Layered solutions for a nonlocal Ginzburg-Landau model with periodic modulation

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Abstract

We study layered solutions in a one-dimensional version of the scalar Ginzburg-Landau equation that involves a mixture of a second spatial derivative and a fractional half-derivative, together with a periodically modulated nonlinearity. This equation appears as the Euler-Lagrange equation of a suitably renormalized fractional Ginzburg-Landau energy with a double-well potential that is multiplied by a periodically varying positive factor bounded away from zero. A priori this energy is not bounded below due to the presence of a nonlocal term in the energy. Nevertheless, through a careful analysis of a minimizing sequence we prove existence of global energy minimizers that connect the two wells at infinity. These minimizers are shown to be the classical solutions of the associated nonlocal Ginzburg-Landau type equation.

keywords: Layered solutions, Nonlocal Ginzburg-Landau, Periodic modulation

1 Introduction

In this paper, we consider minimization of the following nonlocal energy functional:

$$\begin{aligned} J(u) :=& \frac{\alpha}{2}\int_{\mathbb{R}} |u'|^2\,dx + \int_{\mathbb{R}} g(x)\,W(u)\,dx \\ &+\frac{\beta}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{(u(x)-u(y))^2}{(x-y)^2} - \frac{(\eta(x)-\eta(y))^2}{(x-y)^2}\right]dydx \end{aligned} \tag{1.1}$$

in the set

$$\mathcal{A} = \left\{u \in H^1_{\text{loc}}(\mathbb{R}) : u - \eta \in H^1(\mathbb{R})\right\}.$$

Here α, β are positive constants, $g(x)$ is a 1-periodic function satisfying $g(x) \geq \gamma > 0$ for all $x \in \mathbb{R}$, $\eta \in C^\infty(\mathbb{R})$ is a given function satisfying $|\eta| \leq 1$, $\eta(x) = 1$ for $x \geq 1$, $\eta(x) = -1$ for $x \leq -1$, $W(u)$ is a double well potential satisfying

$$W(u) > 0 \text{ if } u \neq \pm 1, \quad W(\pm 1) = W'(\pm 1) = 0 \quad \text{and} \quad W''(\pm 1) > 0. \tag{1.2}$$

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Formally, for $u \in C^2_{loc}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ the Euler-Lagrange equation associated with (1.1) is

$$-\alpha u'' + \beta \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u + g(x) W'(u) = 0 \qquad x \in \mathbb{R}, \tag{1.3}$$

where

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y| \geq \varepsilon} \frac{u(x) - u(y)}{(x-y)^2} dy. \tag{1.4}$$

We are mainly interested in solutions of (1.3) that satisfy

$$\lim_{x \to \pm\infty} u(x) = \pm 1, \tag{1.5}$$

we call such solutions layered solutions.

Equation (1.3) is a special case of the more general equation

$$-\alpha \Delta u + \beta (-\Delta)^s u + g(x) W'(u) = 0 \qquad x \in \mathbb{R}^n, \tag{1.6}$$

where $0 < s < 1$. For $u \in C^2_{loc}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ the operator $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u(x) := C_{n,s} \,\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy = C_{n,s} \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy,$$

where $C_{n,s}$ is a normalization constant to guarantee that the Fourier symbol of the resulting operator is $|\xi|^{2s}$, see e.g. [23], section 3 for more details.

When $g(x) = \gamma$ is a constant, (1.6) reduces to

$$-\alpha \Delta u + \beta (-\Delta)^s u + \gamma W'(u) = 0 \qquad x \in \mathbb{R}^n. \tag{1.7}$$

This type of equation has attracted a lot of attention over the last twenty years (see, e.g., [23–26,42,44,48, 59]). In particular, the structure of layered solutions in the case $\beta = 0$ (Allen-Cahn) or $\alpha = 0$ (fractional Allen-Cahn) is well understood at present. Here a layered solution of (1.7) is a bounded solution which is monotone in one direction. When $\beta = 0$, De Giorgi conjecture posits that the level sets of such a layered solution are hyperplanes for $n \leq 8$. De Giorgi's initial conjecture was for $W(u) = \frac{1}{4}(1-u^2)^2$. This conjecture was proved for any C^2 function $W(u)$ satisfying (1.2) by Ghoussoub and Gui [37] when $n = 2$. When $n = 3$, Ambrosio and Cabré [9] proved the conjecture for a large class of $W(u)$ which includes the original De Giorgi's choice. Later, Alberti, Ambrosio and Cabré [2] extended their results to cover all C^2 function $W(u)$ with the properties specified in (1.2). Under the additional assumption of anti-symmetry of solutions, Ghoussoub and Gui [38] established the De Giorgi conjecture for $n = 4, 5$. Further developments on the conjecture can be found in [11]. De Giorgi conjecture was completely solved by Savin [55, 56] for $4 \leq n \leq 8$ under the additional assumption $\lim_{x_n \to \pm\infty} u(x) = \pm 1$. For dimensions $n \geq 9$, a counter-example was constructed by Del Pino, Kowalczyk and Wei [28]. A weaker version of the De Gigorgi conjecture, known as *Gibbons conjecture,* replaced monotonicity assumption by the stronger condition

$$\lim_{x_n \to \pm\infty} u(x) = \pm 1 \text{ uniformly for } (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}. \tag{1.8}$$

Gibbons conjecture was proved in all dimensions [11, 12, 35].

De Giorgi's conjecture has also been extended to the fractional Allen-Cahn case. The fractional De Giorgi conjecture was proved in [23–25, 58] for the case $n = 2, s \in (0,1)$, and in [20, 21] for $n = 3$ and $s \geq \frac{1}{2}$. Under additional limit conditions, fractional De Giorgi conjecture was proved for $n = 3$ and $s \in \left(0, \frac{1}{2}\right)$ by Dipierro, Serra and Valdinoci in [32] and by Savin in [57] for $4 \leq n \leq 8$ and $s \in [\frac{1}{2}, 1)$. The limit condition is removed in [31] for $n = 3$ and $s \in \left(0, \frac{1}{2}\right)$. Recently, Figalli and Serra [36] solved the De Giorgi conjecture for half-Laplacian when $n = 4$ (such a result is *not* known for the classical case $s = 1$). Layered solutions of Allen-Cahn type equations in the form of a sum of fractional Laplacians of different orders was addressed in [22]. Based on all these results, when $g(x)$ is a constant, solutions to (1.7) satisfying (1.5) reduces to the unique one-dimensional solution (modulo translation) which is monotone and the problem is essentially *one-dimensional.*

When $g(x)$ is not constant, but rather periodic, the continuous translational symmetry of layered solutions of (1.6) is broken and the structure of the set of solutions is much more complex. When $\beta = 0$, the nonautonomous Allen-Cahn equation

$$-\Delta u + W_u(x, u) = 0 \qquad x \in \mathbb{R}^n \tag{1.9}$$

with

$$W(x + k, u) = W(x, u) \qquad \forall k \in \mathbb{Z}^n$$

has been studied extensively over the last three decades. Equation (1.9) is a special case of a model problem initiated by Moser [43] for developing a PDE version of Aubry-Mather theory of monotone twist maps (see [10,13,14,52,53] for related work). A different motivation is to view (1.9) as a model for phase transitions. When $n = 1$ and subject to homogeneous Neumann boundry conditions on the interval of $x \in (0, 1)$, the following results have been proved for (1.9) for various choices of the potential term $W(x, u)$. Angenent, Mallet-Paret and Peletier [1] gave a complete classification of all stable equilibrium solutions to (1.9) for $W_u(x, u) = -u(1 - u)(u - a(x))$. Existence and stability of equilibrium solutions with a single transiton layer is proved in [39] for a general class of $W_u(x, u) = -f(x, u)$, with f satisfying $f(x, 0) = f(x, 1) = 0$ and $f(x + k, u) = f(x, u)$ for some $k > 0$. Nakashima [45] proved existence of stable solutions with multiple transition layers for the case $W_u(x, u) = (u - a(x))(u - b(x))(u - c(x))$. Existence and stability of multilayered solutions were provided in [46] for $W_u(x, u) = h^2(x) f(u)$. Nakashima and Tanaka [47] studied the one-dimensional case with a general potential $W(x, u)$ and obtained existence of solutions with clustering layers. For higher dimensions and the special case of

$$-\Delta u + a(x) W'(u) = 0 \qquad x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \tag{1.10}$$

Alessio, Jeanjean and Montecchiari [5] proved existence of infinitely many solutions which are distinct up to periodic translations and satisfy $\lim_{x_1 \to \pm\infty} u(x_1, x_2) = \pm b$ uniformly in x_2 for the case $n = 2$ when $a(x_1, x_2)$ is a positive, even, periodic function in x_1, x_2, and $W(u)$ is a double well potential vanishing at $u = \pm b$ for some $b > 0$. For the same equation, Alessio and Montecchiari [7] showed existence of brake orbits type solutions, and Alessio, Gui and Montecchiari [3] proved existence and asymptotic behavior of saddle solutions. When $a(x_1, x_2)$ depends only on one variable, existence of two-dimensional solutions was proved by Alessio and Montecchiari [4], and existence of infinitely many solutions can be found in [6, 61]. Alessio and Montecchiari [8] proved existence of infinitely many solutions verifying $\lim_{x_1 \to \pm\infty} u(x_1, x_2, x_3) = \pm 1$ uniformly in (x_2, x_3) for $n = 3$ and $a = a(x_1)$. For results on solutions to (1.9) for $n = 2$ with general potentials, see the papers by Rabinowitz and Stredulinsky [50,51]. Existence of various (multi-layer, mountain pass or higher topological complexity) solutions to (1.10) for general n was obtained in a series of papers by Byeon and Rabinowitz [16–19]. A review on existence results for (1.9) is given in [49] (see book [54] for a more thorough review on extensions of Moser-Bangert theory).

An extensive discussion on moving front solutions for time-dependent inhomogeneous Allen-Cahn equation can also be found in the literature. For example, Xin [60] considered propagating front solutions (which include stationary layered solutions) for

$$u_t = \nabla_x (a(x) \nabla_x u) + b(x) \cdot \nabla_x u + f(u)$$

when $a(x)$, $b(x)$ are periodic and $f(u)$ is bistable. Keener [41] studied propagation of waves in periodic media for the following model:

$$u_t = u_{xx} + \left(1 + g'\left(\frac{x}{L}\right)\right) f(u) - au$$

where $g(x)$ is a 1-periodic function and obtained a nearly complete picture of propagation in perodic medium. In particular, his results show how wave front shape changes when the medium becomes more and more nonuniform, and how propagation failure occurs when the medium becomes sufficiently nonuniform. Pinning and de-pinning phenomena for front propagation in heterogenous media was discussed in [33]. Existence and qualitative properties of pulsating traveling wave solutions is proved in [30] for equation

$$u_t = \left(a\left(\frac{x}{L}\right)u_x\right)_x + f(x,u).$$

where $a(x)$ and $f(x,u)$ are 1-periodic in x.

Studies of layered solution in the fractional case when $g(x)$ is not constant caught less attention. The only work we are aware of is [40] where existence of layered solutions to

$$\left(\frac{-d^2}{dx^2}\right)^s u = g(x)W'(u) \qquad x \in \mathbb{R}$$

was obtained for $s \geq \frac{1}{2}$ when $g > 0$ is an even, periodic function and $W'(u)$ is odd. Another related work is [34] where the authors studied existence of multi-layered solution to the following equation

$$\varepsilon^{2s}\left(-\frac{d^2}{dx^2}\right)^s u - g(x)u\left(1-u^2\right) = 0 \qquad x \in \mathbb{R},$$

when $g(x)$ is not constant and $s > \frac{1}{2}$.

The work in the current paper is partly motivated by a recent work by the authors [27] where we considered the following *renormalized* nonlocal Ginzburg-Landau energy

$$\begin{aligned} E_\varepsilon(u) &= \int_{\mathbb{R}} \varepsilon^2 |u'|^2 dx + \int_{\mathbb{R}} W(u)\,dx \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{(u(x)-u(y))^2}{(x-y)^2} - \frac{(\eta(x)-\eta(y))^2}{(x-y)^2}\right] dydx. \end{aligned} \tag{1.11}$$

We proved existence, regularity, monotonicity and uniqueness (up to translation) of the minimizer of $E_\varepsilon(u)$ in $\mathcal{A}$. Moreover, as $\varepsilon \to 0$ we recovered the solution in [48] as the global minimizer (unique up to translations) of

$$E_0(u) = \int_{\mathbb{R}} W(u)\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{(u(x)-u(y))^2}{(x-y)^2} - \frac{(\eta(x)-\eta(y))^2}{(x-y)^2}\right] dydx.$$

The proof of existence and uniqueness of minimizers in [27] relies on an essential observation that a minimizer of E_ε among all functions satisfying $u - \eta \in W_0^{1,2}(I)$ on any sufficiently large fixed interval I is monotone. Such conclusion follows from the key assumption that (1.11) is translation invariant. For model (1.1) with only discrete translation invariance, this argument fails and we need to seek a new method. The main difficulty to prove the existence of minimizer of (1.1) lies in two parts. Firstly, since $\eta \notin \mathring{H}^{\frac{1}{2}}(\mathbb{R})$, it is not a priori clear that $J(u)$ is bounded from below on $\mathcal{A}$. Secondly, the energy bound does not necessarily imply the boundedness of u in a suitable Sobolev space in general. Therefore, we cannot a priori apply the direct method of calculus of variations to obtain a minimizer. To show that J is bounded from below on $\mathcal{A}$, we divide the real line into the regions where u is close to ± 1 and where u is away from ± 1. By carefully matching the contributions from each region, all negative parts of the potential infinite energy are cancelled out.

To prove the existence of a minimizer, our main idea is as follows. Given an arbitrary minimizing sequence $\{u_n\}$, we replace this sequence by another sequence $\{\overline{u}_n\}$ constructed via reflecting the nagative parts of u_n outside suitable regions. Taking into account our energy estimates from the lower bound argument, we can carefully choose the region where we apply the reflection to u_n so that the energy $J(\overline{u}_n)$ differs only slightly from $J(u_n)$. The sequence $\{\overline{u}_n\}$ satisfies $|\overline{u}_n(x) + sgn(x)| \geq c > 0$ outside a uniformly bounded interval. For such a sequence, boundedness of energy implies boundedness of $\overline{u}_n - \eta$ in $H^1(\mathbb{R})$. From this and a lower semicontinuity argument, we obtain a limit function which attains a minimum of $J(u)$ in $\mathcal{A}$.

Our main result is the following existence and regularity theorem.

