frac{\nu}{2}\sin\vartheta\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\sin\vartheta\right\}\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}}\left(|\vartheta_x|^2 + (\sin\vartheta - h)^2\right)\mathrm{d}x + \frac{\nu}{8\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\left(\sin\vartheta(x) - \sin\vartheta(y)\right)^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y, \qquad (6)
\end{aligned}
$$

where, as usual, $(-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}$ denotes the square root of the one-dimensional negative Laplacian (a linear operator whose Fourier symbol is $|k|$), and, furthermore, we used the identity [31, 32]

$$
\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} u(x) = \frac{1}{\pi}\fint_{\mathbb{R}}\frac{u(x) - u(y)}{(x-y)^2}\,\mathrm{d}y, \qquad (7)
$$

for every x and, say, every $u \in C_c^\infty(\mathbb{R})$, where $\fint$ stands for the principal value of the integral.

We wish to study the minimizers of the energy in (6) among the profiles that connect the two distinct minima of the energy at $x = \pm\infty$. To this end, we need to introduce a suitable admissible class of functions which yields minimizers with the desired properties. We propose to minimize $E(\vartheta;\mathbb{R})$ over the admissible class

$$
\mathcal{A} := \{\vartheta \in H^1_{\mathrm{loc}}(\mathbb{R}) : \vartheta - \eta_h \in H^1(\mathbb{R})\}, \qquad (8)
$$

where $\eta_h \in C^\infty(\mathbb{R};[0,\pi])$ is a fixed non-increasing function such that, setting

$$
\theta_h := \arcsin h \in \left[0, \frac{\pi}{2}\right), \qquad (9)
$$

we have $\eta_h = \pi - \theta_h$ in $(-\infty,-1)$ and $\eta_h = \theta_h$ in $(1,+\infty)$. We point out that the definition of $\mathcal{A}$ does not depend on the choice of η_h: If $\tilde{\eta}_h \in C^\infty(\mathbb{R};[0,1])$ is a different non-increasing function, i.e., $\tilde{\eta}_h \neq \eta_h$, satisfying $\tilde{\eta}_h = \pi - \theta_h$ in $(-\infty,-1)$ and $\tilde{\eta}_h = \theta_h$ in $(1,+\infty)$, then

$$
\mathcal{A} = \{\tilde{\vartheta} \in H^1_{\mathrm{loc}}(\mathbb{R}) : \tilde{\vartheta} - \tilde{\eta}_h \in H^1(\mathbb{R})\}. \qquad (10)
$$

Indeed, any $\vartheta \in \mathcal{A}$ satisfies also $\vartheta - \tilde{\eta}_h \in H^1(\mathbb{R})$ for $\tilde{\eta}_h - \eta_h \in H^1(\mathbb{R})$; vice versa, for the same reason any $\tilde{\vartheta} \in H^1_{\mathrm{loc}}(\mathbb{R})$ with $\tilde{\vartheta} - \tilde{\eta}_h \in H^1(\mathbb{R})$ belongs to $\mathcal{A}$. Note that our choice of the admissible class $\mathcal{A}$ fixes the rotation sense of the Néel wall, and the wall of the opposite rotation sense may be obtained from the minimizer over $\mathcal{A}$ by a reflection about $x = 0$.

It is easy to see that the Euler–Lagrange equation associated with the functional in (6) is given by

$$
-\vartheta_{xx} + \cos\vartheta\sin\vartheta - h\cos\vartheta + \frac{\nu}{2}\cos\vartheta\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\sin\vartheta = 0, \qquad (11)
$$

with the boundary conditions at infinity

$$
\lim_{x\to+\infty}\vartheta(x) = \theta_h, \qquad \lim_{x\to-\infty}\vartheta(x) = \pi - \theta_h. \qquad (12)
$$

The main result of this paper is the following.

Theorem 1 (existence, uniqueness, regularity, strict monotonicity and decay of Néel walls). For every $\nu > 0$ and every $h \in [0,1)$ there exists a minimizer of $E(\vartheta;\mathbb{R})$ in (6) over $\mathcal{A}$ in (8), which is unique (up to translations), strictly decreasing with range equal to $(\theta_h, \pi - \theta_h)$ and is a smooth solution of (11) that satisfies the limit conditions given in (12). Moreover, if $\vartheta^{(0)}:\mathbb{R} \to (\theta_h, \pi - \theta_h)$ is the minimizer of E in the class $\mathcal{A}$ satisfying $\vartheta^{(0)}(0) = \frac{\pi}{2}$, then $\vartheta^{(0)}(x) = \pi - \vartheta^{(0)}(-x)$, and there exists a constant $c > 0$ such that $\lim_{x\to+\infty} x^2(\vartheta^{(0)}(x) - \theta_h) = c$.

4. Some auxiliary lemmas

We start with a few preliminary considerations and lemmas. Let $\vartheta \in \mathcal{A}$. By Morrey's theorem (see [33, theorem 11.34]), $\vartheta - \eta_h \in C^{1/2}(\mathbb{R})$ and $\vartheta - \eta_h \to 0$ as $x \to \pm\infty$; that is, $\vartheta \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and satisfies (12). Furthermore, assuming in addition that $\vartheta(\mathbb{R}) \subset [\theta_h, \pi - \theta_h]$ and $E(\vartheta, \mathbb{R}) < +\infty$, and defining $\rho : \mathbb{R} \to [\theta_h, \frac{\pi}{2}]$ by

$$\rho(x) := \begin{cases} \vartheta(x) & \text{if } \vartheta(x) \in \left[\theta_h, \dfrac{\pi}{2}\right], \\ \pi - \vartheta(x) & \text{if } \vartheta(x) \in \left(\dfrac{\pi}{2}, \pi - \theta_h\right], \end{cases} \tag{13}$$

for every $x \in \mathbb{R}$, we have $\sin\rho = \sin\vartheta$, and since the map $\vartheta \mapsto \rho$ is Lipschitz, we also have $|\vartheta_x| = |\rho_x|$ almost everywhere on $\mathbb{R}$. Thus

$$\begin{aligned} +\infty > E(\vartheta, \mathbb{R}) = E(\rho, \mathbb{R}) &= \frac{1}{2}\int_{\mathbb{R}} |\rho_x|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} (\sin\rho - h)^2\,\mathrm{d}x + \frac{\nu}{4}\|\sin\rho\|^2_{\mathring{H}^{1/2}(\mathbb{R})} \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}} |\rho_x|^2\,\mathrm{d}x + \frac{c_h}{4}\int_{\mathbb{R}} \left(\rho - \theta_h\right)^2\mathrm{d}x + \frac{\nu}{4}\|\sin\rho\|^2_{\mathring{H}^{1/2}(\mathbb{R})} \\ &\geqslant \frac{c_h}{4}\left\|\rho - \theta_h\right\|^2_{H_1} + \frac{\nu}{4}\|\sin\rho\|^2_{\mathring{H}^{1/2}(\mathbb{R})}, \end{aligned} \tag{14}$$

where $c_h := \cos^2\left(\frac{\pi}{4} + \frac{\theta_h}{2}\right) > 0$ for all $\theta_h < \frac{\pi}{2}$. In the above inequality, we used the fact that, since $\rho(\mathbb{R}) \subset [\theta_h, \frac{\pi}{2}]$, $\theta_h \in [0, \frac{\pi}{2})$, and $\sin z \geqslant z/\sqrt{2}$ for all $z \in [0, \frac{\pi}{4}]$, we have

$$\sin\rho - h = \sin\rho - \sin\theta_h = 2\cos\left(\frac{\rho + \theta_h}{2}\right)\sin\left(\frac{\rho - \theta_h}{2}\right) \geqslant \frac{\sqrt{c_h}}{\sqrt{2}}(\rho - \theta_h). \tag{15}$$

Lemma 2 (restriction of rotations). *Let $\vartheta \in \mathcal{A}$ such that $E(\vartheta) < +\infty$. Then there exists $\tilde{\vartheta} \in \mathcal{A}$ such that $\tilde{\vartheta}(\mathbb{R}) \subset [\theta_h, \pi - \theta_h]$ and $E(\tilde{\vartheta}) \leqslant E(\vartheta)$, with strict inequality unless $\vartheta(\mathbb{R}) \subset [\theta_h, \pi - \theta_h]$.*

Proof.
Step 1. We show first that there exists $\vartheta^\tau \in \mathcal{A}$ such that $\vartheta^\tau(\mathbb{R}) \subset [0, \pi]$, $\sin\vartheta^\tau = |\sin\vartheta|$, and $E(\vartheta^\tau) \leqslant E(\vartheta)$. Let $\vartheta^\tau : \mathbb{R} \to [0, \pi]$ be defined by

$$\vartheta^\tau(x) := \begin{cases} \vartheta(x) - 2k\pi & \text{if } \vartheta(x) \in [2k\pi, (2k+1)\pi), \\ 2k\pi - \vartheta(x) & \text{if } \vartheta(x) \in [(2k-1)\pi, 2k\pi), \end{cases} \tag{16}$$

for every $x \in \mathbb{R}$. The definition is well-posed since $\{[2k\pi, (2k+1)\pi), [(2k-1)\pi, 2k\pi) : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$. Notice that ϑ^τ is obtained by means of a translation by $-2k\pi$ in each interval of the form $[2k\pi, (2k+1)\pi)$, and by means of a reflection with respect to the origin and a translation by $2k\pi$ in each interval of the form $[(2k-1)\pi, 2k\pi)$. By construction, $\vartheta^\tau \in [0, \pi]$ and $\sin\vartheta \leqslant |\sin\vartheta| = \sin\vartheta^\tau$ so that $\|\sin\vartheta\|_{L^2(\mathbb{R})} = \|\sin\vartheta^\tau\|_{L^2(\mathbb{R})}$ and $-h\int_{\mathbb{R}} \sin\vartheta^\tau\,\mathrm{d}x \leqslant -h\int_{\mathbb{R}} \sin\vartheta\,\mathrm{d}x$, implying that $\int_{\mathbb{R}}(\sin\vartheta^\tau - h)^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}}(\sin\vartheta - h)^2\,\mathrm{d}x$. Furthermore, since the map $\vartheta \mapsto \vartheta^\tau$ is Lipschitz, we also have $\int_{\mathbb{R}} \vartheta_x^2\,\mathrm{d}x = \int_{\mathbb{R}} \left(\vartheta_x^\tau\right)^2\mathrm{d}x$.

The conclusion now comes from the fact that

$$\begin{aligned} \int_{\mathbb{R}} u\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} u\,\mathrm{d}x &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left(u(x) - u(y)\right)^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left(|u(x)| - |u(y)|\right)^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} |u|\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}|u|\,\mathrm{d}x. \end{aligned} \tag{17}$$

Step 2. Without loss of generality, in view of step 1, we can assume that $\vartheta \in \mathcal{A}$ is such that $\vartheta(\mathbb{R}) \subset [0, \pi]$ and $E(\vartheta) < +\infty$. Set $I := \{x \in \mathbb{R} : \vartheta(x) \in [0, \theta_h) \cup (\pi - \theta_h, \pi]\}$ and let $\tilde{\vartheta}: \mathbb{R} \to [\theta_h, \pi - \theta_h]$ be defined by

$$\tilde{\vartheta}(x) := \begin{cases} \theta_h & \text{if } \vartheta(x) < \theta_h, \\ \vartheta(x) & \text{if } \vartheta(x) \in [\theta_h, \pi - \theta_h] = \mathbb{R} \setminus I, \\ \pi - \theta_h & \text{if } \vartheta(x) > \pi - \theta_h \end{cases} \tag{18}$$

for every $x \in \mathbb{R}$.

Notice that, since $\vartheta = \tilde{\vartheta}$ in $\mathbb{R} \setminus I$, we have

$$\| \sin \vartheta - h \|^2_{L^2(\mathbb{R})} - \| \sin \tilde{\vartheta} - h \|^2_{L^2(\mathbb{R})} = \| \sin \vartheta - h \|^2_{L^2(I)} \geqslant 0. \tag{19}$$

Moreover, since $\tilde{\vartheta}$ is constant on I, we have

$$\begin{aligned} \| \sin \tilde{\vartheta} \|^2_{\mathring{H}^{1/2}(\mathbb{R})} &= \frac{1}{2\pi} \int_{\mathbb{R}\setminus I} \int_{\mathbb{R}\setminus I} \frac{\big(\sin \vartheta(x) - \sin \vartheta(y) \big)^2}{(x-y)^2} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \frac{1}{\pi} \int_{\mathbb{R}\setminus I} \int_I \frac{\big(h - \sin \vartheta(y) \big)^2}{(x-y)^2} \, \mathrm{d}x \, \mathrm{d}y, \end{aligned} \tag{20}$$

and so

$$\begin{aligned} \| \sin \vartheta \|^2_{\mathring{H}^{1/2}(\mathbb{R})} - \| \sin \tilde{\vartheta} \|^2_{\mathring{H}^{1/2}(\mathbb{R})} &= \frac{1}{2\pi} \int_I \int_I \frac{\big(\sin \vartheta(x) - \sin \vartheta(y) \big)^2}{(x-y)^2} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \frac{1}{\pi} \int_{\mathbb{R}\setminus I} \int_I \frac{\big(\sin \vartheta(x) - \sin \vartheta(y) \big)^2 - \big(h - \sin \vartheta(y) \big)^2}{(x-y)^2} \, \mathrm{d}x \, \mathrm{d}y \\ &\geqslant \frac{1}{\pi} \int_{\mathbb{R}\setminus I} \int_I \frac{\big(\sin \vartheta(x) - h \big) \big(h + \sin \vartheta(x) - 2 \sin \vartheta(y) \big)}{(x-y)^2} \, \mathrm{d}x \, \mathrm{d}y > 0, \end{aligned} \tag{21}$$

where in the second to last inequality we have used the identity $A^2 - B^2 = (A - B)(A + B)$ and in the last one the fact that for every $x \in I$ and every $y \in \mathbb{R} \setminus I$ the following inequalities hold: $\sin \vartheta(x) < h \leqslant \sin \vartheta(y)$. Then we conclude that $E(\tilde{\vartheta}) < E(\vartheta)$. □

The following rearrangement property is a consequence of lemma 7.17 in [34].