Theorem 1.1 *Let α, β be positive constants. Assume $g \in C^\infty(\mathbb{R})$ is a positive 1-periodic function, $\eta \in C^\infty(\mathbb{R})$ is a given function satisfying $|\eta| \leq 1$, $\eta(x) = 1$ for $x \geq 1$, $\eta(x) = -1$ for $x \leq -1$, $W(u)$ is a double well potential satisfying (1.2). Then there exists a minimizer u_0 of $J(u)$ over $\mathcal{A}$. Moreover, $u_0 \in C^{2,\frac{1}{2}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and satisfies the Euler-Lagrange equation*

$$-\alpha u_0'' + g(x) W'(u_0) + \beta \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_0 = 0,$$

and the condition at infinity

$$\lim_{x \to \pm\infty} u_0(x) = \pm 1.$$

Here the fractional operator $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}}$ is defined by (1.4).

We prove Theorem 1.1 in three steps. We first check that $J(u)$ is bounded from below. Let

$$F(u) := \int_{\mathbb{R}} g(x) W(u(x))\, dx + \frac{\beta}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{(u(x) - u(y))^2}{(x-y)^2} - \frac{(\eta(x) - \eta(y))^2}{(x-y)^2} \right] dy\, dx. \tag{1.12}$$

We show that F is bounded from below in section 2. In the second step, we construct a global minimizer of J in $\mathcal{A}$ in section 3. Regularity is treated in section 4 and follows from a bootstrap argument, since $u_0 = v_0 + \eta$ with $v_0 \in H^1(\mathbb{R})$. However, a priori it is not clear whether $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_0 \in L^2(\mathbb{R})$, and we handle this term separately when deriving the Euler-Lagrange equation.

2 Lower bound on $J(u)$

Let $J(u)$, $F(u)$ be defined by (1.1) and (1.12). We shall prove the following lower bound in this section.

Proposition 2.1 *There exists a constant $C > 0$ such that $F(u) > -C$ for any $u \in \mathcal{A}$.*

The lower bound on $J(u)$ follows directly from Proposition 2.1.

2.1 Overview

Recall that by our assumptions there exists $\gamma > 0$ such that $g(x) \geq \gamma$ for all $x \in \mathbb{R}$.

First we observe that replacing u by ± 1 whenever $|u| > 1$, the energy is only getting smaller. Here for the nonlocal term, a direct calculation shows $(u(x) - u(y))^2 \geq (\tilde{u}(x) - \tilde{u}(y))^2$ where $\tilde{u}(x) = \max\{\min(u(x), 1), -1\})$. Without loss of generality, we shall assume $|u(x)| \leq 1$ on $\mathbb{R}$ throughout the paper. Moreover, the following Lemma can be proved by the same argument as in the proof of Lemma 2.2 from [27].

Lemma 2.2 *Given $u \in \mathcal{A}$, there exists a sequence $u_n \in \mathcal{A}$ such that $u_n - \eta \in C_0^\infty(\mathbb{R})$ and $J(u_n) \to J(u)$ as $n \to \infty$.*

We introduce the following subset of $\mathcal{A}$:

$$\mathcal{A}_0 := \{u \in \mathcal{A} : |u(x)| \leq 1 \text{ on } \mathbb{R} \text{ and } u - \eta \text{ is compactly supported in } \mathbb{R}\}.$$

Letting

$$f(x,y) := \frac{\beta}{4\pi}\left[\frac{(u(x)-u(y))^2}{(x-y)^2} - \frac{(\eta(x)-\eta(y))^2}{(x-y)^2}\right],$$

we can write $F(u)$ as

$$\begin{aligned} F(u) &= \int_1^\infty g(x) W(u) dx + \int_{-\infty}^{-1} g(x) W(u) dx + \int_{-1}^{1} g(x) W(u) dx \\ &\quad + \int_1^\infty \int_1^\infty f(x,y)\, dydx + \int_{-\infty}^{-1}\int_{-\infty}^{-1} f(x,y)\, dydx \\ &\quad +2\int_1^\infty \int_{-\infty}^{-1} f(x,y)\, dydx + 2\int_1^\infty \int_{-1}^{1} f(x,y)\, dydx \\ &\quad +2\int_{-\infty}^{-1}\int_{-1}^{1} f(x,y)\, dydx + \int_{-1}^{1}\int_{-1}^{1} f(x,y)\, dydx. \end{aligned} \tag{2.1}$$

Direct calculation shows that the integrals

$$\int_1^\infty \int_{-1}^{1} \frac{(\eta(x)-\eta(y))^2}{(x-y)^2} dydx, \quad \int_{-\infty}^{-1}\int_{-1}^{1} \frac{(\eta(x)-\eta(y))^2}{(x-y)^2} dydx, \quad \int_{-1}^{1}\int_{-1}^{1} \frac{(\eta(x)-\eta(y))^2}{(x-y)^2} dydx$$

are all bounded. To show $F(u)$ is bounded from below, the question reduces to showing that

$$\begin{aligned} &\int_1^\infty \int_1^\infty f(x,y)\, dydx + \int_{-\infty}^{-1}\int_{-\infty}^{-1} f(x,y)\, dydx + 2\int_1^\infty \int_{-\infty}^{-1} f(x,y)\, dydx \\ &+\int_1^\infty g(x) W(u) dx + \int_{-\infty}^{-1} g(x) W(u) dx > -C, \end{aligned} \tag{2.2}$$

for some $C > 0$ independent of $u \in \mathcal{A}_0$. Since $\eta \notin \mathring{H}^{1/2}(\mathbb{R})$, the term $\int_1^\infty \int_{-\infty}^{-1} f(x,y)\, dydx$ could potentially be negative infinity. In particular, if we choose a sequence $u_n(x)$ which oscillates between 1 and -1 on intervals which get larger and larger, it is not clear that we can have a uniform lower bound on the left-hand side in (2.2). Our idea is the following: if $|u|$ stays away from 1 on a big portion of $\mathbb{R}$, the term

$$\int_1^\infty g(x) W(u)\, dx + \int_{-\infty}^{-1} g(x) W(u)\, dx$$

would dominate

$$\int_1^\infty \int_{-\infty}^{-1} f(x,y)\, dydx.$$

On the other hand, if $|u| \sim 1$ on $\mathbb{R}$ and u oscillates between 1 and -1, the sum

$$\int_1^\infty \int_1^\infty \frac{(u(x)-u(y))^2}{(x-y)^2} dydx + \int_{-\infty}^{-1}\int_{-\infty}^{-1} \frac{(u(x)-u(y))^2}{(x-y)^2} dydx + 2\int_1^\infty \int_{-\infty}^{-1} \frac{(u(x)-u(y))^2}{(x-y)^2} dydx$$

would approach infinity at the same order as

$$\int_1^\infty \int_{-\infty}^{-1} \frac{(\eta(x)-\eta(y))^2}{(x-y)^2} dydx,$$

thus eventually canceling out the potential negative infinite energy. In both cases, we obtain a finite lower bound on $F(u)$.

To explain our ideas more precisely, recall that $u-\eta \in H^1(\mathbb{R})$ and has compact support for every $u \in \mathcal{A}_0$. By Sobolev embedding theorem, $u-\eta$ and, therefore, u are continuous. Given any $\delta > 0$, we define the following decomposition of $(-\infty,-1] \cup [1,\infty)$ with respect to u:

$$\begin{aligned} I_\delta^+ &:= \{x \geq 1 : -1 \leq u(x) \leq -1+\delta\}, \\ II_\delta^+ &:= \{x \geq 1 : 1-\delta \leq u(x) \leq 1\}, \\ III_\delta^+ &:= \{x \geq 1 : -1+\delta < u(x) < 1-\delta\}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} I_\delta^- &:= \{x \leq -1 : -1 \leq u(x) \leq -1+\delta\}, \\ II_\delta^- &:= \{x \leq -1 : 1-\delta \leq u(x) \leq 1\}, \\ III_\delta^- &:= \{x \leq -1 : -1+\delta < u(x) < 1-\delta\}. \end{aligned} \tag{2.4}$$

Under these notations, we observe I_δ^+, II_δ^-, III_δ^+ and III_δ^- are all bounded sets. We show that there exists a constant $C = C(\delta,\gamma,\beta,W) > 0$ and independent of $u \in \mathcal{A}_0$ such that

$$\begin{aligned} &\int_1^\infty \int_{-\infty}^{-1} f(x,y)\,dydx + \frac{\gamma}{4}\int_1^\infty W(u)\,dx + \frac{\gamma}{4}\int_{-\infty}^{-1} W(u)\,dx \\ \geq\ & -\int_{I_\delta^+}\int_{I_\delta^-} \frac{\beta}{\pi(x-y)^2}dydx - \int_{II_\delta^+}\int_{II_\delta^-} \frac{\beta}{\pi(x-y)^2}dydx - C, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\int_1^\infty \int_1^\infty f(x,y)\,dydx + \int_{-\infty}^{-1}\int_{-\infty}^{-1} f(x,y)\,dydx + \frac{\gamma}{2}\int_1^\infty W(u)\,dx \\ &+\frac{\gamma}{2}\int_{-\infty}^{-1} W(u)\,dx - \int_{I_\delta^+}\int_{I_\delta^-} \frac{2\beta}{\pi(x-y)^2}dydx - \int_{II_\delta^+}\int_{II_\delta^-} \frac{2\beta}{\pi(x-y)^2}dydx \\ >\ & -C. \end{aligned} \tag{2.6}$$

Throughout the paper, we will use C to represent a generic constant independent of $u \in \mathcal{A}_0$, and depending only on δ, γ, β and W, which might change from line to line. A lower bound in (2.2) follows from (2.5) and (2.6).

Since

$$\begin{aligned} &\int_1^\infty \int_1^\infty f(x,y)\,dydx + \int_{-\infty}^{-1}\int_{-\infty}^{-1} f(x,y)\,dydx \\ \geq\ & \frac{\beta}{2\pi}\int_{I_\delta^+}\int_{II_\delta^+} \frac{(u(x)-u(y))^2}{(x-y)^2}dydx + \frac{\beta}{2\pi}\int_{I_\delta^-}\int_{II_\delta^-} \frac{(u(x)-u(y))^2}{(x-y)^2}dydx, \end{aligned}$$

the proof of (2.6) reduces to the following main technical inequalities:

$$\begin{aligned} J_\delta^+(u) &:= \frac{\beta}{4\pi}\int_{I_\delta^+}\int_{II_\delta^+} \frac{(u(x)-u(y))^2}{(x-y)^2}dydx - \int_{I_\delta^+}\int_{I_\delta^-} \frac{\beta}{\pi(x-y)^2}dydx + \frac{\gamma}{4}\int_1^\infty W(u)\,dx \\ &> -C, \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
J_{\delta}^{-}(u) &:= \frac{\beta}{4\pi}\int_{I_{\delta}^{-}}\int_{II_{\delta}^{-}}\frac{(u(x)-u(y))^{2}}{(x-y)^{2}}dydx-\int_{II_{\delta}^{+}}\int_{II_{\delta}^{-}}\frac{\beta}{\pi(x-y)^{2}}dydx+\frac{\gamma}{4}\int_{-\infty}^{-1}W(u)\,dx \\
&> -C. \qquad (2.8)
\end{aligned}
$$

The proof of (2.7) and (2.8) uses a contradiction argument. We prove one bound, the other one can be proved similarly. Assume $J_{\delta}^{+}(u_n)\to-\infty$ for some sequence (u_n). Representing the decomposition of $(-\infty,-1]\cup[1,\infty)$ with respect to u_n by adding index n in (2.3) and (2.4), decompose $I_{\delta,n}^{+}$, $I_{\delta,n}^{-}$ and $II_{\delta,n}^{+}$ into union of disjoint intervals. We can estimate

$$
\int_{I_{\delta,n}^{+}}\int_{II_{\delta,n}^{+}}\frac{(u_n(x)-u_n(y))^{2}}{(x-y)^{2}}dydx \qquad (2.9)
$$

and

$$
\int_{I_{\delta,n}^{+}}\int_{I_{\delta,n}^{-}}\frac{4}{(x-y)^{2}}dydx \qquad (2.10)
$$

in terms of summation of integral over those intervals. In particular, $J_{\delta}^{+}(u_n)\to-\infty$ implies $I_{\delta,n}^{+}\subset[1,R_n]$ where $R_n\to\infty$ and (2.10) goes to infinity at most logarithmically in R_n. If $|u_n|$ is bounded away from 1 on a large portion of $[1,\infty)$, then the term $\int_1^\infty W(u_n)$ dominates (2.10). If $u_n\approx-1$ on a large portion of $[1,\infty)$, then (2.9) would approach infinity at the same logarithmic order as (2.10). In either case, we can always conclude that $J_{\delta}^{+}(u_n)$ is bounded from below, a contradiction.

2.2 $F(u)$ is bounded from below on $\mathcal{A}_0$

We prove Proposition 2.1 in several steps.

2.2.1 Preliminaries

We first state the following lemma.

Lemma 2.3 *Given $u\in\mathcal{A}_0$, , the following bounds hold:*

$$
\int_{-1}^{1}g(x)W(u)\,dx\le C,\qquad \int_{-1}^{1}\int_{-1}^{1}\frac{(\eta(x)-\eta(y))^{2}}{(x-y)^{2}}dydx\le 4\|\eta'\|_{\infty}^{2} \qquad (2.11)
$$

$$
\int_{1}^{\infty}\int_{-1}^{1}\frac{(\eta(x)-\eta(y))^{2}}{(x-y)^{2}}dydx\le 2\|\eta'\|_{\infty}^{2},\qquad \int_{-\infty}^{-1}\int_{-1}^{1}\frac{(\eta(x)-\eta(y))^{2}}{(x-y)^{2}}dydx\le 2\|\eta'\|_{\infty}^{2} \qquad (2.12)
$$

Proof. The bounds in (2.11) are straightforward. By the definition of η, we have

$$
\int_{1}^{\infty}\int_{-1}^{1}\frac{(\eta(x)-\eta(y))^{2}}{(x-y)^{2}}dydx=\int_{-1}^{1}\frac{(\eta(x)-1)^{2}}{1-x}dx\le 2\|\eta'\|_{\infty}^{2}.
$$

The second inequality in (2.12) follows from a similar argument. ■

Lemma 2.3 implies that the terms involving integration on $[-1,1]$ in (2.1) are all bounded from below. The boundedness of $F(u)$ from below would then follow from the following lemma.