Lemma 3. *Let $u \in L^2(\mathbb{R})$ be a non-negative function and let u^* be its symmetric decreasing rearrangement, i.e.*

$$u^*(x) := \int_0^{+\infty} \chi_{\{u>t\}^*}(x) \, \mathrm{d}t, \tag{22}$$

where for a Borel set $A \subset \mathbb{R}$ the rearranged set A^* is the interval with measure $\mathcal{L}^1(A)$ centred at the origin. Then

$$\|u^*\|^2_{\mathring{H}^{1/2}(\mathbb{R})} = \int_{\mathbb{R}} u^* \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u^* \, \mathrm{d}x \leqslant \int_{\mathbb{R}} u \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u \, \mathrm{d}x = \|u\|^2_{\mathring{H}^{1/2}(\mathbb{R})}, \tag{23}$$

with equality only if u is a translation of a symmetric decreasing function.

To prove our main result, we first need the following preliminary lemma. The idea goes back to [22], except that here our variable is the angle function rather than the first component of the magnetization vector. Let us point out that the rearrangement argument of lemma 4 only yields the non-increasing property of ϑ for the minimizer. To prove strict decrease, one needs an additional argument presented in step 3 of the proof of the main theorem below.

Lemma 4 (rearrangement). *Let* $\vartheta \in \mathcal{A}$ *be such that* $\vartheta(\mathbb{R}) \subset [\theta_h, \pi - \theta_h]$ *and* $E(\vartheta) < +\infty$. *Then there exists a function* $\vartheta^o(x): \mathbb{R} \to [\theta_h, \pi - \theta_h]$ *satisfying (12) and the following properties:*

$$\vartheta^o(0) = \frac{\pi}{2}, \quad \vartheta^o(x) = \pi - \vartheta^o(-x), \quad \vartheta^o_x \leqslant 0 \qquad \text{on } \mathbb{R} \quad \text{and} \quad E(\vartheta^o) \leqslant E(\vartheta), \tag{24}$$

where the equality in the latter expression holds only if $\sin\vartheta$ *is a translation of a symmetric decreasing function.*

Proof. Let $\rho: \mathbb{R} \to [\theta_h, \frac{\pi}{2}]$ be defined as in (13) for every $x \in \mathbb{R}$. Then, from the discussion at the beginning of section 4 we have $E(\vartheta, \mathbb{R}) = E(\rho, \mathbb{R})$. Now, define $\rho^o : \mathbb{R} \to [\theta_h, \frac{\pi}{2}]$ by setting

$$\rho^o(x) := \theta_h + \big(\rho(x) - \theta_h\big)^*, \tag{25}$$

where given a function f, f^* stands for the symmetric rearrangement of f. This implies that ρ^o is even, $(\rho^*)_x \leqslant 0$ on $\mathbb{R}^+$, and $\rho^o(x) \to \theta_h$ as $|x| \to +\infty$. Moreover, the level sets of ρ^o are simply the rearrangement of the level sets of ρ, i.e.,

$$\{x : \rho^o(x) > t\} = \{x : \rho(x) > t\}^*. \tag{26}$$

A consequence of this is the equimeasurability of the functions ρ^o and ρ, i.e.,

$$\mathcal{L}^1\big(\{x : \rho^o(x) > t\}\big) = \mathcal{L}^1\big(\{x : \rho(x) > t\}\big) \tag{27}$$

for every $t > 0$. This, together with the layer cake representation theorem 1.13 in [34], yields

$$\int_{\mathbb{R}} \phi\big(\rho^o(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}} \phi(\rho(x))\,\mathrm{d}x \tag{28}$$

for every monotone, absolutely continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(0) = 0$. Choosing $\phi(z) = (\sin(\min(z, \frac{\pi}{2})) - h)^2_+$, we see that

$$\int_{\mathbb{R}} \big(\sin\rho^o(x) - h\big)^2\,\mathrm{d}x = \int_{\mathbb{R}} \big(\sin\rho - h\big)^2\,\mathrm{d}x. \tag{29}$$

We recall now the following property of rearrangements of any given Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ vanishing at infinity:

$$(\Phi \circ |f|)^* = \Phi \circ f^* \tag{30}$$

for every $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ non-decreasing. Applying the above property to $f := \rho - \theta_h : \mathbb{R} \to [0, \frac{\pi}{2} - \theta_h]$ and $\Phi : f \mapsto \sin(\theta_h + f)$, which is increasing for every $f \in [0, \frac{\pi}{2} - \theta_h]$, we get

$$(\sin\rho)^* = \sin\big(\theta_h + (\rho - \theta_h)^*\big) = \sin\rho^o. \tag{31}$$

In view of the above identity and using also lemma 3, we have

$$\begin{aligned} \int_{\mathbb{R}} \sin\rho^o \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} \sin\rho^o\,\mathrm{d}x &= \int_{\mathbb{R}} (\sin\rho)^* \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} (\sin\rho)^*\,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}} \sin\rho \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} \sin\rho\,\mathrm{d}x = \int_{\mathbb{R}} \sin\vartheta \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} \sin\vartheta\,\mathrm{d}x, \end{aligned} \tag{32}$$

where the equality holds only if $\sin\vartheta$ is a translation of a symmetric decreasing function. Finally, by lemma 7.17 in [34],

$$\int_{\mathbb{R}} |\rho^o_x|^2\,\mathrm{d}x = \int_{\mathbb{R}} \Big[\big(\rho(x) - \theta_h\big)^*_x\Big]^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}} |\rho_x|^2\,\mathrm{d}x = \int_{\mathbb{R}} |\vartheta_x|^2\,\mathrm{d}x. \tag{33}$$

Define $\vartheta^o : \mathbb{R} \to [\theta_h, \pi - \theta_h]$ by setting

$$\vartheta^o(x) := \begin{cases} \rho^o(x) & \text{if } x \geqslant 0, \\ \pi - \vartheta^o(-x) = \pi - \rho^o(x) & \text{if } x < 0. \end{cases} \tag{34}$$

Since $\sin\vartheta^o = \sin\rho^o$, by (29), (32) and (33) we conclude. □

We now investigate the decay of monotone solutions of (11) satisfying (12). This information, combined with the properties of the fundamental solution of the linearization of (11) around θ_h (see, e.g., [16, section 5.1]), will be used to establish the precise asymptotics behaviour of the minimizers of E over $\mathcal{A}$ as $x \to \pm\infty$.