Lemma 2.4 *There exists a constant $C=C(\delta,\gamma,\beta,W)>0$ such that for all $u\in\mathcal{A}_0$, the following lower bound holds:*

$$
\begin{aligned}
&\int_{1}^{\infty}\int_{1}^{\infty}f(x,y)\,dydx+\int_{-\infty}^{-1}\int_{-\infty}^{-1}f(x,y)\,dydx+2\int_{1}^{\infty}\int_{-\infty}^{-1}f(x,y)\,dydx \\
&+\int_{1}^{\infty}g(x)W(u)\,dx+\int_{-\infty}^{-1}g(x)W(u)\,dx>-C.
\end{aligned}
$$

Lemma 2.4 is proved in two steps. Under decompositions (2.3) and (2.4), we can write

$$\begin{aligned}
&\int_1^\infty \int_{-\infty}^{-1} f(x,y)\,dydx \\
=\ &\int_{I_\delta^+}\int_{I_\delta^-} f(x,y)\,dydx + \int_{I_\delta^+}\int_{II_\delta^-} f(x,y)\,dydx + \int_{I_\delta^+}\int_{III_\delta^-} f(x,y)\,dydx \\
&+ \int_{II_\delta^+}\int_{I_\delta^-} f(x,y)\,dydx + \int_{II_\delta^+}\int_{II_\delta^-} f(x,y)\,dydx + \int_{II_\delta^+}\int_{III_\delta^-} f(x,y)\,dydx \\
&+ \int_{III_\delta^+}\int_{I_\delta^-} f(x,y)\,dydx + \int_{III_\delta^+}\int_{II_\delta^-} f(x,y)\,dydx + \int_{III_\delta^+}\int_{III_\delta^-} f(x,y)\,dydx. \qquad (2.13)
\end{aligned}$$

The following lower bound will be used in the proof of Lemma 2.4.

Lemma 2.5 *Let $A \subset [1,\infty)$ and $B \subset (-\infty,-1]$. Assume either A or B is bounded, then there exists a constant $C = C(\beta) > 0$ such that for all $u \in \mathcal{A}_0$, the following bounds hold:*

$$\int_A\int_B f(x,y)\,dydx \geq -\varepsilon\int_A (1-u)^2 - \varepsilon\int_B (1+u)^2 - \frac{C}{\varepsilon}, \qquad (2.14)$$

and

$$\int_A\int_B f(x,y)\,dydx \geq -\varepsilon\int_A (1+u)^2 - \varepsilon\int_B (1-u)^2 - \frac{C}{\varepsilon} \qquad (2.15)$$

for any $\varepsilon > 0$.

Proof. We only show how to obtain (2.14), as the other inequality follows by a similar argument. Since $(\eta(x)-\eta(y))^2 = 4$ for $x \in A$ and $y \in B$, we have

$$f(x,y) = \frac{\beta}{4\pi}\left[\frac{(u(x)-u(y))^2-4}{(x-y)^2}\right] = \frac{\beta}{4\pi}\left[\frac{(u(x)-u(y)-2)^2+4(u(x)-u(y)-2)}{(x-y)^2}\right].$$

Hence we deduce

$$\begin{aligned}
&\int_A\int_B f(x,y)\,dydx \\
=\ &\frac{\beta}{4\pi}\left[\int_A\int_B \frac{(u(x)-u(y)-2)^2}{(x-y)^2}dydx - 4\int_A\int_B \frac{(1-u(x))}{(x-y)^2}dydx - 4\int_A\int_B \frac{(1+u(y))}{(x-y)^2}dydx\right] \\
\geq\ &\frac{\beta}{4\pi}\left[\int_A\int_B \frac{(u(x)-u(y)-2)^2}{(x-y)^2}dydx - 4\int_A \frac{1-u(x)}{x+1}dx - 4\int_B \frac{1+u(y)}{1-y}dy\right] \\
\geq\ &\frac{\beta}{4\pi}\int_A\int_B \frac{(u(x)-u(y)-2)^2}{(x-y)^2}dydx - \varepsilon\int_A (1-u)^2 - \varepsilon\int_B (1+u)^2 - \frac{C}{\varepsilon},
\end{aligned}$$

where we applied Hölder's inequality in the last line. Notice that when either A or B is bounded, the integral $\int_A\int_B \frac{(u(x)-u(y)-2)^2}{(x-y)^2}dydx$ is finite for any $u \in \mathcal{A}_0$, justifying the spliting of the integrals in the calculation above. ■

The first step to prove Lemma 2.4 is the following Lemma.

Lemma 2.6 *For any $0 < \delta \ll 1$, there exists a constant $C = C(\delta,\gamma,\beta,W)$ such that for all $u \in \mathcal{A}_0$*

$$\begin{aligned}
&\int_1^\infty\int_{-\infty}^{-1} f(x,y)\,dydx + \frac{\gamma}{4}\int_1^\infty W(u)dx + \frac{\gamma}{4}\int_{-\infty}^{-1} W(u)\,dx \\
\geq\ &-\int_{I_\delta^+}\int_{I_\delta^-}\frac{\beta}{\pi(x-y)^2}dydx - \int_{II_\delta^+}\int_{II_\delta^-}\frac{\beta}{\pi(x-y)^2}dydx - C
\end{aligned}$$

Proof. Recall that I_δ^+, II_δ^-, III_δ^+ and III_δ^+ are all bounded sets. By estimate (2.14) in Lemma 2.5, we have

$$\int_{II_\delta^+}\int_{I_\delta^-} f(x,y)\,dydx \geq -\varepsilon\int_{II_\delta^+}(1-u)^2\,dx - \varepsilon\int_{I_\delta^-}(1+u)^2 - \frac{C}{\varepsilon}, \tag{2.16}$$

$$\int_{II_\delta^+}\int_{III_\delta^-} f(x,y)\,dydx \geq -\varepsilon\int_{II_\delta^+}(1-u)^2 - \varepsilon\int_{III_\delta^-}(1+u)^2 - \frac{C}{\varepsilon}, \tag{2.17}$$

$$\int_{III_\delta^+}\int_{I_\delta^-} f(x,y)\,dydx \geq -\varepsilon\int_{III_\delta^+}(1-u)^2\,dx - \varepsilon\int_{I_\delta^-}(1+u)^2 - \frac{C}{\varepsilon}, \tag{2.18}$$

and

$$\int_{III_\delta^+}\int_{III_\delta^-} f(x,y)\,dydx \geq -\varepsilon\int_{III_\delta^+}(1-u)^2 - \varepsilon\int_{III_\delta^-}(1+u)^2 - \frac{C}{\varepsilon}. \tag{2.19}$$

Here for (2.16) we used the fact that the integral $\int_{II_\delta^+}\int_{I_\delta^-} f(x,y)\,dydx$ can be written as a sum of integrals of the form $\int_A\int_B f(x,y)\,dydx$, where either A or B is bounded when $u\in\mathcal{A}_0$.

By estimate (2.15) from Lemma 2.5, we have

$$\int_{I_\delta^+}\int_{II_\delta^-} f(x,y)\,dydx \geq -\varepsilon\int_{I_\delta^+}(1+u)^2 - \varepsilon\int_{II_\delta^-}(1-u)^2 - \frac{C}{\varepsilon}, \tag{2.20}$$

$$\int_{I_\delta^+}\int_{III_\delta^-} f(x,y)\,dydx \geq -\varepsilon\int_{I_\delta^+}(1+u)^2 - \varepsilon\int_{III_\delta^-}(1-u)^2 - \frac{C}{\varepsilon}, \tag{2.21}$$

and

$$\int_{III_\delta^+}\int_{II_\delta^-} f(x,y)\,dydx \geq -\varepsilon\int_{III_\delta^+}(1+u)^2 - \varepsilon\int_{II_\delta^-}(1-u)^2 - \frac{C}{\varepsilon}. \tag{2.22}$$

Summing up (2.16) – (2.22) yields

$$\begin{aligned}
&\int_1^\infty\int_{-\infty}^{-1} f(x,y)\,dydx + \frac{\gamma}{4}\int_1^\infty W(u)\,dx + \frac{\gamma}{4}\int_{-\infty}^{-1} W(u)\,dy \\
\geq\ & -\int_{I_\delta^+}\int_{I_\delta^-}\frac{\beta}{\pi(x-y)^2}dydx - \int_{II_\delta^+}\int_{II_\delta^-}\frac{\beta}{\pi(x-y)^2}dydx - \frac{C}{\varepsilon} \\
& -2\varepsilon\int_{I_\delta^+\cup I_\delta^-\cup III_\delta^-}(1+u)^2 - 2\varepsilon\int_{II_\delta^+\cup II_\delta^-\cup III_\delta^+}(1-u)^2 - \varepsilon\int_{III_\delta^-}(1-u)^2\,dy - \varepsilon\int_{III_\delta^+}(1+u)^2\,dx \\
& +\frac{\gamma}{4}\int_{I_\delta^+\cup I_\delta^-} W(u)\,dx + \frac{\gamma}{4}\int_{II_\delta^+\cup II_\delta^-} W(u)\,dx + \frac{\gamma}{4}\int_{III_\delta^+\cup III_\delta^-} W(u)\,dx.
\end{aligned}$$

Recall that $W(\pm1) = W'(\pm1) = 0$, $W(u) > 0$ for $|u| < 1$ and $W''(\pm1) > 0$. Picking $\delta \ll 1$, we have the following estimates

$$W(u(x)) = \frac{1}{2}W''(-(1-\theta(x)) + \theta(x)u(x))(1+u)^2 \geq \frac{1}{4}W''(-1)(1+u)^2 \text{ for } x\in I_\delta^+\cup I_\delta^-, \tag{2.23}$$

$$W(u(x)) = \frac{1}{2}W''((1-\theta(x)) + \theta(x)u(x))(1-u)^2 \geq \frac{1}{4}W''(1)(1-u)^2 \text{ for } x\in II_\delta^+\cup II_\delta^-, \tag{2.24}$$

Since

$$W(u(x)) \geq \min_{|u|\leq 1-\delta} W(u) \text{ for } x\in III_\delta^+\cup III_\delta^-,$$

the conclusion follows by taking $\varepsilon = \min\left(\frac{\gamma}{48}\min_{|u|\leq1-\delta} W(u), \frac{\gamma}{32}W''(-1), \frac{\gamma}{32}W''(1)\right)$. ■

The second step to prove Lemma 2.4 is the following Lemma.

Lemma 2.7 *There exists a constant $C = C(\delta, \gamma, \beta, W) > 0$ such that for all $u \in \mathcal{A}_0$ we have*

$$\begin{aligned}
& \int_1^\infty \int_1^\infty + \int_{-\infty}^{-1} \int_{-\infty}^{-1} f(x,y)\,dydx + \frac{\gamma}{2}\int_1^\infty W(u)\,dx + \frac{\gamma}{2}\int_{-\infty}^{-1} W(u)\,dx \\
& - \int_{I_\delta^+} \int_{I_\delta^-} \frac{2\beta}{\pi(x-y)^2}dydx - \int_{II_\delta^+} \int_{II_\delta^-} \frac{2\beta}{\pi(x-y)^2}dydx \\
> \quad & -C.
\end{aligned}$$

Lemma 2.6 and Lemma 2.7 imply Lemma 2.4.

2.2.2 Proof of Lemma 2.7

Decompositions and some basic estimates The proof of Lemma 2.7 is rather long and technical. First we decompose each set into intervals. By our assumption, for any given $\delta \in (0,1)$, there exists $R_1(u) > 0$ and $R_2(u) > 0$ such that

$$u(x) = 1 \text{ for all } x \geq R_1(u) \tag{2.25}$$

and

$$u(x) = -1 \text{ for all } x \leq -R_2(u). \tag{2.26}$$

It follows that III_δ^+ and III_δ^- are open subsets of $(1, R_1(u))$ and $(-R_2(u), -1)$ respectively. By the structure theorem of open sets in $\mathbb{R}$ and choosing u to be the continuous representative, there exist indices $N^\pm$, and positive numbers $\alpha_i^\pm$, $\beta_i^\pm$ such that we can write III_δ^+ and III_δ^- as unions of disjoint open intervals in the following form.

$$III_\delta^+ \cap (1, R_1(u)) = \bigcup_{i=1}^{N^+} \left(\alpha_i^+, \beta_i^+\right), \quad III_\delta^- \cap (-R_2(u), -1) = \bigcup_{j=1}^{N^-} \left(-\beta_j^-, -\alpha_j^-\right). \tag{2.27}$$

Without loss of generality, we can assume $N^\pm$ are finite and $\alpha_i < \beta_i < \alpha_{i+1} < \beta_{i+1}$ for all i. In fact, recall that $u - \eta$ in $H_0^1(\mathbb{R})$, write $R = \max(R_1(u), R_2(u))$, we can obtain an approximation u_1 in $C^\infty(\mathbb{R})$ that is equal to η for $|x| > R + 1$ and arbitrarily close to u in $H^1(\mathbb{R})$. Taking a linear interpolant u_2 of u_1 over a sufficiently fine partition X of $[-1-R, 1+R]$, we get a function that is arbitrarily close to u_1 in $W^{1,\infty}(\mathbb{R})$. Finally, shifting the values of the function u_2 at the (finitely many) points of X by arbitrarily small amounts if necessary, we get a function u_3 that is arbitrarily close to u_2 in $W^{1,\infty}(\mathbb{R})$ and u_3' is non-zero a.e. in $\mathbb{R}$. Hence on every interval of the partition X there is at most one point at which $|u_3| = 1 - \delta$. From this, we conclude that each interval of the partition X intersecst III_δ^+ (or III_δ^-) at most once. Relabling if necessary, we thus find finitely many disjoint intervals (α_k, β_k) (merge $(\alpha_k, \beta_k) \cup (\alpha_{k+1}, \beta_{k+1})$ into (α_k, β_{k+1}) if $\beta_k = \alpha_{k+1}$) where

$$-1 + \delta < u(x) < 1 - \delta \text{ on each } (\alpha_k, \beta_k).$$

Renaming our endpoints we find disjoint intervals $[a_i, b_i]$, $[c_j, d_j] \subset [1, R_1(u)]$ and indices K, L such that

$$I_\delta^+ = \bigcup_{i=1}^{K} [a_i, b_i] \tag{2.28}$$

and

$$II_\delta^+ = \bigcup_{j=1}^{L} [c_j, d_j] \tag{2.29}$$

with

$$-1 \leq u(x) \leq -1 + \delta \text{ on } I_\delta^+,$$

$$1-\delta \leq u(x) \leq 1 \text{ on } II_\delta^+,$$

and

$$-1+\delta < u(x) < 1-\delta \quad \text{on} \quad [1, R_1(u)] \setminus \left(I_\delta^+ \cup II_\delta^+\right) = III_\delta^+.$$

Here by a slight abuse of notation, we denote $[c_l, d_l] = [c_l, \infty)$, and if $a_1 = 1$ we replace $[a_1, b_1]$ by $(1, b_1]$ in (2.28), or if $c_1 = 1$, replace $[c_1, d_1]$ by $(1, d_1]$ in (2.29). Similarly we write II_δ^- and I_δ^- as unions of disjoint intervals. For the rest of the paper, we write

$$I_\delta^+(u) = \cup_{i=1}^K [a_i, b_i], \; II_\delta^+(u) = \cup_{j=1}^L [c_j, d_j]; \tag{2.30}$$

$$I_\delta^-(u) = \cup_{i=1}^{\widetilde{K}} \left[-\widetilde{b}_i, -\widetilde{a}_i\right], \; II_\delta^-(u) = \cup_{j=1}^{\widetilde{L}} \left[-\widetilde{d}_j, -\widetilde{c}_j\right]. \tag{2.31}$$

with the understanding that $\left[-\widetilde{d}_{\widetilde{L}}, -\widetilde{c}_{\widetilde{L}}\right] = \left(-\infty, -\widetilde{c}_{\widetilde{L}}\right]$ and $\left[-\widetilde{b}_1, -\widetilde{a}_1\right] = \left[-\widetilde{b}_1, -1\right)$ if $\widetilde{a}_1 = 1$, or $\left[-\widetilde{d}_1, -\widetilde{c}_1\right] = \left[-\widetilde{d}_1, -1\right)$ if $\widetilde{c}_1 = 1$. Note that in this form, all $a_i, b_i,\ c_j, d_j,\ \widetilde{a}_i, \widetilde{b}_i,\ \widetilde{c}_j, \widetilde{d}_j$ are greater or equal to 1.

We first state some basic estimates.

Lemma 2.8 *The following estimates hold:*

$$\int_{I_\delta^+} \int_{I_\delta^-} \frac{1}{(x-y)^2} dy dx = \ln\left(\prod_{i=1}^K \prod_{j=1}^{\widetilde{L}} \frac{a_i + \widetilde{d}_j}{b_i + \widetilde{d}_j} \cdot \frac{b_i + \widetilde{c}_j}{a_i + \widetilde{c}_j}\right), \tag{2.32}$$

$$\int_{I_\delta^+} \int_{II_\delta^+} \frac{1}{(x-y)^2} dy dx = \ln\left(\prod_{i=1}^K \prod_{j=1}^{L} \frac{b_i - d_j}{a_i - d_j} \cdot \frac{a_i - c_j}{b_i - c_j}\right) \tag{2.33}$$

$$\int_{I_\delta^-} \int_{II_\delta^-} \frac{1}{(x-y)^2} dy dx = \ln\left(\prod_{i=1}^{\widetilde{K}} \prod_{j=1}^{\widetilde{L}} \frac{\widetilde{b}_i - \widetilde{d}_j}{\widetilde{a}_i - \widetilde{d}_j} \cdot \frac{\widetilde{a}_i - \widetilde{c}_j}{\widetilde{b}_i - \widetilde{c}_j}\right) \tag{2.34}$$

$$\int_{II_\delta^+} \int_{II_\delta^-} \frac{1}{(x-y)^2} dy dx = \ln\left(\prod_{i=1}^{\widetilde{K}} \prod_{j=1}^{L} \frac{\widetilde{a}_i + d_j}{\widetilde{b}_i + d_j} \cdot \frac{\widetilde{b}_i + c_j}{\widetilde{a}_i + c_j}\right), \tag{2.35}$$

Proof. By (2.30) and (2.31), we have

$$\begin{aligned}
\int_{I_\delta^+} \int_{I_\delta^-} \frac{1}{(x-y)^2} dy dx &= \sum_{i=1}^K \sum_{j=1}^{\widetilde{L}} \int_{a_i}^{b_i} \int_{-\widetilde{d}_j}^{-\widetilde{c}_j} \frac{1}{(x-y)^2} dy dx = \sum_{i=1}^K \sum_{j=1}^{\widetilde{L}} \int_{a_i}^{b_i} \left(\frac{1}{x+\widetilde{c}_j} - \frac{1}{x+\widetilde{d}_j}\right) dx \\
&= \sum_{i=1}^K \sum_{j=1}^{\widetilde{L}} \left(\ln \frac{b_i + \widetilde{c}_j}{a_i + \widetilde{c}_j} - \ln \frac{b_i + \widetilde{d}_j}{a_i + \widetilde{d}_j}\right) = \ln\left(\prod_{i=1}^K \prod_{j=1}^{\widetilde{L}} \frac{a_i + \widetilde{d}_j}{b_i + \widetilde{d}_j} \cdot \frac{b_i + \widetilde{c}_j}{a_i + \widetilde{c}_j}\right).
\end{aligned}$$

(2.33), (2.34) and (2.35) are proved similarly. ■

An immediate corollary of Lemmas 2.8 is the following.

Corollary 2.9 *We have the following bounds.*

$$\int_{I_\delta^+} \int_{I_\delta^-} \frac{1}{(x-y)^2} dy dx < \begin{cases} \ln 2 & \text{if } \widetilde{c}_1 > b_K \\ \ln 2 \frac{b_K}{a_1} & \text{if } \widetilde{c}_1 < b_K \end{cases}, \tag{2.36}$$

$$\int_{II_\delta^+} \int_{II_\delta^-} \frac{1}{(x-y)^2} dy dx < \begin{cases} \ln 2 & \text{if } c_1 > \widetilde{b}_{\widetilde{K}} \\ \ln 2 \frac{\widetilde{b}_{\widetilde{K}}}{\widetilde{a}_1} & \text{if } c_1 < \widetilde{b}_{\widetilde{K}} \end{cases}. \tag{2.37}$$

Proof.

$$\begin{aligned}\int_{I_\delta^+}\int_{I_\delta^-}\frac{1}{(x-y)^2}dydx &= \sum_{i=1}^{K}\sum_{j=1}^{\widetilde{L}}\int_{a_i}^{b_i}\int_{-\widetilde{d}_j}^{-\widetilde{c}_j}\frac{1}{(x-y)^2}dydx \\ &\le \int_{a_1}^{b_K}\int_{-d_{\widetilde{L}}}^{-\widetilde{c}_1}\frac{1}{(x-y)^2}dydx \\ &= \ln\left(\frac{b_K+\widetilde{c}_1}{a_1+\widetilde{c}_1}\cdot\frac{a_1+d_{\widetilde{L}}}{b_K+d_{\widetilde{L}}}\right) \\ &\le \ln\left(\frac{b_K+\widetilde{c}_1}{a_1+\widetilde{c}_1}\right)\end{aligned}$$

If $\widetilde{c}_1 > b_K$,

$$\frac{\widetilde{c}_1+b_K}{\widetilde{c}_1+a_1} < 2,$$

if $\widetilde{c}_1 < b_K$,

$$\frac{\widetilde{c}_1+b_K}{\widetilde{c}_1+a_1} < \frac{2b_K}{a_1},$$

so (2.36) follows. The estimate (2.37) follows from a similar argument. ■

Assume $I_\delta^\pm, II_\delta^\pm$ are written as unions of intervals in the form (2.30) and (2.31). Let $\cup_{i=1}^{n}[a_i,b_i]\subset I_\delta^+$ and $\cup_{j=1}^{m}[c_j,d_j]\subset II_\delta^+$, $\cup_{i=1}^{\widetilde{n}}\left[-\widetilde{b}_i,-\widetilde{a}_i\right]\subset I_\delta^-$ and $\cup_{j=1}^{\widetilde{m}}\left[-\widetilde{d}_j,-\widetilde{c}_j\right]\subset II_\delta^-$. If $d_j=\infty$, we write $[c_j,d_j]=[c_j,\infty)$ and $\left[-\widetilde{d}_j,-\widetilde{c}_j\right]=(-\infty,\widetilde{c}_j]$ if $\widetilde{d}_j=\infty$. Assume also

$$0<a_1<b_1<a_2<\cdots<b_{n-1}<a_n<b_n<c_1<d_1<c_2<\cdots<d_m$$

and

$$0<\widetilde{a}_1<\widetilde{b}_1<\widetilde{a}_2<\cdots<\widetilde{a}_{\widetilde{n}}<\widetilde{b}_{\widetilde{n}}<\widetilde{c}_1<\widetilde{d}_1<\widetilde{c}_2<\cdots<\widetilde{d}_{\widetilde{m}}.$$

We have the following estimates.

Lemma 2.10 *Assume $|c_1-b_n|\ge 1$, $\left|\widetilde{c}_1-\widetilde{b}_{\widetilde{n}}\right|\ge 1$ for all i and j. For $\delta\ll 1$, there exists $C=C(\delta,\gamma,\beta,W)>0$ such that the following estimates hold for any $u\in\mathcal{A}_0$.*

$$\begin{aligned}&\frac{\beta}{4\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{a_i}^{b_i}\int_{c_j}^{d_j}\frac{(u(x)-u(y))^2}{(x-y)^2}-\int_{I_\delta^+}\int_{I_\delta^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u)\,dx \qquad (2.38)\\ \ge\ &\frac{\beta}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\ln\frac{c_j-a_i}{c_j-b_i}\cdot\frac{d_j-b_i}{d_j-a_i}-\frac{\beta}{\pi}\ln\frac{2b_K}{a_1}+\frac{\gamma\min_{|u|\le 1-\delta}W(u)}{4}\left|III_\delta^+\right|-C.\end{aligned}$$

$$\begin{aligned}&\frac{\beta}{4\pi}\sum_{i=1}^{\widetilde{n}}\sum_{j=1}^{\widetilde{m}}\int_{-\widetilde{b}_i}^{-\widetilde{a}_i}\int_{-\widetilde{d}_j}^{-\widetilde{c}_j}\frac{(u(x)-u(y))^2}{(x-y)^2}-\int_{II_\delta^+}\int_{II_\delta^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_{-\infty}^{-1}W(u)\,dx \qquad (2.39)\\ \ge\ &\frac{\beta}{\pi}\sum_{i=1}^{\widetilde{n}}\sum_{j=1}^{\widetilde{m}}\ln\frac{\widetilde{c}_j-\widetilde{b}_i}{\widetilde{c}_j-\widetilde{a}_i}\cdot\frac{\widetilde{d}_j-\widetilde{a}_i}{\widetilde{d}_j-\widetilde{b}_i}-\frac{\beta}{\pi}\ln\frac{2\widetilde{b}_{\widetilde{K}}}{\widetilde{a}_1}+\frac{\gamma\min_{|u|\le 1-\delta}W(u)}{4}\left|III_\delta^-\right|-C.\end{aligned}$$

Proof. We prove (2.38), (2.39) follows from a similar argument. By Corollary 2.9 and $(2.23-2.24)$,

$$
\begin{aligned}
& \frac{\beta}{4\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{a_i}^{b_i}\int_{c_j}^{d_j}\frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left(x-y\right)^2}dydx-\int_{I_\delta^+}\int_{I_\delta^-}\frac{\beta}{\pi\left(x-y\right)^2}dydx+\frac{\gamma}{4}\int_1^{\infty}W\left(u\right)dx \\
= \; & \frac{\beta}{4\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{a_i}^{b_i}\int_{c_j}^{d_j}\frac{\left(u\left(x\right)+1+1-u\left(y\right)-2\right)^2-4}{\left(x-y\right)^2}dydx \\
& +\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{a_i}^{b_i}\int_{c_j}^{d_j}\frac{\beta}{\pi\left(x-y\right)^2}dydx-\int_{I_\delta^+}\int_{I_\delta^-}\frac{\beta}{\pi\left(x-y\right)^2}dydx+\frac{\gamma}{4}\int_1^{\infty}W\left(u\right)dx \\
\geq \; & -\frac{\beta}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{a_i}^{b_i}\int_{c_j}^{d_j}\frac{\left(u\left(x\right)+1\right)}{\left(x-y\right)^2}dydx-\frac{\beta}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{a_i}^{b_i}\int_{c_j}^{d_j}\frac{\left(1-u\left(y\right)\right)}{\left(x-y\right)^2}dydx \\
& +\frac{\beta}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\ln\left(\frac{c_j-a_i}{c_j-b_i}\cdot\frac{d_j-b_i}{d_j-a_i}\right)-\frac{\beta}{\pi}\ln\left(2\frac{b_K}{a_1}\right)+\frac{\gamma\min_{|u|\leq 1-\delta}W\left(u\right)}{4}\left|III_\delta^+\right|+\frac{\gamma}{4}\int_{I_\delta^+\cup II_\delta^+}W\left(u\right)dx \\
\geq \; & -\frac{\beta}{\pi}\sum_{i=1}^{n}\int_{a_i}^{b_i}\left(u\left(x\right)+1\right)\sum_{j=1}^{m}\left(\frac{1}{c_j-x}-\frac{1}{d_j-x}\right)dx-\frac{\beta}{\pi}\sum_{j=1}^{m}\int_{c_j}^{d_j}\left(1-u\left(y\right)\right)\sum_{i=1}^{n}\left(\frac{1}{y-a_i}-\frac{1}{y-b_i}\right)dy \\
& +\frac{\beta}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\ln\left(\frac{c_j-a_i}{c_j-b_i}\cdot\frac{d_j-b_i}{d_j-a_i}\right)-\frac{\beta}{\pi}\ln\left(2\frac{b_K}{a_1}\right)+\frac{\gamma\min_{|u|\leq 1-\delta}W\left(u\right)}{4}\left|III_\delta^+\right| \\
& +\frac{\gamma W''\left(-1\right)}{16}\int_{I_\delta^+}\left(1+u\right)^2dx+\frac{\gamma W''(1)}{16}\int_{II_\delta^+}\left(1-u\right)^2dx \\
\geq \; & -\frac{\gamma W''\left(-1\right)}{16}\sum_{i=1}^{n}\int_{a_i}^{b_i}\left(u\left(x\right)+1\right)^2dx-\frac{4\beta^2}{\pi^2\gamma W''\left(-1\right)}\sum_{i=1}^{n}\int_{a_i}^{b_i}\frac{1}{\left(c_1-x\right)^2}dx \\
& -\frac{\gamma W''(1)}{16}\sum_{j=1}^{m}\int_{c_j}^{d_j}\left(1-u\left(y\right)\right)^2dy-\frac{4\beta^2}{\pi^2\gamma W''(1)}\sum_{j=1}^{m}\int_{c_j}^{d_j}\frac{1}{\left(y-a_1\right)^2}dy \\
& +\frac{\beta}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\ln\left(\frac{c_j-a_i}{c_j-b_i}\cdot\frac{d_j-b_i}{d_j-a_i}\right)-\frac{\beta}{\pi}\ln\left(2\frac{b_K}{a_1}\right)+\frac{\gamma\min_{|u|\leq 1-\delta}W\left(u\right)}{4}\left|III_\delta^+\right| \\
& +\frac{\gamma W''\left(-1\right)}{16}\int_{I_\delta^+}\left(1+u\right)^2+\frac{\gamma W''(1)}{16}\int_{II_\delta^+}\left(1-u\right)^2 \\
\geq \; & \frac{\beta}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{m}\ln\left(\frac{c_j-a_i}{c_j-b_i}\cdot\frac{d_j-b_i}{d_j-a_i}\right)-\frac{\beta}{\pi}\ln\left(2\frac{b_K}{a_1}\right)+\frac{\gamma\min_{|u|\leq 1-\delta}W\left(u\right)}{4}\left|III_\delta^+\right|-C
\end{aligned}
$$