Lemma 5. *Let $\vartheta \in C^\infty(\mathbb{R})$ be a non-increasing solution of (11) satisfying (12), with $\vartheta(-x) = \pi - \vartheta(x)$ for all $x \in \mathbb{R}$ and all derivatives vanishing as $x \to \pm\infty$. Let $u = \sin\vartheta - h$ and assume that there exist $c > 0$ and $\alpha \in (0, 2]$ such that $\|u\|_{W^{2,\infty}(\mathbb{R})} \leqslant c$ and*

$$u(x) \leqslant \frac{c}{1 + |x|^\alpha} \qquad \forall x \in \mathbb{R}. \tag{35}$$

Then there exists $C = C(c, \alpha, h, \nu) > 0$ such that

$$|u_x(x)|,\, |u_{xx}(x)|,\, \left| \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u(x) \right| \leqslant \frac{C}{1 + |x|^\alpha} \qquad \forall x \in \mathbb{R}. \tag{36}$$

Proof. Throughout the proof we assume that $x > 0$ is sufficiently large depending only on c, α and h. All the constants in the estimates are also assumed to depend only on c, α, h and ν.

Equation (11) written in terms of u reads

$$\frac{u_{xx}}{1 - h^2 - 2uh - u^2} + \frac{(h + u)u_x^2}{(1 - h^2 - 2uh - u^2)^2} = u + \frac{\nu}{2} \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u. \tag{37}$$

In particular, since $u(x)$ goes to zero as $x \to \infty$ together with all its derivatives, we also have that $\lim_{x\to\infty} M(x) = 0$, where

$$M(x) := \max_{[x,\infty)} \left| \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u \right|. \tag{38}$$

Now, multiply (11) by ϑ_x and integrate over (x, ∞). Together with the monotonicity of $u(x)$, this yields

$$\vartheta_x^2(x) \leqslant u^2(x) + \nu u(x) M(x). \tag{39}$$

Using the fact that $u_x = \vartheta_x \cos\vartheta$, from (37) and (39) we obtain

$$|u_{xx}(x)| \leqslant C(u(x) + M(x)). \tag{40}$$

On the other hand, for every $\delta > 0$ sufficiently small we have by (7)

$$\begin{aligned} \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u(x) &= \frac{1}{\pi} \int_{-\infty}^{-\frac{1}{2}x} \frac{u(x) - u(y)}{(x - y)^2}\, \mathrm{d}y + \frac{1}{\pi} \int_{-\frac{1}{2}x}^{\frac{1}{2}x} \frac{u(x) - u(y)}{(x - y)^2}\, \mathrm{d}y \\ &\quad + \frac{1}{\pi} \int_{\frac{1}{2}x}^{x-\delta} \frac{u(x) - u(y)}{(x - y)^2}\, \mathrm{d}y + \frac{1}{\pi} \fint_{x-\delta}^{x+\delta} \frac{u(x) - u(y)}{(x - y)^2}\, \mathrm{d}y + \frac{1}{\pi} \int_{x+\delta}^{+\infty} \frac{u(x) - u(y)}{(x - y)^2}\, \mathrm{d}y. \end{aligned} \tag{41}$$

Using the symmetric decreasing property of $u(x)$, we can then estimate the left-hand side of (41) as

$$\left| \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u(x) \right| \leqslant C \left(\delta^{-1} u(\tfrac{1}{2}x) + \delta \max_{[x-\delta, x+\delta]} |u_{xx}| + \int_{-\frac{1}{2}x}^{\frac{1}{2}x} \frac{u(y)}{(x - y)^2}\, \mathrm{d}y \right), \tag{42}$$

where to estimate the fourth term in the right-hand side of (41) we used Taylor formula $u(y) = u(x) + u_x(x)(y-x) + \frac{1}{2}u_{xx}(\bar{x}(y))(y-x)^2$, for some $\bar{x}(y)$ lying between x and y, and noted that the linear term in $y - x$ does not contribute to the principal value of the integral. Combining the estimate in (42) with (40) and the assumption on the decay of u in (35) then yields

$$\left| \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u(x) \right| \leqslant C \left(\frac{\delta^{-1}}{x^\alpha} + \delta M(x-\delta) + \frac{1}{x^2} + \frac{1}{x^{1+\alpha}} \right). \tag{43}$$

Now, choosing $\delta > 0$ sufficiently small, the last estimate implies

$$\left| \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} u(x) \right| \leqslant Cx^{-\alpha} + \frac{1}{2} M(x-\delta), \tag{44}$$

for every $\alpha \leqslant 2$. Therefore, by monotonicity of $M(x)$, we also have

$$M(x) \leqslant Cx^{-\alpha} + \tfrac{1}{2} M(x-\delta), \tag{45}$$

for all $x > 0$ sufficiently large.

Let us show that (45) implies the same kind of bound as in (35) for $M(x)$. We use an induction argument. Let $x_n := x_0 + n\delta$ and $M_n := M(x_n)$ for $n \in \mathbb{N}$ and some $x_0 > 0$ to be fixed shortly. Clearly, by (45) we have $M_1 \leqslant cx_1^{-\alpha}$ for $c = 4C + \frac{1}{2}M(x_0)(x_0+\delta)^\alpha$. We claim that if also $M_{n-1} \leqslant cx_{n-1}^{-\alpha}$, then (45) implies $M_n \leqslant cx_n^{-\alpha}$, provided that x_0 is chosen to be sufficiently large. Indeed, by (45) and the assumption $M_{n-1} \leqslant cx_{n-1}^{-\alpha}$ we have

$$M_n \leqslant \left(C + \frac{cx_n^\alpha}{2(x_n - \delta)^\alpha} \right) x_n^{-\alpha} \leqslant (C + \tfrac{3}{4}c)x_n^{-\alpha} \leqslant cx_n^{-\alpha}, \tag{46}$$

provided that $x_0 \geqslant 6\delta$, and the claim follows. Once we established the bound for M_n, the estimate $M(x) \leqslant C/(1+x^\alpha)$ for all $x \in \mathbb{R}$ follows by monotonicity and boundedness of $M(x)$.

To conclude the proof of the lemma, we combine the estimate for $M(x)$ just obtained with (39) and (40). □

5. Proof of the main result

We are now in a position to prove the main result of this paper, theorem 1.

Step 1: existence. Let $\{\vartheta_n\} \subset \mathcal{A}$ be a minimizing sequence, i.e.

$$\lim_{n \to +\infty} E(\vartheta_n) = \inf\{E(\vartheta) : \vartheta \in \mathcal{A}\} < +\infty. \tag{47}$$

By translation invariance we may assume that $\vartheta_n(0) = \frac{\pi}{2}$ and by lemma 4 that each ϑ_n satisfies

$$\vartheta_n(\mathbb{R}) \subset [\theta_h, \pi - \theta_h], \quad \vartheta_n(x) = \pi - \vartheta_n(-x), \quad \lim_{x \to +\infty} \vartheta_n(x) = \theta_h, \quad (\vartheta_n)_x \leqslant 0 \quad \text{on } \mathbb{R}. \tag{48}$$

For each n, let $\rho_n : \mathbb{R} \to [0, \frac{\pi}{2}]$ be defined as in (13) (after replacing ϑ with ϑ_n). By (14) we have

$$\|\rho_n - \theta_h\|^2_{H^1(\mathbb{R})} + \|\sin \rho_n\|^2_{\mathring{H}^{1/2}(\mathbb{R})} \leqslant C < +\infty. \tag{49}$$

In view of Banach-Alaoglu–Bourbaki's theorem, there exist a subsequence of $\{\rho_n\}$, not relabelled, a function $v \in H^1(\mathbb{R})$ and a function $u \in H^{1/2}(\mathbb{R})$ such that

$$\rho_n - \theta_h \rightharpoonup v \quad \text{weakly in } H^1(\mathbb{R}) \quad \text{and} \quad \sin \rho_n \rightharpoonup u \quad \text{weakly in} \quad H^{1/2}(\mathbb{R}). \tag{50}$$

Let us fix $k \in \mathbb{N}$. By the compact embedding $H^1(-k,k) \subset\subset L^2(-k,k)$, we may find a subsequence, not relabeled, such that $\rho_n - \theta_h \to v$ strongly in $L^2(-k,k)$ and $\mathcal{L}^1$-a.e. in $(-k,k)$. Thus, $\rho_n \to \rho$ $\mathcal{L}^1$-a.e. in $(-k,k)$, where $\rho := v + \theta_h$. Moreover, since

$$\sup_n \|\sin\rho_n\|^2_{\mathring{H}^{1/2}(-k,k)} = \sup_n \left(\frac{1}{2\pi} \int_{-k}^{k} \int_{-k}^{k} \frac{\big(\sin\rho_n(x) - \sin\rho_n(y)\big)^2}{(x-y)^2} \,\mathrm{d}x\,\mathrm{d}y \right) < +\infty, \tag{51}$$

it follows by the fractional compact embedding (see for instance section 8.6 of [34]) that $\{\sin\rho_n - h\}$ is precompact in $L^2(-k,k)$, that is, up to a subsequence, not relabeled, $\sin\rho_n \to u$ strongly in $L^2(-k,k)$ and $\mathcal{L}^1$-a.e. in $(-k,k)$. Then by the uniqueness of the limits, $u = \sin\rho$ $\mathcal{L}^1$-a.e. in $(-k,k)$ for every k.

Finally, by the lower semicontinuity of the L^2 and $H^{1/2}$ norms with respect to their weak convergences and Fatou lemma applied to the sequence $\{(\sin\rho_n - h)^2\}$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{-k}^{k} |\rho_x|^2\,\mathrm{d}x + \frac{1}{2}\int_{-k}^{k} (\sin\rho - h)^2\,\mathrm{d}x + \frac{\nu}{8\pi}\int_{-k}^{k}\int_{-k}^{k} \frac{\big(\sin\rho(x) - \sin\rho(y)\big)^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant \liminf_{n\to+\infty} \frac{1}{2}\int_{-k}^{k} |(\rho_n)_x|^2\,\mathrm{d}x + \frac{1}{2}\int_{-k}^{k} (\sin\rho_n - h)^2\,\mathrm{d}x \\
&\qquad + \frac{\nu}{8\pi}\int_{-k}^{k}\int_{-k}^{k} \frac{\big(\sin\rho_n(x) - \sin\rho_n(y)\big)^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y \leqslant \liminf_{n\to+\infty} E(\rho_n, \mathbb{R}).
\end{aligned} \tag{52}$$

Applying Lebesgue's monotone convergence theorem to the sequences $\{\chi_{(-k,k)}[(\rho_x)^2 + (\sin\rho - h)^2]\}$ and $\left\{\chi_{(-k,k)^2} \frac{(\sin\vartheta(x) - \sin\vartheta(y))^2}{(x-y)^2}\right\}$, we then obtain

$$E(\rho, \mathbb{R}) \leqslant \liminf_{n\to+\infty} E(\rho_n, \mathbb{R}) = \liminf_{n\to+\infty} E(\vartheta_n, \mathbb{R}). \tag{53}$$

Given such a ρ, let us define $\vartheta : \mathbb{R} \to \mathbb{R}$ by setting

$$\vartheta^{(0)}(x) := \begin{cases} \rho(x) & \text{if } x \geqslant 0, \\ \pi - \rho(x) & \text{if } x < 0, \end{cases} \tag{54}$$

for every $x \in \mathbb{R}$. We claim that $\vartheta^{(0)}$ satisfies the following properties:

$$\vartheta^{(0)}(0) = \frac{\pi}{2}, \quad \vartheta^{(0)}(x) = \pi - \vartheta^{(0)}(-x), \quad \lim_{x\to+\infty} \vartheta^{(0)}(x) = \theta_h, \quad \vartheta^{(0)}_x \leqslant 0 \quad \text{on } \mathbb{R}. \tag{55}$$

Since $\rho - \theta_h \in H^1(\mathbb{R})$, by Morrey's theorem we have $\lim_{x\to+\infty} \vartheta^{(0)}(x) = \lim_{x\to+\infty} \rho(x) = \theta_h$ and $\vartheta^{(0)} \in C(\mathbb{R})$. Finally, since weak convergence in $H^1(\mathbb{R})$ implies pointwise and uniform convergence on compacts (see for instance theorem 8.6 in [34]), by the first of (50) we have $\rho_n \to \rho$ locally uniformly on $\mathbb{R}$. Therefore, the following properties

$$\rho_n(0) = \frac{\pi}{2}, \quad \rho_n(x) = \rho_n(-x), \quad \rho_n(x_1) \geqslant \rho_n(x_2) \qquad \text{if } 0 \leqslant x_1 \leqslant x_2, \tag{56}$$

are preserved when taking the limit as $n \to +\infty$ which in turn implies by construction that the properties in (55) are satisfied.

Since by construction $E(\vartheta^{(0)}, \mathbb{R}) = E(\rho, \mathbb{R})$, by (53) and (55), we conclude that $\vartheta^{(0)}$ is a minimizer for E in the class $\mathcal{A}$.

Step 2: Euler–Lagrange equation. Since the first variation of E at any global minimizer of E is zero, we have that the minimizer $\vartheta^{(0)}$ of E, as well as any other minimizer of E, satisfies

$$\begin{aligned}
0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(\vartheta^{(0)} + t\varphi, \mathbb{R}) = \int_{\mathbb{R}} \Bigg(&\varphi_x \vartheta^{(0)}_x + \varphi\cos\vartheta^{(0)}\sin\vartheta^{(0)} - h\varphi\cos\vartheta^{(0)} \\
&+ \frac{\nu}{2}\varphi\cos\vartheta^{(0)} \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} \sin\vartheta^{(0)} \Bigg)\,\mathrm{d}x,
\end{aligned} \tag{57}$$

for every $\varphi \in H^1(\mathbb{R})$. In other words, $\phi = \vartheta^{(0)}$ is a weak solution of the ordinary differential equation