The last two steps follow from

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{a_i}^{b_i}(u(x)+1)\sum_{j=1}^{m}\left(\frac{1}{c_j-x}-\frac{1}{d_j-x}\right)dx\\
\le\ & \frac{\epsilon}{2}\sum_{i=1}^{n}\int_{a_i}^{b_i}(u(x)+1)^2\,dx+\frac{1}{2\epsilon}\sum_{i=1}^{n}\int_{a_i}^{b_i}\left(\sum_{j=1}^{m}\left(\frac{1}{c_j-x}-\frac{1}{d_j-x}\right)\right)^2 dx\\
\le\ & \frac{\epsilon}{2}\int_{I_\delta^+}(u(x)+1)^2+\frac{1}{2\epsilon}\sum_{i=1}^{n}\int_{a_i}^{b_i}\left(\frac{1}{c_1-x}\right)^2 dx\\
\le\ & \frac{\epsilon}{2}\int_{I_\delta^+}(u(x)+1)^2+\frac{1}{2\epsilon}\int_{a_1}^{b_n}\left(\frac{1}{c_1-x}\right)^2 dx\\
\le\ & \frac{\epsilon}{2}\int_{I_\delta^+}(u(x)+1)^2+\frac{1}{2\epsilon}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{m}\int_{c_j}^{d_j}(1-u(y))\sum_{i=1}^{n}\left(\frac{1}{y-a_i}-\frac{1}{y-b_i}\right)dy\\
\le\ & \frac{\epsilon}{2}\int_{II_\delta^+}(1-u(y))^2+\frac{1}{2\epsilon}.
\end{aligned}$$

■

Proof of the main technical lemma Observe that

$$\begin{aligned}
&\int_1^\infty\int_1^\infty f(x,y)\,dydx+\int_{-\infty}^{-1}\int_{-\infty}^{-1}f(x,y)\,dydx\\
=\ & \frac{\beta}{4\pi}\int_1^\infty\int_1^\infty\frac{(u(x)-u(y))^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{-\infty}^{-1}\int_{-\infty}^{-1}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx\\
=\ & \frac{\beta}{4\pi}\int_{I_\delta^+}\int_{I_\delta^+}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{II_\delta^+}\int_{II_\delta^+}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx\\
&+\frac{\beta}{4\pi}\int_{III_\delta^+}\int_{III_\delta^+}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{I_\delta^-}\int_{I_\delta^-}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx\\
&+\frac{\beta}{4\pi}\int_{II_\delta^-}\int_{II_\delta^-}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{III_\delta^-}\int_{III_\delta^-}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx\\
&+\frac{\beta}{2\pi}\int_{I_\delta^+}\int_{II_\delta^+}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx+\frac{\beta}{2\pi}\int_{II_\delta^+}\int_{III_\delta^+}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx\\
&+\frac{\beta}{2\pi}\int_{I_\delta^+}\int_{III_\delta^+}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx+\frac{\beta}{2\pi}\int_{I_\delta^-}\int_{II_\delta^-}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx\\
&+\frac{\beta}{2\pi}\int_{II_\delta^-}\int_{III_\delta^-}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx+\frac{\beta}{2\pi}\int_{I_\delta^-}\int_{III_\delta^-}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx, \qquad (2.40)
\end{aligned}$$

Lemma 2.7 would follow from (2.40) and the following Lemma.

Lemma 2.11 *There exists a constant $C = C(\delta, \gamma, \beta, W) > 0$ such that*

$$\begin{aligned} & \frac{\beta}{4\pi}\int_{I_\delta^+}\int_{II_\delta^+}\frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left(x-y\right)^2}dydx+\frac{\beta}{4\pi}\int_{I_\delta^-}\int_{II_\delta^-}\frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left(x-y\right)^2}dydx \\ & +\frac{\gamma}{4}\int_1^\infty W\left(u\right)dx+\frac{\gamma}{4}\int_{-\infty}^{-1}W\left(u\right)dx-\int_{I_\delta^+}\int_{I_\delta^-}\frac{\beta}{\pi\left(x-y\right)^2}dydx-\int_{II_\delta^+}\int_{II_\delta^-}\frac{\beta}{\pi\left(x-y\right)^2}dydx \\ > & -C \end{aligned}$$

Lemma 2.11 is a direct corollary of the following Lemma.

Lemma 2.12 *There exists a constant $C = C(\delta, \gamma, \beta, W) > 0$ such that for all $u \in \mathcal{A}_0$ the following bounds hold:*

$$\frac{\beta}{4\pi}\int_{I_\delta^+}\int_{II_\delta^+}\frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left(x-y\right)^2}dydx-\int_{I_\delta^+}\int_{I_\delta^-}\frac{\beta}{\pi\left(x-y\right)^2}dydx+\frac{\gamma}{4}\int_1^\infty W\left(u\right)dx \quad > \quad -C, \tag{2.41}$$

$$\frac{\beta}{4\pi}\int_{I_\delta^-}\int_{II_\delta^-}\frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left(x-y\right)^2}dydx-\int_{II_\delta^+}\int_{II_\delta^-}\frac{\beta}{\pi\left(x-y\right)^2}dydx+\frac{\gamma}{4}\int_{-\infty}^{-1}W\left(u\right)dx \quad > \quad -C. \tag{2.42}$$

Proof. We prove (2.41), as the proof of (2.42) is similar. We argue by contradiction and let

$$J_\delta^+\left(u\right):=\frac{\beta}{4\pi}\int_{I_\delta^+}\int_{II_\delta^+}\frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left(x-y\right)^2}dydx-\int_{I_\delta^+}\int_{I_\delta^-}\frac{\beta}{\pi\left(x-y\right)^2}dydx+\frac{\gamma}{4}\int_1^\infty W\left(u\right)dx$$

We show that

$$J_\delta^+\left(u_n\right)\to-\infty$$

implies

$$\limsup_{n\to\infty}\min_{1\le l\le K_n}\frac{a_l^n}{b_l^n}=1 \tag{2.43}$$

On the other hand, (2.43) and a diagonal argument would imply $J_\delta^+\left(u_n\right)$ is bounded from below, a contradiction. We prove (2.43) in a series of steps. We will only explain the first three steps in detail. The remaining steps can be proved similarly.

Assume (2.41) fails. Then there would exist a sequence $\{u_n\}$ such that $J_\delta^+\left(u_n\right) < -n$. We denote the decomposition of $I_\delta^\pm\left(u_n\right)$ and $II_\delta^\pm\left(u_n\right)$ as follows.

$$\begin{aligned} I_{\delta,n}^+ &= \cup_{i=1}^{K_n}\left[a_i^n, b_i^n\right], \quad II_{\delta,n}^+=\cup_{i=1}^{L_n}\left[c_j^n, d_j^n\right], \\ I_{\delta,n}^- &= \cup_{i=1}^{\widetilde{K}_n}\left[-\widetilde{b}_i^n, -\widetilde{a}_i^n\right], \quad II_{\delta,n}^-=\cup_{i=1}^{\widetilde{L}_n}\left[-\widetilde{d}_j^n, -\widetilde{c}_j^n\right] \end{aligned}$$

Here we have $d_{L_n}^n=\infty$ and $c_{L_n}^n > b_{K_n}^n$. We assume

$$\begin{aligned} \left(b_{K_n}^n, c_{L_n}^n\right)\cap II_{\delta,n}^+ &= \cup_{k=j_1^n}^{L_n-1}\left[c_k^n, d_k^n\right], \\ \left(b_{i-1}^n, a_i^n\right)\cap II_{\delta,n}^+ &= \cup_{k=j_i^n}^{j_{i-1}^n-1}\left[c_k^n, d_k^n\right] \quad \text{for } i=2,\cdots,K_n. \end{aligned}$$

Step 1: **$J_\delta^+\left(u_n\right)\to-\infty$ implies there exists i_1^n satisfying $j_1^n\le i_1^n\le L_n$ such that**

$$\liminf_{n\to\infty}\frac{c_{i_1^n}^n}{b_{K_n}^n}=\limsup_{n\to\infty}\frac{a_{K_n}^n}{b_{K_n}^n}=1. \tag{2.44}$$

First we observe

$$\begin{aligned}-n &> J_\delta^+(u_n)=\frac{\beta}{4\pi}\int_{I_{\delta,n}^+}\int_{II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\\ &\geq -\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx\geq-\frac{\beta}{\pi}\ln\left(2\frac{b_{K_n}^n}{a_1^n}\right).\end{aligned}$$

Therefore, $J_\delta^+(u_n)\to-\infty$ implies

$$\limsup_{n\to\infty} b_{K_n}^n=\infty. \tag{2.45}$$

Case I: Assume that for a subsequence, $c_{L_n}^n-b_{K_n}^n<1$ (without relabeling for simplicity of notations). Then by Lemma 2.10,

$$\begin{aligned}-n &> J_\delta^+(u_n)\\ &\geq \frac{\beta}{4\pi}\int_{a_{K_n}^n}^{b_{K_n}^n}\int_{c_{L_n}^n+1}^\infty\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx\\ &\geq \frac{\beta}{\pi}\ln\frac{a_{K_n}^n-c_{L_n}^n-1}{b_{K_n}^n-c_{L_n}^n-1}-\frac{\beta}{\pi}\ln\left(2\frac{b_{K_n}^n}{a_1^n}\right)+\frac{\gamma\min_{|u|\leq1-\delta}W(u)}{4}\left|\left(b_{K_n}^n,c_{L_n}^n\right)\cap III_{\delta,n}^+\right|-C\\ &\geq \frac{\beta}{\pi}\ln\frac{c_{L_n}^n-a_{K_n}^n}{b_{K_n}^n}-\frac{\beta}{\pi}\ln 2+\frac{\gamma\min_{|u|\leq1-\delta}W(u)}{4}\left|\left(b_{K_n}^n,c_{L_n}^n\right)\cap III_{\delta,n}^+\right|-C\\ &\geq \frac{\beta}{\pi}\ln\frac{c_{L_n}^n-a_{K_n}^n}{b_{K_n}^n}-C.\end{aligned}$$

Taking liminf on both sides, we have

$$\liminf_{n\to\infty}\frac{c_{L_n}^n}{b_{K_n}^n}=\limsup_{n\to\infty}\frac{a_{K_n}^n}{b_{K_n}^n}=1$$

For the remaining cases, we shall always assume $c_{L_n}^n-b_{K_n}^n\geq1$. Also whenever we need to work on a subsequence, we always use the original sequence for simplicity of notations.

Case II: Assume for a subsequence that there exists $i_1^n\in\{\,j_1^n,\ j_1^n+1,\ \cdots,L_n-1\}$ **such that** $c_{i_1^n}^n-b_{K_n}^n<1$. We also assume $b_{K_n}^n-a_{K_n}^n>1$ (otherwise (2.44) follows directly). Then by

$$g_{a,b}(x)=ax-b\ln x\geq b\left(1-\ln\frac{b}{a}\right),\ \text{ for } x>0, \tag{2.46}$$

and Lemma 2.10, we bound $J_\delta^+(u_n)$ as follows.

$$\begin{aligned}
-n &> J_\delta^+(u_n) \\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n}^n}^{b_{K_n}^n}\int_{c_{L_n}^n}^{\infty}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{a_{K_n}^n}^{b_{K_n}^n}\int_{(b_{K_n}^n,\ c_{L_n}^n)\cap II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx \\
&\quad -\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n}^n}^{b_{K_n}^n-1}\int_{c_{i_1^n}^n}^{\infty}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{a_{K_n}^n}^{b_{K_n}^n}\int_{d_{i_1^n}^n}^{d_{i_1^n}^n+\left|(b_{K_n}^n,\ c_{L_n}^n)\cap III_{\delta,n}^+\right|}\frac{\beta}{\pi(x-y)^2}dydx \\
&\quad -\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-a_{K_n}^n}{c_{i_1^n}^n+1-b_{K_n}^n}-\frac{\beta}{\pi}\ln\left(\frac{d_{i_1^n}^n-a_{K_n}^n}{d_{i_1^n}^n-b_{K_n}^n}\cdot\frac{d_{i_1^n}^n+\left|(b_{K_n}^n,c_{L_n}^n)\cap III_{\delta,n}^+\right|-b_{K_n}^n}{d_{i_1^n}^n+\left|(b_{K_n}^n,c_{L_n}^n)\cap III_{\delta,n}^+\right|-a_{K_n}^n}\right)-C \\
&\quad -\frac{\beta}{\pi}\ln\left(2\frac{b_{K_n}^n}{a_1^n}\right)+\frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|(b_{K_n}^n,c_{L_n}^n)\cap III_{\delta,n}^+\right| \\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-a_{K_n}^n}{b_{K_n}^n}-\frac{\beta}{\pi}\ln\left(1+\left|(b_{K_n}^n,c_{L_n}^n)\cap III_{\delta,n}^+\right|\right) \\
&\quad +\frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|(b_{K_n}^n,c_{L_n}^n)\cap III_{\delta,n}^+\right|-C \\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-a_{K_n}^n}{b_{K_n}^n}-C.
\end{aligned}$$

Taking liminf on both sides, we must have

$$\liminf\frac{c_{i_1^n}^n}{b_{K_n}^n}=\limsup\frac{a_{K_n}^n}{b_{K_n}^n}=1.$$

Case III: Assume no such i_1^n from case II exists, then we must have $c_{j_1^n}^n - b_{K_n}^n \geq 1$. Thus

$$\begin{aligned}
-n &> J_\delta^+ (u_n) \\
&\geq \frac{\beta}{4\pi} \int_{a_{K_n}^n}^{b_{K_n}^n} \int_{\left(c_{j_1^n}^n,\, \infty\right) \cap II_{\delta,n}^+} \frac{(u_n(x) - u_n(y))^2}{(x-y)^2} dydx - \int_{I_{\delta,n}^+} \int_{I_{\delta,n}^-} \frac{\beta}{\pi (x-y)^2} dydx + \frac{\gamma}{4} \int_1^\infty W(u_n)\, dx \\
&\geq \frac{\beta}{4\pi} \int_{a_{K_n}^n}^{b_{K_n}^n} \int_{c_{j_1^n}^n}^{\infty} \frac{(u_n(x) - u_n(y))^2}{(x-y)^2} dydx - \int_{a_{K_n}^n}^{b_{K_n}^n} \int_{d_{j_1^n}^n}^{d_{j_1^n}^n + \left|\left(b_{K_n}^n,\, c_{L_n}^n\right) \cap III_{\delta,n}^+\right|} \frac{\beta}{\pi (x-y)^2} dydx \\
&\quad - \int_{I_{\delta,n}^+} \int_{I_{\delta,n}^-} \frac{\beta}{\pi (x-y)^2} dydx + \frac{\gamma}{4} \int_1^\infty W(u_n)\, dx \\
&\geq \frac{\beta}{\pi} \ln \frac{c_{j_1^n}^n - a_{K_n}^n}{c_{j_1^n}^n - b_{K_n}^n} - \frac{\beta}{\pi} \ln \left(\frac{d_{j_1^n}^n - a_{K_n}^n}{d_{j_1^n}^n - b_{K_n}^n} \cdot \frac{d_{j_1^n}^n + \left| \left(b_{K_n}^n, c_{L_n}^n\right) \cap III_{\delta,n}^+ \right| - b_{K_n}^n}{d_{j_1^n}^n + \left| \left(b_{K_n}^n, c_{L_n}^n\right) \cap III_{\delta,n}^+ \right| - a_{K_n}^n} \right) \\
&\quad - \frac{\beta}{\pi} \ln \left(2 \frac{b_{K_n}^n}{a_1^n} \right) + \frac{\gamma \min_{|u| \leq 1-\delta} W(u)}{4} \left| \left(b_{K_n}^n, c_{L_n}^n\right) \cap III_{\delta,n}^+ \right| - C \\
&\geq \frac{\beta}{\pi} \ln \frac{c_{j_1^n}^n - a_{K_n}^n}{b_{K_n}^n} - \frac{\beta}{\pi} \ln \left(c_{j_1^n}^n - b_{K_n}^n \right) - \frac{\beta}{\pi} \ln \left(1 + \left| \left(b_{K_n}^n, c_{L_n}^n\right) \cap III_{\delta,n}^+ \right| \right) \\
&\quad + \frac{\gamma \min_{|u| \leq 1-\delta} W(u)}{4} \left| \left(b_{K_n}^n, c_{L_n}^n\right) \cap III_{\delta,n}^+ \right| - C \\
&\geq \frac{\beta}{\pi} \ln \frac{c_{j_1^n}^n - a_{K_n}^n}{b_{K_n}^n} - C.
\end{aligned}$$