$$-\phi_{xx} + b(x)\cos\phi = 0, \tag{58}$$

where $b(x) := \sin\vartheta^{(0)} - h + \frac{1}{2}\nu(-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}\sin\vartheta^{(0)}$. Since $\vartheta_x^{(0)} \in L^2(\mathbb{R})$ and the sine function is Lipschitz, we have $(\sin\vartheta^{(0)})_x = \vartheta_x^{(0)}\cos\vartheta^{(0)} \in L^2(\mathbb{R})$, and by the estimate

$$\left\| u + \frac{\nu}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} u \right\|_{L^2(\mathbb{R})} \leqslant C\|u\|_{H^1(\mathbb{R})}, \tag{59}$$

applied to $u := \sin\vartheta^{(0)} - h$, we conclude that $b(x) \in L^2(\mathbb{R})$. By (58), the weak second derivative of $\vartheta^{(0)}$ is given by $\vartheta_{xx}^{(0)} = b(x)\cos\vartheta^{(0)}$ and is, hence, in $L^2(\mathbb{R})$. Therefore, $\vartheta_x^{(0)} \in H^1(\mathbb{R})$ and by Morrey's theorem, together with the fact that $\vartheta^{(0)} \in C(\mathbb{R})$, we have that $\vartheta_x^{(0)}$ is continuous and bounded on $\mathbb{R}$ as well. Using the interpolation inequality (here we follow the arguments of [23])

$$\|\vartheta_x^{(0)}\|_{L^4(\mathbb{R})} \leqslant \|\vartheta_x^{(0)}\|_{L^\infty(\mathbb{R})}^{1/2}\|\vartheta_x^{(0)}\|_{L^2(\mathbb{R})}^{1/2} < +\infty, \tag{60}$$

we have $(\sin\vartheta^{(0)})_{xx} = \vartheta_{xx}^{(0)}\cos\vartheta^{(0)} - (\vartheta_x^{(0)})^2\sin\vartheta^{(0)} \in L^2(\mathbb{R})$. Thus, by (59) and the fact that $(\mathrm{d}/\mathrm{d}x)(-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}u = (-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}u_x$, readily verified via the Fourier representation of the fractional Laplacian [32], we conclude that $b(x) \in H^1(\mathbb{R})$ and so by Morrey's theorem $b(x)$ is continuous and bounded. This in turn implies that $\vartheta_{xx}^{(0)} = b(x)\cos\vartheta^{(0)} \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, that is, $\vartheta^{(0)} \in C^2(\mathbb{R})$ is a classical solution of (58). Finally, bootstrapping these regularity arguments, we conclude that $\vartheta^{(0)} \in C^\infty(\mathbb{R})$, and, moreover, all the derivatives are bounded uniformly in $\mathbb{R}$. Together with boundedness of their L^2 norms, the latter implies that all the derivatives of $\vartheta^{(0)}$ vanish at infinity.

Step 3: strict monotonicity. First we show that $\vartheta_x^{(0)}(0) < 0$ employing the same argument as in [23]. Since $\vartheta^{(0)} \in C^2(\mathbb{R})$ is a classical solution of (58), if we assume that $\vartheta^{(0)}(0) = \frac{\pi}{2}$ and $\vartheta_x^{(0)}(0) = 0$, the uniqueness theorem for ordinary differential equations implies that $\vartheta^{(0)}$ must be identically equal to $\frac{\pi}{2}$, a contradiction.

Now we prove $\vartheta_x^{(0)} < 0$ on $\mathbb{R}^+$. We argue by contradiction and assume that there exist $\bar{x} > 0$ such that $\vartheta_x^{(0)}(\bar{x}) = 0$. But then also $\vartheta_{xx}^{(0)}$ must be zero at $\bar{x}$, because otherwise $\vartheta^{(0)}$ will be either strictly convex or strictly concave at $\bar{x}$, contradicting the fact that $\vartheta_x^{(0)} \leqslant 0$ on $\mathbb{R}$. Moreover, by differentiating the Euler-Lagrange equation (11) and taking into account that $\vartheta_x(\bar{x}) = \vartheta_{xx}(\bar{x}) = 0$, we get

$$\vartheta_{xxx}^{(0)}(\bar{x}) = \frac{\nu}{2}\cos\vartheta^{(0)}(\bar{x})g(\bar{x}), \qquad g := \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\left(\vartheta_x^{(0)}\cos\vartheta^{(0)}\right). \tag{61}$$

Now we observe that $\cos\vartheta^{(0)}(\bar{x}) > 0$ because $\vartheta^{(0)}(\mathbb{R}^+) \subset [\theta_h, \frac{\pi}{2}]$, and, recalling (7), taking into account that $\vartheta_x(\bar{x}) = \vartheta_{xx}(\bar{x}) = 0$, and noticing that the integral converges near $\bar{x}$ in the usual sense, we have

$$\begin{aligned} g(\bar{x}) &= -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\vartheta_x^{(0)}(y)\cos\vartheta^{(0)}(y)}{(\bar{x}-y)^2}\,\mathrm{d}y \\ &= -\frac{1}{\pi}\int_{\mathbb{R}^+} \frac{\vartheta_x^{(0)}(y)\cos\vartheta^{(0)}(y)}{(\bar{x}-y)^2}\,\mathrm{d}y - \frac{1}{\pi}\int_{\mathbb{R}^+} \frac{\vartheta_x^{(0)}(y)\cos\vartheta^{(0)}(y)}{(\bar{x}+y)^2}\,\mathrm{d}y > 0, \end{aligned} \tag{62}$$

where we have used the fact that $\vartheta_x^{(0)}(-x) = \vartheta_x^{(0)}(x) \leqslant 0$, and the inequality is strict on a set of positive measure. Therefore, $\vartheta_{xxx}^{(0)}(\bar{x}) > 0$, which implies that $\vartheta^{(0)}$ is increasing at $\bar{x}$, contradicting the fact that $\vartheta_x^{(0)} \leqslant 0$ on $\mathbb{R}$. Hence, $\vartheta^{(0)}$ has to be strictly decreasing on $\mathbb{R}^+$.

Since $\vartheta^{(0)}(x) = \pi - \vartheta^{(0)}(-x)$ for every $x < 0$, it follows that $\vartheta_x^{(0)} < 0$ also on $\mathbb{R}^-$.

Step 4: uniqueness (up to translations). It is clear that in view of the translational invariance the function $\vartheta^{(x_0)}(x) = \vartheta^{(0)}(x - x_0)$ with any $x_0 \in \mathbb{R}$ still belongs to $\mathcal{A}$ and satisfies $\vartheta^{(x_0)}(x_0) = \frac{\pi}{2}$. To conclude, we have to show that every minimizers of E in $\mathcal{A}$ is of the form $\vartheta^{(0)}(x - x_0)$ for some $x_0 \in \mathbb{R}$. Our argument is related to the strict convexity of the integrand in (6) written as a function of $u = \sin\vartheta$ and its derivative noted in [23].

We employ the strict monotonicity of minimizers, which implies that for every minimizer there is a unique point at which $\vartheta = \frac{\pi}{2}$. Let $\vartheta^{(1)}$ and $\vartheta^{(2)}$ be two different minimizers, which, after a suitable translation, satisfy $\vartheta^{(1)}(0) = \frac{\pi}{2}$ and $\vartheta^{(2)}(0) = \frac{\pi}{2}$. Define $\tilde{\vartheta}(x) := \arcsin\left(\frac{\sin\vartheta^{(1)}(x) + \sin\vartheta^{(2)}(x)}{2}\right)$ for $x \geqslant 0$ and $\tilde{\vartheta}(x) := \pi - \arcsin\left(\frac{\sin\vartheta^{(1)}(x) + \sin\vartheta^{(2)}(x)}{2}\right)$ for $x < 0$ (the function $\sin\tilde{\vartheta}$ is symmetric decreasing by step 1). We claim that for all $x \neq 0$ we have

$$\left(\tilde{\vartheta}_x\right)^2 = \frac{\left(\vartheta_x^{(1)}\cos\vartheta^{(1)} + \vartheta_x^{(2)}\cos\vartheta^{(2)}\right)^2}{4 - (\sin\vartheta^{(1)} + \sin\vartheta^{(2)})^2} \leqslant \frac{\left(\vartheta_x^{(1)}\right)^2 + \left(\vartheta_x^{(2)}\right)^2}{2}. \tag{63}$$

Once the claim is proved, we get that $\tilde{\vartheta}_x \in L^2(\mathbb{R})$ and, hence, $\tilde{\vartheta} \in \mathcal{A}$. Moreover, since the anisotropy and the stray-field terms in the energy are quadratic in $\sin\vartheta$, we get $E(\tilde{\vartheta}, \mathbb{R}) < \frac{1}{2}[E(\vartheta^{(1)}, \mathbb{R}) + E(\vartheta^{(2)}, \mathbb{R})]$, which contradicts the minimality of $\vartheta^{(1)}$ and $\vartheta^{(2)}$.

Let us come to the proof of (63). Observe that by two-dimensional Cauchy–Schwarz inequality

$$\frac{\left(\vartheta_x^{(1)}\cos\vartheta^{(1)} + \vartheta_x^{(2)}\cos\vartheta^{(2)}\right)^2}{4 - (\sin\vartheta^{(1)} + \sin\vartheta^{(2)})^2} \leqslant \frac{\left(\left(\vartheta_x^{(1)}\right)^2 + \left(\vartheta_x^{(2)}\right)^2\right)\left(\cos^2\vartheta^{(1)} + \cos^2\vartheta^{(2)}\right)}{4 - (\sin\vartheta^{(1)} + \sin\vartheta^{(2)})^2}. \tag{64}$$

On the other hand, we have

$$\begin{aligned}\frac{2\cos^2\vartheta^{(1)} + 2\cos^2\vartheta^{(2)}}{4 - (\sin\vartheta^{(1)} + \sin\vartheta^{(2)})^2} &= \frac{4 - 2\sin^2\vartheta^{(1)} - 2\sin^2\vartheta^{(2)}}{4 - (\sin\vartheta^{(1)} + \sin\vartheta^{(2)})^2} \\ &\leqslant \frac{4 - \sin^2\vartheta^{(1)} - \sin^2\vartheta^{(2)} - 2\sin\vartheta^{(1)}\sin\vartheta^{(2)}}{4 - (\sin\vartheta^{(1)} + \sin\vartheta^{(2)})^2} = 1.\end{aligned} \tag{65}$$

Combing (65) with (64) then yeilds (63).

Step 5: decay. We claim that by the results of the previous steps the unique minimizer $\vartheta^{(0)}$ of E in $\mathcal{A}$ satisfies the assumptions of lemma 5 with $\alpha = \frac{1}{2}$. Indeed, $\vartheta^{(0)}$ is a smooth decreasing solution of (11) satisfying (12) with all derivatives vanishing at infinity and obeying the required symmetry property. Furthermore, since $u = \sin\vartheta^{(0)} - h \in L^2(\mathbb{R})$ is symmetric decreasing, by an elementary property of monotone functions (see, e.g., [35, lemma A.IV]) we have that $u(x) \leqslant |x|^{-1/2}\|u\|_{L^2(\mathbb{R})}$. Therefore, the conclusions of lemma 5 apply to $\vartheta^{(0)}$. We now claim that this, in turn, implies the same kind of estimates for $\rho(x) - \theta_h$, where ρ is defined by (13) with $\vartheta = \vartheta^{(0)}$. Indeed, since $u_x = \vartheta_x^{(0)}\cos\vartheta^{(0)}$ and $u_{xx} = \vartheta_{xx}^{(0)}\cos\vartheta^{(0)} - (\vartheta_x^{(0)})^2\sin\vartheta^{(0)}$, the estimates for the derivatives follow from (36), and to obtain the estimate for $(-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}\rho$, one can use (42) with u replaced by ρ.

We now rewrite (11) in the following form:

$$L(\rho(x) - \theta_h) = f(x), \qquad f(x) := f_1(x) + f_2(x) + f_3(x), \tag{66}$$

where

$$L := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\nu\cos^2\theta_h\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} + \cos^2\theta_h \tag{67}$$

is a linear operator that can be viewed as a map from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$, and

$$\begin{aligned}f_1(x) := {} & \cos\theta_h(\cos\theta_h - \cos\rho(x))(\rho(x) - \theta_h) \\ & + \cos\rho(x)(\cos\theta_h(\rho(x) - \theta_h) - \sin\rho(x) + \sin\theta_h),\end{aligned} \tag{68}$$

$$f_2(x) := \frac{1}{2}\nu\cos\theta_h\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\cos\theta_h(\rho(x)-\theta_h)-\sin\rho(x)+\sin\theta_h), \tag{69}$$

$$f_3(x) := \frac{1}{2}\nu(\cos\theta_h-\cos\rho(x))\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\sin\rho(x)-\sin\theta_h). \tag{70}$$

Note that L represents the operator that generates the linearization of (11) around $\vartheta = \theta_h$, and all the terms in the definition of f are of 'quadratic order' in $\rho - \theta_h$. Also note that the fundamental solution $G(x)$ associated with L, i.e. the solution of $LG(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta-function, is a positive, continuous, piecewise-smooth function with a jump of the derivative at the origin and a decay $G(x) \sim |x|^{-2}$ at infinity (see lemma A.1). Moreover, L is invertible, and we have

$$\rho - \theta_h = L^{-1}f(x) = \int_{\mathbb{R}} G(x-y)f(y)\,\mathrm{d}y. \tag{71}$$

Observe that since *a priori* $u(x) \leqslant c/(1+|x|^{1/2})$, using Taylor expansion in $\rho-\theta_h$ we have $|f_1(x)| \leqslant C/(1+|x|^{1/2})^2$, and by lemma 5 we also have $|f_3(x)| \leqslant C/(1+|x|^{1/2})^2$. To prove that $|f_2(x)| \leqslant C/(1+|x|^{1/2})^2$ as well, we use the estimate in (42) once again, taking into account that $|(\rho\cos\theta_h - \sin\rho)_{xx}| \leqslant |\cos\theta_h - \cos\rho|\,|\rho_{xx}| + |\rho_x|^2\sin\rho \leqslant C/(1+|x|^{1/2})^2$ for $x \neq 0$ and arguing as in lemma 5. Thus, we have $|f(x)| \leqslant C/(1+|x|)$, and hence using (71), we obtain

$$\rho(x) - \theta_h \leqslant \int_{\mathbb{R}} G(x-y)|f(y)|\,\mathrm{d}y \leqslant C\int_{\mathbb{R}}\frac{|f(y)|}{1+|x-y|^2}\,\mathrm{d}y \leqslant \frac{C'}{1+|x|}. \tag{72}$$