The last step used (2.46) and $\left(b_{K_n}^n, c_{j_1^n}^n\right) \subset \left(b_{K_n}^n, c_{L_n}^n\right) \cap III_{\delta,n}^+$. Taking liminf on both sides of the equation above, we must have

$$\liminf \frac{c_{j_1^n}^n}{b_{K_n}^n} = \limsup \frac{a_{K_n}^n}{b_{K_n}^n} = 1.$$

Step 2: $J_\delta^+ (u_n) \to -\infty$ implies there exists i_2^n satisfying $j_2^n \leq i_2^n \leq i_1^n$ such that

$$\liminf_{n\to\infty} \frac{c_{i_2^n}^n}{b_{K_n-1}^n} = \limsup_{n\to\infty} \frac{a_{K_n-1}^n}{b_{K_n-1}^n} = 1 \tag{2.47}$$

First we observe

$$\begin{aligned}
-n &> J_\delta^+ (u_n) \geq - \int_{I_{\delta,n}^+} \int_{I_{\delta,n}^-} \frac{\beta}{\pi (x-y)^2} dydx + \frac{\gamma}{4} \int_1^\infty W(u_n) \\
&\geq -\frac{\beta}{\pi} \ln \left(2 \frac{b_{K_n-1}^n}{a_1^n} \right) - \frac{\beta}{\pi} \ln \frac{b_{K_n}^n}{a_{K_n}^n}.
\end{aligned}$$

Therefore, by Step 1 we have that $J_\delta^+ (u_n) \to -\infty$ implies

$$\limsup b_{K_n-1}^n = \infty.$$

Case I: If $\liminf_{n\to\infty} \frac{a_{K_n}^n - b_{K_n-1}^n}{c_{i_1^n}^n - b_{K_n-1}^n} = 0$, **then** $\limsup_{n\to\infty} \frac{c_{i_1^n}^n - a_{K_n}^n}{c_{i_1^n}^n - b_{K_n-1}^n} = 1$. In this case, then we can replace $\left(a_{K_n-1}^n, b_{K_n-1}^n\right)\cup\left(a_{K_n}^n, b_{K_n}^n\right)$ by $\left(a_{K_n-1}^n, b_{K_n}^n\right)$ and repeat our argument in Step 1 as follows.

$$\begin{aligned}
-n &> J_\delta^+(u_n)\\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{\left(b_{K_n}^n\infty\right)\cap II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{a_{K_n}^n}^{b_{K_n}^n}\int_{\left(b_{K_n}^n\infty\right)\cap II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx\\
&\quad-\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx\\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n}^n}\int_{c_{i_1^n}^n}^\infty\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{a_{K_n-1}^n}^{b_{K_n}^n}\int_{d_{i_1^n}^n}^{d_{i_1^n}^n+\left|\left(b_{K_n}^n,\,c_{L_n}^n\right)\cap III_{\delta,n}^+\right|}\frac{\beta}{\pi(x-y)^2}dydx\\
&\quad-\int_{b_{K_n-1}^n}^{a_{K_n}^n}\int_{c_{i_1^n}^n}^\infty\frac{\beta}{\pi(x-y)^2}dydx-\frac{\beta}{\pi}\ln 2\frac{b_{K_n-1}^n}{a_1^n}-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx\\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n}^n}\int_{c_{i_1^n}^n}^\infty\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{a_{K_n-1}^n}^{b_{K_n}^n}\int_{d_{i_1^n}^n}^{d_{i_1^n}^n+\left|\left(b_{K_n}^n,C_{L_n}^n\right)\cap III_{\delta,n}^+\right|}\frac{\beta}{\pi(x-y)^2}dydx\\
&\quad-\frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-b_{K_n-1}^n}{c_{i_1^n}^n-a_{K_n}^n}-\frac{\beta}{\pi}\ln 2\frac{b_{K_n-1}^n}{a_1^n}-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx\\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-a_{K_n-1}^n}{c_{i_1^n}^n-b_{K_n}^n}-\frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-b_{K_n-1}^n}{c_{i_1^n}^n-a_{K_n}^n}-\frac{\beta}{\pi}\ln\left(\frac{d_{i_1^n}^n-a_{K_n-1}^n}{d_{i_1^n}^n-b_{K_n}^n}\cdot\frac{d_{i_1^n}^n+\left|\left(b_{K_n-1}^n,a_{K_n}^n\right)\cap III_{\delta,n}^+\right|-b_{K_n}^n}{d_{i_1^n}^n+\left|\left(b_{K_n-1}^n,a_{K_n}^n\right)\cap III_{\delta,n}^+\right|-a_{K_n-1}^n}\right)\\
&\quad-\frac{\beta}{\pi}\ln 2\frac{b_{K_n-1}^n}{a_1^n}-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx-C\\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-a_{K_n-1}^n}{b_{K_n-1}^n}-\frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-b_{K_n-1}^n}{c_{i_1^n}^n-a_{K_n}^n}-\frac{\beta}{\pi}\ln\left(c_{i_1^n}^n-b_{K_n}^n\right)-\frac{\beta}{\pi}\ln\left(1+\left|\left(b_{K_n-1}^n,a_{K_n}^n\right)\cap III_{\delta,n}^+\right|\right)\\
&\quad-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}+\frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III_{\delta,n}^+\right|-C\\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-a_{K_n-1}^n}{b_{K_n-1}^n}-\frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-b_{K_n-1}^n}{c_{i_1^n}^n-a_{K_n}^n}-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}-C.
\end{aligned}$$

Taking liminf on both sides, we get

$$\liminf\frac{c_{i_1^n}^n}{b_{K_n-1}^n}=\limsup\frac{a_{K_n-1}^n}{b_{K_n-1}^n}=1.$$

Case II: If $\liminf_{n\to\infty}\frac{a_{K_n}^n-b_{K_n-1}^n}{c_{i_1^n}^n-b_{K_n-1}^n}>0$, **There are three cases.**

Case II-i: If $\left(b^n_{K_n-1}, a^n_{K_n}\right) \cap II^+_{\delta,n} = \emptyset$, then $\left(b^n_{K_n-1}, a^n_{K_n}\right) \subset III^+_{\delta,n}$,

$$
\begin{aligned}
-n &> J^+_\delta(u_n) \\
&\geq \frac{\beta}{4\pi}\int_{a^n_{K_n-1}}^{b^n_{K_n-1}} \int_{\left(b^n_{K_n},\, \infty\right)\cap II^+_{\delta,n}} \frac{\left(u_n(x)-u_n(y)\right)^2}{(x-y)^2}dydx + \frac{\beta}{4\pi}\int_{a^n_{K_n}}^{b^n_{K_n}} \int_{\left(b^n_{K_n},\infty\right)\cap II^+_{\delta,n}} \frac{\left(u_n(x)-u_n(y)\right)^2}{(x-y)^2}dydx \\
&\quad -\int_{I^+_{\delta,n}}\int_{I^-_{\delta,n}} \frac{\beta}{\pi(x-y)^2}dydx + \frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{4\pi}\int_{a^n_{K_n-1}}^{b^n_{K_n-1}}\int_{c^n_{i^n_1}}^\infty \frac{\left(u_n(x)-u_n(y)\right)^2}{(x-y)^2}dydx - \int_{a^n_{K_n-1}}^{b^n_{K_n-1}}\int_{d^n_{i^n_1}}^{d^n_{i^n_1}+\left|\left(b^n_{K_n},\, c^n_{L_n}\right)\cap III^+_{\delta,n}\right|} \frac{\beta}{\pi(x-y)^2}dydx \\
&\quad +\frac{\beta}{4\pi}\int_{a^n_{K_n}}^{b^n_{K_n}}\int_{c^n_{i^n_1}}^\infty \frac{\left(u_n(x)-u_n(y)\right)^2}{(x-y)^2}dydx - \int_{a^n_{K_n}}^{b^n_{K_n}}\int_{d^n_{i^n_1}}^{d^n_{i^n_1}+\left|\left(b^n_{K_n},\, c^n_{L_n}\right)\cap III^+_{\delta,n}\right|} \frac{\beta}{\pi(x-y)^2}dydx \\
&\quad -\frac{\beta}{\pi}\ln 2\frac{b^n_{K_n-1}}{a^n_1} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} + \frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n-1}}{c^n_{i^n_1}-b^n_{K_n-1}} - \frac{\beta}{\pi}\ln\left(\frac{d^n_{i^n_1}-a^n_{K_n-1}}{d^n_{i^n_1}-b^n_{K_n-1}}\cdot\frac{d^n_{i^n_1}+\left|\left(b^n_{K_n}, c^n_{L_n}\right)\cap III^+_{\delta,n}\right| - b^n_{K_n-1}}{d^n_{i^n_1}+\left|\left(b^n_{K_n}, c^n_{L_n}\right)\cap III^+_{\delta,n}\right| - a^n_{K_n-1}}\right) \\
&\quad +\frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n}}{c^n_{i^n_1}-b^n_{K_n}} - \frac{\beta}{\pi}\ln\left(\frac{d^n_{i^n_1}-a^n_{K_n}}{d^n_{i^n_1}-b^n_{K_n}}\cdot\frac{d^n_{i^n_1}+\left|\left(b^n_{K_n}, c^n_{L_n}\right)\cap III^+_{\delta,n}\right| - b^n_{K_n}}{d^n_{i^n_1}+\left|\left(b^n_{K_n}, c^n_{L_n}\right)\cap III^+_{\delta,n}\right| - a^n_{K_n}}\right) \\
&\quad -\frac{\beta}{\pi}\ln 2\frac{b^n_{K_n-1}}{a^n_1} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right| - C \\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n-1}}{b^n_{K_n-1}} + \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}+1-a^n_{K_n}}{c^n_{i^n_1}+1-b^n_{K_n-1}} - \frac{\beta}{\pi}\ln\left(c^n_{i^n_1}-b^n_{K_n}\right) - \frac{2\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n}, c^n_{L_n}\right)\cap III^+_{\delta,n}\right|\right) \\
&\quad -\frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right| - C \\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n-1}}{b^n_{K_n-1}} - 4\ln\left(1+\left(a^n_{K_n}-b^n_{K_n-1}\right)\right) - \frac{\beta}{\pi}\ln\left(c^n_{i^n_1}-b^n_{K_n}\right) - \frac{2\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n}, c^n_{L_n}\right)\cap III^+_{\delta,n}\right|\right) \\
&\quad -\frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right| - C \\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n-1}}{b^n_{K_n-1}} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} - C, \qquad (2.48)
\end{aligned}
$$

where we used (2.46) and the facts

$$
\left(a^n_{K_n}-b^n_{K_n-1}\right) \leq \left|III^+_{\delta,n}\right|, \quad \left(c^n_{i^n_1}-b^n_{K_n}\right) \leq \max\left(1, \left|\left(b^n_{K_n}, c^n_{L_n}\right)\cap III^+_{\delta,n}\right|\right).
$$

Taking liminf on both sides of (2.48), we conclude

$$
\liminf_{n\to\infty}\frac{c^n_{i^n_1}}{b^n_{K_n-1}} = \limsup_{n\to\infty}\frac{a^n_{K_n-1}}{b^n_{K_n-1}} = 1.
$$

Case II-ii: If $\left(b^n_{K_n-1}, a^n_{K_n}\right) \cap II^+_{\delta,n} \neq \emptyset$, **there are two cases.** If $\liminf_{n\to\infty}\frac{c^n_{i^n_1}}{a^n_{K_n-1}} = 1$, then (2.47) follows directly. We therefore assume $\liminf_{n\to\infty}\frac{c^n_{i^n_1}}{a^n_{K_n-1}} > 1$ for the remaining two cases.

Case II-ii-a: There exists $i_2^n \in \{j_2^n,\ j_2^n+1,\ \cdots,\ i_1^n\}$ such that $c_{i_2^n}^n - b_{K_n-1}^n < 1$. We assume $b_{K_n-1}^n - a_{K_n-1}^n > 1$ without loss of generality. Applying Corollary 2.9, Lemma 2.10 and (2.46), we bound $J_\delta^+(u_n)$ as follows.

$$\begin{aligned}
-n &> J_\delta^+(u_n) \\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{(b_{K_n-1}^n,\ a_{K_n}^n)\cap II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx + \frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{(b_{K_n}^n,\infty)\cap II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx \\
&\quad -\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx + \frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n-1}\int_{c_{i_2^n}^n}^{a_{K_n}^n}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx - \int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{d_{i_2^n}^n}^{d_{i_2^n}^n+\left|(b_{K_n-1}^n,\ a_{K_n}^n)\cap III_{\delta,n}^+\right|}\frac{\beta}{\pi(x-y)^2}dydx \\
&\quad +\frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{c_{i_1^n}^n}^{\infty}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx - \int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{d_{i_1^n}^n}^{d_{i_1^n}^n+\left|(b_{K_n-1}^n,\ c_{L_n}^n)\cap III_{\delta,n}^+\right|}\frac{\beta}{\pi(x-y)^2}dydx \\
&\quad -\frac{\beta}{\pi}\ln 2\frac{b_{K_n-1}^n}{a_1^n} - \frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n} + \frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{\pi}\ln\left(\frac{c_{i_2^n}^n - a_{K_n-1}^n}{c_{i_2^n}^n + 1 - b_{K_n-1}^n}\cdot\frac{a_{K_n}^n - b_{K_n-1}^n + 1}{a_{K_n}^n - a_{K_n-1}^n}\right) - \frac{\beta}{\pi}\ln\left(\frac{d_{i_2^n}^n - a_{K_n-1}^n}{d_{i_2^n}^n - b_{K_n-1}^n}\cdot\frac{d_{i_2^n}^n + \left|\left(b_{K_n-1}^n, a_{K_n}^n\right)\cap III_{\delta,n}^+\right| - b_{K_n-1}^n}{d_{i_2^n}^n + \left|\left(b_{K_n-1}^n, a_{K_n}^n\right)\cap III_{\delta,n}^+\right| - a_{K_n-1}^n}\right) \\
&\quad +\frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n - a_{K_n-1}^n}{c_{i_1^n}^n - b_{K_n-1}^n} - \frac{\beta}{\pi}\ln\left(\frac{d_{i_1^n}^n - a_{K_n-1}^n}{d_{i_1^n}^n - b_{K_n-1}^n}\cdot\frac{d_{i_1^n}^n + \left|\left(b_{K_n}^n, c_{L_n}^n\right)\cap III_{\delta,n}^+\right| - b_{K_n-1}^n}{d_{i_1^n}^n + \left|\left(b_{K_n}^n, c_{L_n}^n\right)\cap III_{\delta,n}^+\right| - a_{K_n-1}^n}\right) \\
&\quad -\frac{\beta}{\pi}\ln 2\frac{b_{K_n-1}^n}{a_1^n} - \frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n} - C + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III_{\delta,n}^+\right| \\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_2^n}^n - a_{K_n-1}^n}{b_{K_n-1}^n} + \frac{\beta}{\pi}\ln\left(\frac{c_{i_1^n}^n - a_{K_n-1}^n}{c_{i_1^n}^n - b_{K_n-1}^n}\cdot\frac{a_{K_n}^n - b_{K_n-1}^n}{a_{K_n}^n - a_{K_n-1}^n}\right) - \frac{\beta}{\pi}\ln\left(1 + \left|\left(b_{K_n-1}^n, a_{K_n}^n\right)\cap III_{\delta,n}^+\right|\right) \\
&\quad -\frac{\beta}{\pi}\ln\left(1 + \left|\left(b_{K_n}^n, c_{L_n}^n\right)\cap III_{\delta,n}^+\right|\right) - \frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n} - C \\
&\quad -\frac{\beta}{\pi}\ln\left(1 + c_{i_2^n}^n - b_{K_n-1}^n\right) + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III_{\delta,n}^+\right| \\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_2^n}^n - a_{K_n-1}^n}{b_{K_n-1}^n} + \frac{\beta}{\pi}\ln\frac{a_{K_n}^n - b_{K_n-1}^n}{c_{i_1^n}^n - b_{K_n-1}^n} - \frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n} - C.
\end{aligned}$$

Taking liminf on both sides, by Step 1 and $\liminf_{n\to\infty}\frac{a_{K_n}^n - b_{K_n-1}^n}{a_{K_n}^n - a_{K_n-1}^n} > 0$, we must have

$$\liminf\frac{c_{i_2^n}^n}{b_{K_n-1}^n} = \limsup\frac{a_{K_n-1}^n}{b_{K_n-1}^n} = 1.$$

Case II-i-b: No such i_2^n exists, then we must have $c_{j_2^n}^n - b_{K_n}^n \geq 1$. Applying Corollary 2.9, Lemma 2.10 and (2.46) , we bound $J_{\delta}^{+}(u_n)$ as follows.

$$
\begin{aligned}
-n &> J_{\delta}^{+}(u_n) \\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{\left(b_{K_n-1}^n,\ a_{K_n}^n\right)\cap II_{\delta,n}^{+}}\frac{\left(u_n(x)-u_n(y)\right)^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{\left(b_{K_n}^n\infty\right)\cap II_{\delta,n}^{+}}\frac{\left(u_n(x)-u_n(y)\right)^2}{(x-y)^2}dydx \\
&\quad -\int_{I_{\delta,n}^{+}}\int_{I_{\delta,n}^{-}}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^{\infty}W(u_n)\,dx \\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{c_{j_2^n}^n}^{a_{K_n}^n}\frac{\left(u_n(x)-u_n(y)\right)^2}{(x-y)^2}dydx-\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{d_{j_2^n}^n}^{d_{j_2^n}^n+\left|\left(b_{K_n-1}^n,\ a_{K_n}^n\right)\cap III_{\delta,n}^{+}\right|}\frac{\beta}{\pi(x-y)^2}dydx \\
&\quad +\frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{c_{i_1^n}^n}^{\infty}\frac{\left(u_n(x)-u_n(y)\right)^2}{(x-y)^2}dydx-\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{d_{i_1^n}^n}^{d_{i_1^n}^n+\left|\left(b_{K_n-1}^n,\ c_{L_n}^n\right)\cap III_{\delta,n}^{+}\right|}\frac{\beta}{\pi(x-y)^2}dydx \\
&\quad -\frac{\beta}{\pi}\ln 2\frac{b_{K_n-1}^n}{a_1^n}-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}+\frac{\gamma}{4}\int_1^{\infty}W(u_n)\,dx \\
&\geq \frac{\beta}{\pi}\ln\left(\frac{c_{j_2^n}^n-a_{K_n-1}^n}{c_{j_2^n}^n-b_{K_n-1}^n}\cdot\frac{a_{K_n}^n-b_{K_n-1}^n}{a_{K_n}^n-a_{K_n-1}^n}\right)-\frac{\beta}{\pi}\ln\left(\frac{d_{j_2^n}^n-a_{K_n-1}^n}{d_{j_2^n}^n-b_{K_n-1}^n}\cdot\frac{d_{j_2^n}^n+\left|\left(b_{K_n-1}^n,a_{K_n}^n\right)\cap III_{\delta,n}^{+}\right|-b_{K_n-1}^n}{d_{j_2^n}^n+\left|\left(b_{K_n-1}^n,a_{K_n}^n\right)\cap III_{\delta,n}^{+}\right|-a_{K_n-1}^n}\right) \\
&\quad +\frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-a_{K_n-1}^n}{c_{i_1^n}^n-b_{K_n-1}^n}-\frac{\beta}{\pi}\ln\left(\frac{d_{i_1^n}^n-a_{K_n-1}^n}{d_{i_1^n}^n-b_{K_n-1}^n}\cdot\frac{d_{i_1^n}^n+\left|\left(b_{K_n}^n,c_{L_n}^n\right)\cap III_{\delta,n}^{+}\right|-b_{K_n-1}^n}{d_{i_1^n}^n+\left|\left(b_{K_n}^n,c_{L_n}^n\right)\cap III_{\delta,n}^{+}\right|-a_{K_n-1}^n}\right) \\
&\quad -\frac{\beta}{\pi}\ln 2\frac{b_{K_n-1}^n}{a_1^n}-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}-C+\frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III_{\delta,n}^{+}\right| \\
&\geq \frac{\beta}{\pi}\ln\frac{c_{j_2^n}^n-a_{K_n-1}^n}{b_{K_n-1}^n}+\frac{\beta}{\pi}\ln\left(\frac{c_{i_1^n}^n-a_{K_n-1}^n}{c_{i_1^n}^n-b_{K_n-1}^n}\cdot\frac{a_{K_n}^n-b_{K_n-1}^n}{a_{K_n}^n-a_{K_n-1}^n}\right)-\frac{\beta}{\pi}\ln\left(1+\left|\left(b_{K_n-1}^n,a_{K_n}^n\right)\cap III_{\delta,n}^{+}\right|\right) \\
&\quad -\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}-C-\frac{\beta}{\pi}\ln\left(c_{j_2^n}^n-b_{K_n-1}^n\right)+\frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III_{\delta,n}^{+}\right| \\
&\geq \frac{\beta}{\pi}\ln\frac{c_{j_2^n}^n-a_{K_n-1}^n}{b_{K_n-1}^n}+\frac{\beta}{\pi}\ln\frac{a_{K_n}^n-b_{K_n-1}^n}{c_{i_1^n}^n-b_{K_n-1}^n}-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}-C.
\end{aligned}
$$

Taking liminf on both sides, we must have

$$
\liminf\frac{c_{j_2^n}^n}{b_{K_n-1}^n}=\limsup\frac{a_{K_n-1}^n}{b_{K_n-1}^n}=1.
$$

Step 3: $J_{\delta}^{+}(u_n)\to-\infty$ implies there exists i_3^n satisfying $j_3^n\leq i_3^n\leq i_2^n$ such that

$$
\liminf_{n\to\infty}\frac{c_{i_3^n}^n}{b_{K_n-2}^n}=\limsup_{n\to\infty}\frac{a_{K_n-2}^n}{b_{K_n-2}^n}=1 \tag{2.49}
$$

First we observe

$$\begin{aligned}
-n &> J_\delta^+(u_n) \\
&= \frac{\beta}{\pi}\int_{I_{\delta,n}^+}\int_{II_{\delta,n}^+}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx-\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq -\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx \\
&\geq -\frac{\beta}{\pi}\ln\left(2\frac{b_{K_n-2}^n}{a_1^n}\right)-\frac{\beta}{\pi}\ln\frac{b_{K_n}^n}{a_{K_n}^n}-\frac{\beta}{\pi}\ln\frac{b_{K_n-1}^n}{a_{K_n-1}^n}.
\end{aligned}$$

Therefore

$$\limsup b_{K_n-2}^n=\infty.$$

Case I: $\liminf\frac{a_{K_n}^n-b_{K_n-2}^n}{c_{i_1^n}^n-b_{K_n-2}^n}=0$. This implies

$$\liminf\frac{c_{i_1^n}^n-b_{K_n-2}^n}{c_{i_1^n}^n-a_{K_n}^n}=1. \tag{2.50}$$

In this case, we can replace $\left(a_{K_n-2}^n,b_{K_n-2}^n\right)\cup\left(a_{K_n-1}^n,b_{K_n-1}^n\right)\cup\left(a_{K_n}^n,b_{K_n}^n\right)$ by $\left(a_{K_n-2}^n,b_{K_n}^n\right)$, and repeat our argument in step 1.

We estimate $J_\delta^+(u_n)$ as follows.

$$\begin{aligned}
-n &> J_\delta^+(u_n) \\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-2}^n}^{b_{K_n-2}^n}\int_{(b_{K_n}^n,\infty)\cap II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{a_{K_n-1}^n}^{b_{K_n-1}^n}\int_{(b_{K_n}^n,\infty)\cap II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx \\
&\quad+\frac{\beta}{\pi}\int_{a_{K_n}^n}^{b_{K_n}^n}\int_{(b_{K_n}^n,\infty)\cap II_{\delta,n}^+}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{4\pi}\int_{a_{K_n-2}^n}^{b_{K_n}^n}\int_{c_{i_1^n}^n}^{\infty}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{a_{K_n-2}^n}^{b_{K_n}^n}\int_{d_{i_1^n}^n}^{d_{i_1^n}^n+\left|(b_{K_n}^n,\infty)\cap III_{\delta,n}^+\right|}\frac{\beta}{\pi(x-y)^2}dydx \\
&\quad-\int_{b_{K_n-2}^n}^{a_{K_n}^n}\int_{c_{i_1^n}^n}^{\infty}\frac{\beta}{\pi(x-y)^2}dydx-\int_{I_{\delta,n}^+}\int_{I_{\delta,n}^-}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-a_{K_n-2}^n}{c_{i_1^n}^n-b_{K_n}^n}-\frac{\beta}{\pi}\ln\left[\frac{d_{i_1^n}^n-a_{K_n-2}^n}{d_{i_1^n}^n-b_{K_n}^n}\cdot\frac{d_{i_1^n}^n+\left|(b_{K_n}^n,\infty)\cap III_{\delta,n}^+\right|-b_{K_n}^n}{d_{i_1^n}^n+\left|(b_{K_n}^n,\infty)\cap III_{\delta,n}^+\right|-a_{K_n-2}^n}\right] \\
&\quad-\frac{\beta}{\pi}\ln\frac{c_{i_1^n}^n-b_{K_n-2}^n}{c_{i_1^n}^n-a_{K_n}^n}-\frac{\beta}{\pi}\ln 2\frac{b_{K_n}^n}{a_1^n}+\frac{\gamma\min_{|u|\leq 1-\delta}}{4}\left|III_{\delta,n}^+\right|.
\end{aligned}$$

By (2.50), we are back to the situation in Step 1 with $\left(a_{K_n}^n,b_{K_n}^n\right)$ replaced by $\left(a_{K_n-2}^n,b_{K_n}^n\right)$ and we conclude

$$\liminf\frac{c_{i_1^n}^n}{b_{K_n}^n}=\limsup\frac{a_{K_n-2}^n}{b_{K_n}^n}=\limsup\frac{a_{K_n-2}^n}{b_{K_n-2}^n}=1.$$

Case II: $\liminf\frac{a_{K_n}^n-b_{K_n-2}^n}{c_{i_1^n}^n-b_{K_n-2}^n}>0$ **and** $\liminf\frac{a_{K_n-1}^n-b_{K_n-2}^n}{c_{i_2^n}^n-b_{K_n-2}^n}=0$. This implies

$$\limsup\frac{c_{i_2^n}^n-a_{K_n-1}^n}{c_{i_2^n}^n-b_{K_n-2}^n}=1. \tag{2.51}$$

In this case, we replace $\left(a_{K_n-2}^n, b_{K_n-2}^n\right) \cup \left(a_{K_n-1}^n, b_{K_n-1}^n\right)$ by $\left(a_{K_n-2}^n, b_{K_n-1}^n\right)$ and repeat our argument in step 2.