In view of (72), we have now improved the estimate for $u(x)$ to the one in (35) with $\alpha = 1$. Repeating the argument above, we then conclude that $u(x) \leqslant C'/(1+|x|^2)$. Let us show that this estimate is, in fact, optimal, and obtain the precise asymptotics of the decay of the profile. In view of the estimate just mentioned and arguing as in lemma 5, we have $|f(x)| \leqslant C/(1+|x|^4)$. Therefore, using the multipole expansion in (71) and the decay property of $G(x)$ from lemma A.1, we conclude that

$$\rho - \theta_h = \frac{a}{|x|^2} + o(|x|^{-2}), \qquad a = \frac{\nu}{2\pi\cos^2\theta_h}\int_{\mathbb{R}} f(y)\,\mathrm{d}y. \tag{73}$$

Clearly, $a \geqslant 0$ in (73), and to complete the proof we need to show that $a > 0$. To establish this fact, we first note that $\int_{\mathbb{R}} f_1(x)\,\mathrm{d}x > 0$ and $\int_{\mathbb{R}} f_2(x)\,\mathrm{d}x = 0$. Indeed, it is easy to see that $f_1(x) > 0$ for all $x \in \mathbb{R}$, and the second equality follows from the fact that the operator $(-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}$ is self-adjoint and that constants lie in its kernel. To show that $\int_{\mathbb{R}} f_3(x)\,\mathrm{d}x > 0$, we use (7) and symmetrize the integral to obtain

$$\begin{aligned}\int_{\mathbb{R}} f_3(x)\,\mathrm{d}x &= -\frac{\nu}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\sin\rho(x)-\sin\rho(y))(\cos\rho(x)-\cos\rho(y))}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{\nu}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\sin\big(\rho(x)+\rho(y)\big)\sin^2\left(\frac{\rho(x)-\rho(y)}{2}\right)}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{74}$$

where we used trigonometric identities to arrive at the last line. In view of the fact that $\rho \in (\theta_h, \frac{1}{2}\pi)$, the right-hand side of (74) is positive, and the claim follows.

This concludes the proof of the theorem. □

Let us remark that the arguments in the proof of uniqueness above, with the test function $\vartheta^t(x) := \arcsin(t\sin\vartheta^{(1)}(x)+(1-t)\sin\vartheta^{(2)}(x))$ for $x \geqslant 0$ and $\vartheta^t = \pi - \arcsin(t\sin\vartheta^{(1)}(x)+(1-t)\sin\vartheta^{(2)}(x))$ for $x < 0$, where $t \in [0,1]$, could also be used to prove uniqueness of the *critical points* of the energy E in the class of solutions of (11) with values in $(\vartheta_h, \pi-\vartheta_h)$, obeying (12) and crossing the value of $\frac{\pi}{2}$ only once at the origin (for a similar treatment see [36]). However, since the computations in this case become exceedingly tedious and the precise behaviour of the solutions at infinity may be needed, we have not pursued this question any further in this paper. Nevertheless, establishing such a uniqueness result would be helpful for interpreting the results of the numerical solution of (11) as the Néel wall profiles.

It would also be interesting to see if the one-dimensional Néel wall profiles considered in this paper are the only minimizers (or even critical points) of the two-dimensional thin film energy in (4) with respect to perturbations with compact support that have the asymptotic behaviour given by (12). We note that in the case $\nu = 0$ this problem reduces to the famous problem of De Giorgi, whose solution in two space dimensions is now well understood [37] (see also [38] for a recent overview). Whether such a result remains valid in the presence of a non-local term ($\nu > 0$) remains to be seen (one result in this direction was obtained in [24]). Let us note that while for the local problem a continuous family of solutions obtained by rotations of the one-dimensional profile exists, in the non-local problem the orientation of the wall is expected to be fixed by the condition that the net charge across the wall be zero. The latter only allows walls that are parallel to the easy axis.

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Appendix.

The following lemma establishes the basic properties of the fundamental solution for the operator L (see also [16, section 5.1]).

Lemma A.1. *Let $G(x)$ be the fundamental solution for the operator L defined in (67). Then*

$$G(x) = \frac{2\nu}{\pi} \int_0^\infty \frac{t\,\mathrm{e}^{-t|x|\cos\theta_h}}{\nu^2 t^2 \cos^2\theta_h + 4(t^2-1)^2}\,\mathrm{d}t. \tag{A.1}$$

In particular, $G \in C^\infty(\mathbb{R}\backslash\{0\}) \cap \mathrm{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $G > 0$, $G(x) = G(-x)$, and

$$G(x) = \frac{\nu}{2\pi\cos^2\theta_h}|x|^{-2} + O(|x|^{-4}). \tag{A.2}$$

Proof. The proof is a simple application of Fourier transform and contour integration techniques, which we present here for completeness. Observe first that the Fourier transform $\widehat{G}(k) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}kx} G(x)\,\mathrm{d}x$ of $G(x)$ is well-defined and is given by

$$\widehat{G}(k) = \frac{1}{|k|^2 + \frac{1}{2}\nu\cos^2\theta_h|k| + \cos^2\theta_h}. \tag{A.3}$$

Interpreting $|k| = \sqrt{k^2}$ as an analytic function of k in the complex plane with a branch cut on the imaginary axis, i.e., defining $\sqrt{(x+iy)^2} = |x| + iy\,\mathrm{sgn}\,x$ for $x \neq 0$, we can write the formula for inverting the Fourier transform of G as

$$G(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2 + \frac{1}{2}\nu\cos^2\theta_h\sqrt{k^2} + \cos^2\theta_h}\,\mathrm{d}k, \tag{A.4}$$

and treat it as an integral along the real axis in the complex plane. In particular, $G(x)$ is even, and in the following it suffices to consider only $x \geqslant 0$.

It is easy to see that with our choice of the analytic branch the integrand in (A.4) has no poles. Furthermore, since the integral in (A.4) over a semicircle of radius $R > 0$ in the positive imaginary half-plane vanishes for $x \geqslant 0$ as $R \to \infty$, we can deform the contour of integration to run back and forth along the positive imaginary axis. After some algebra, we then find that the integral in (A.4) coincides with that in (A.1).

From the representation in (A.1), one immediately concludes that $G(x)$ is positive, bounded, smooth for all $x \neq 0$ and Lipschitz-continuous at $x = 0$. The estimate in (A.2) is then obtained by an elementary asymptotic analysis of the integral in (A.1). $\square$

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