We bound $J_\delta^+ (u_n)$ as follows.

$$\begin{aligned}
-n &> J_\delta^+ (u_n) \\
&\geq \frac{\beta}{4\pi} \int_{a_{K_n-2}^n}^{b_{K_n-2}^n} \int_{II_{\delta,n}^+} \frac{\left(u_n(x) - u_n(y)\right)^2}{(x-y)^2} dydx + \frac{\beta}{4\pi} \int_{a_{K_n-1}^n}^{b_{K_n-1}^n} \int_{II_{\delta,n}^+} \frac{\left(u_n(x) - u_n(y)\right)^2}{(x-y)^2} dydx \\
&\quad + \frac{\beta}{4\pi} \int_{a_{K_n}^n}^{b_{K_n}^n} \int_{II_{\delta,n}^+} \frac{\left(u_n(x) - u_n(y)\right)^2}{(x-y)^2} dydx - \int_{I_{\delta,n}^+} \int_{I_{\delta,n}^-} \frac{\beta}{\pi (x-y)^2} dydx + \frac{\gamma}{4} \int_1^\infty W(u_n)\, dx \\
&\geq \frac{\beta}{4\pi} \int_{a_{K_n-2}^n}^{b_{K_n-1}^n} \int_{II_{\delta,n}^+ \cap \left(c_{i_2^n}^n, \infty\right)} \frac{\left(u_n(x) - u_n(y)\right)^2}{(x-y)^2} dydx - \int_{b_{K_n-2}^n}^{a_{K_n-1}^n} \int_{II_{\delta,n}^+ \cap \left(c_{i_2^n}^n, \infty\right)} \frac{\beta}{\pi (x-y)^2} dydx \\
&\quad + \frac{\beta}{\pi} \int_{a_{K_n}^n}^{b_{K_n}^n} \int_{II_{\delta,n}^+} \frac{\left(u_n(x) - u_n(y)\right)^2}{(x-y)^2} dydx - \int_{I_{\delta,n}^+} \int_{I_{\delta,n}^-} \frac{\beta}{\pi (x-y)^2} dydx + \frac{\gamma}{4} \int_1^\infty W(u_n)\, dx \\
&\geq \frac{\beta}{\pi} \int_{a_{K_n-2}^n}^{b_{K_n-1}^n} \int_{II_{\delta,n}^+ \cap \left(c_{i_2^n}^n, \infty\right)} \frac{\left(u_n(x) - u_n(y)\right)^2}{(x-y)^2} dydx + \frac{\beta}{4\pi} \int_{a_{K_n}^n}^{b_{K_n}^n} \int_{II_{\delta,n}^+} \frac{\left(u_n(x) - u_n(y)\right)^2}{(x-y)^2} dydx \\
&\quad - \int_{I_{\delta,n}^+} \int_{I_{\delta,n}^-} \frac{\beta}{\pi (x-y)^2} dydx + \frac{\gamma}{4} \int_1^\infty W(u_n)\, dx - \frac{\beta}{\pi} \ln \frac{c_{i_2^n}^n - b_{K_n-2}^n}{c_{i_2^n}^n - a_{K_n-1}^n}.
\end{aligned}$$

By (2.51), we are back to the case of step 2 with $\left(a_{K_n-1}^n, b_{K_n-1}^n\right)$ replaced by $\left(a_{K_n-2}^n, b_{K_n-1}^n\right)$ and we can conclude

$$\liminf \frac{c_{i_2^n}^n}{b_{K_n-1}^n} = \limsup \frac{a_{K_n-2}^n}{b_{K_n-1}^n} = \limsup \frac{a_{K_n-2}^n}{b_{K_n-2}^n} = 1.$$

Case III: $\liminf \frac{a_{K_n}^n - b_{K_n-2}^n}{c_{i_1^n}^n - b_{K_n-2}^n} > 0$ **and** $\liminf \frac{a_{K_n-1}^n - b_{K_n-2}^n}{c_{i_2^n}^n - b_{K_n-2}^n} > 0$. We discuss several cases.

Case III-i: $\left(b_{K_n-2}^n, a_{K_n-1}^n\right) \cap II_{\delta,n}^+ = \emptyset$. Then $\left(b_{K_n-2}^n, a_{K_n-1}^n\right) \subset III_{\delta,n}^+$. We also assume $\liminf_{n\to\infty} \frac{a_{K_n}^n - b_{K_n-1}^n + 1}{c_{i_1^n}^n - b_{K_n-1}^n} > 0$.

We have

$$
\begin{aligned}
-n &> J_\delta^+(u_n) \\
&\geq \frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{\left(b^n_{K_n-1},a^n_{K_n}\right)\cap II^+_{\delta,n}} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx + \frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{\left(b^n_{K_n}\infty\right)\cap II^+_{\delta,n}} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx \\
&\quad + \frac{\beta}{4\pi}\int_{a^n_{K_n-1}}^{b^n_{K_n-1}}\int_{\left(b^n_{K_n-1},\, a^n_{K_n}\right)\cap II^+_{\delta,n}} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx + \frac{\beta}{4\pi}\int_{a^n_{K_n-1}}^{b^n_{K_n-1}}\int_{\left(b^n_{K_n}\infty\right)\cap II^+_{\delta,n}} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx \\
&\quad - \int_{I^+_{\delta,n}}\int_{I^-_{\delta,n}} \frac{\beta}{\pi(x-y)^2}dydx + \frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{c^n_{i^n_2}}^{a^n_{K_n}} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx - \int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{d^n_{i^n_2}}^{d^n_{i^n_2}+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|} \frac{\beta}{\pi(x-y)^2}dydx \\
&\quad + \frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{c^n_{i^n_1}}^{\infty} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx - \int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{d^n_{i^n_1}}^{d^n_{i^n_1}+\left|\left(b^n_{K_n},c^n_{L_n}\right)\cap III^+_{\delta,n}\right|} \frac{\beta}{\pi(x-y)^2}dydx \\
&\quad + \frac{\beta}{4\pi}\int_{a^n_{K_n-1}}^{b^n_{K_n-1}-1}\int_{c^n_{i^n_2}}^{a^n_{K_n}} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx - \int_{a^n_{K_n-1}}^{b^n_{K_n-1}}\int_{d^n_{i^n_2}}^{d^n_{i^n_2}+\left|\left(b^n_{K_n-1},\, a^n_{K_n}\right)\cap III^+_{\delta,n}\right|} \frac{\beta}{\pi}(x-y)^2dydx \\
&\quad - \frac{\beta}{\pi}\ln 2\frac{b^n_{K_n-2}}{a^n_1} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n-1}}{a^n_{K_n-1}} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} + \frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{\pi}\ln\left(\frac{c^n_{i^n_2}-a^n_{K_n-2}}{c^n_{i^n_2}-b^n_{K_n-2}}\cdot\frac{a^n_{K_n}-b^n_{K_n-2}}{a^n_{K_n}-a^n_{K_n-2}}\right) - \frac{\beta}{\pi}\ln\left(\frac{d^n_{i^n_2}-a^n_{K_n-2}}{d^n_{i^n_2}-b^n_{K_n-2}}\cdot\frac{d^n_{i^n_2}+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|-b^n_{K_n-2}}{d^n_{i^n_2}+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|-a^n_{K_n-2}}\right) \\
&\quad + \frac{\beta}{\pi}\ln\left(\frac{c^n_{i^n_2}-a^n_{K_n-1}}{c^n_{i^n_2}+1-b^n_{K_n-1}}\cdot\frac{a^n_{K_n}-b^n_{K_n-1}+1}{a^n_{K_n}-a^n_{K_n-1}}\right) - \frac{\beta}{\pi}\ln\left(\frac{d^n_{i^n_2}-a^n_{K_n-1}}{d^n_{i^n_2}-b^n_{K_n-1}}\cdot\frac{d^n_{i^n_2}+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|-b^n_{K_n-1}}{d^n_{i^n_2}+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|-a^n_{K_n-1}}\right) \\
&\quad + \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n-2}}{c^n_{i^n_1}-b^n_{K_n-2}} + \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n-1}}{c^n_{i^n_1}-b^n_{K_n-1}} - \frac{\beta}{\pi}\ln\left(\frac{d^n_{i^n_1}-a^n_{K_n-2}}{d^n_{i^n_1}-b^n_{K_n-2}}\cdot\frac{d^n_{i^n_1}+\left|\left(b^n_{K_n},c^n_{L_n}\right)\cap III^+_{\delta,n}\right|-b^n_{K_n-2}}{d^n_{i^n_1}+\left|\left(b^n_{K_n},c^n_{L_n}\right)\cap III^+_{\delta,n}\right|-a^n_{K_n-2}}\right) \\
&\quad - \frac{\beta}{\pi}\ln 2\frac{b^n_{K_n-2}}{a^n_1} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n-1}}{a^n_{K_n-1}} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} - C + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right| \\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i^n_2}-a^n_{K_n-2}}{b^n_{K_n-2}} + \frac{\beta}{\pi}\ln\left(\frac{c^n_{i^n_2}-a^n_{K_n-1}}{c^n_{i^n_2}-b^n_{K_n-2}}\cdot\frac{a^n_{K_n}-b^n_{K_n-2}}{a^n_{K_n}-a^n_{K_n-2}}\right) + \frac{\beta}{\pi}\ln\left(\frac{c^n_{i^n_1}-a^n_{K_n-2}}{c^n_{i^n_1}-b^n_{K_n-2}}\cdot\frac{a^n_{K_n}-b^n_{K_n-1}+1}{a^n_{K_n}-a^n_{K_n-1}}\right) \\
&\quad + \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n-1}}{c^n_{i^n_1}-b^n_{K_n-1}} - \frac{\beta}{\pi}\ln\left(c^n_{i^n_2}-b^n_{K_n-1}+1\right) - \frac{2\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|\right) \\
&\quad - \frac{\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n},c^n_{L_n}\right)\cap III^+_{\delta,n}\right|\right) - \frac{\beta}{\pi}\ln\frac{b^n_{K_n-1}}{a^n_{K_n-1}} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} - C + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right| \\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i^n_2}-a^n_{K_n-2}}{b^n_{K_n-2}} + \frac{\beta}{\pi}\ln\left(\frac{c^n_{i^n_2}-a^n_{K_n-1}}{c^n_{i^n_2}-b^n_{K_n-2}}\right) + \frac{\beta}{\pi}\ln\left(\frac{a^n_{K_n}-b^n_{K_n-2}}{c^n_{i^n_1}-b^n_{K_n-2}}\right) + \frac{\beta}{\pi}\ln\left(\frac{a^n_{K_n}-b^n_{K_n-1}+1}{c^n_{i^n_1}-b^n_{K_n-1}}\right) \\
&\quad - \frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}} - \frac{\beta}{\pi}\ln\frac{b^n_{K_n-1}}{a^n_{K_n-1}} - C.
\end{aligned}
$$

Taking liminf on both sides, we get

$$\liminf \frac{c^n_{i^n_2}}{b^n_{K_n-1}} = \limsup \frac{a^n_{K_n-2}}{b^n_{K_n-2}} = 1.$$

If $\liminf_{n\to\infty} \frac{a^n_{K_n}-b^n_{K_n-1}+1}{c^n_{i^n_1}-b^n_{K_n-1}} = 0$, then $\limsup_{n\to\infty} \frac{c^n_{i^n_1}-a^n_{K_n}}{c^n_{i^n_1}-b^n_{K_n-1}} = 1$. We can modify the argument above by replacing $\left(a^n_{K_n-1}, b^n_{K_n-1}\right) \cup \left(a^n_{K_n}, b^n_{K_n}\right)$ by $\left(a^n_{K_n-1}, b^n_{K_n}\right)$ and same conclusion follows.

Case III-ii: $\left(b^n_{K_n-2}, a^n_{K_n-1}\right) \cap II^+_{\delta,n} \neq \emptyset$. There are two cases.

Case III-ii-a: There exists $i^n_3 \in \{j^n_3,\ j^n_3+1,\ \cdots, j^n_2-1\}$ **such that**

$$c^n_{i^n_3} - b^n_{K_n-2} < 1. \tag{2.52}$$

We bound $J_\delta^+(u_n)$ as follows.

$$\begin{aligned}
-n &> J_\delta^+(u_n)\\
&\geq \frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{\left(b^n_{K_n-2},a^n_{K_n-1}\right)\cap II^+_{\delta,n}}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx+\frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{\left(b^n_{K_n-1},a^n_{K_n}\right)\cap II^+_{\delta,n}}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx\\
&\quad+\frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{\left(b^n_{K_n}\infty\right)\cap II^+_{\delta,n}}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{I^+_{\delta,n}}\int_{I^-_{\delta,n}}\frac{\beta}{\pi(x-y)^2}dydx+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx\\
&\geq \frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}-1}\int_{c^n_{i_3^n}}^{a^n_{K_n-1}}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{d^n_{i_3^n}}^{d^n_{i_3^n}+\left|\left(b^n_{K_n-2},a^n_{K_n-1}\right)\cap III^+_{\delta,n}\right|}\frac{\beta}{\pi(x-y)^2}dydx\\
&\quad+\frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{c^n_{i_2^n}}^{a^n_{K_n}}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{d^n_{i_2^n}}^{d^n_{i_2^n}+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|}\frac{\beta}{\pi(x-y)^2}dydx\\
&\quad+\frac{\beta}{4\pi}\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{c^n_{i_1^n}}^{\infty}\frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx-\int_{a^n_{K_n-2}}^{b^n_{K_n-2}}\int_{d^n_{i_1^n}}^{d^n_{i_1^n}+\left|\left(b^n_{K_n},c^n_{L_n}\right)\cap III^+_{\delta,n}\right|}\frac{\beta}{\pi(x-y)^2}dydx\\
&\quad-\frac{\beta}{\pi}\ln 2\frac{b^n_{K_n-2}}{a^n_1}-\frac{\beta}{\pi}\ln\frac{b^n_{K_n-1}}{a^n_{K_n-1}}-\frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}}+\frac{\gamma}{4}\int_1^\infty W(u_n)\,dx\\
&\geq \frac{\beta}{\pi}\ln\left(\frac{c^n_{i_3^n}-a^n_{K_n-2}}{c^n_{i_3^n}-b^n_{K_n-2}+1}\cdot\frac{a^n_{K_n-1}-b^n_{K_n-2}}{a^n_{K_n-1}-a^n_{K_n-2}}\right)-\frac{\beta}{\pi}\ln\left(\frac{d^n_{i_3^n}-a^n_{K_n-2}}{d^n_{i_3^n}-b^n_{K_n-2}}\cdot\frac{d^n_{i_3^n}+\left|\left(b^n_{K_n-2},a^n_{K_n-1}\right)\cap III^+_{\delta,n}\right|-b^n_{K_n-2}}{d^n_{i_3^n}+\left|\left(b^n_{K_n-2},a^n_{K_n-1}\right)\cap III^+_{\delta,n}\right|-a^n_{K_n-2}}\right)\\
&\quad+\frac{\beta}{\pi}\ln\left(\frac{c^n_{i_2^n}-a^n_{K_n-2}}{c^n_{i_2^n}-b^n_{K_n-2}}\cdot\frac{a^n_{K_n}-b^n_{K_n-2}}{a^n_{K_n}-a^n_{K_n-2}}\right)-\frac{\beta}{\pi}\ln\left(\frac{d^n_{i_2^n}-a^n_{K_n-2}}{d^n_{i_2^n}-b^n_{K_n-2}}\cdot\frac{d^n_{i_2^n}+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|-b^n_{K_n-2}}{d^n_{i_2^n}+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|-a^n_{K_n-2}}\right)\\
&\quad+\frac{\beta}{\pi}\ln\frac{c^n_{i_1^n}-a^n_{K_n-2}}{c^n_{i_1^n}-b^n_{K_n-2}}-\frac{\beta}{\pi}\ln\left(\frac{d^n_{i_1^n}-a^n_{K_n-2}}{d^n_{i_1^n}-b^n_{K_n-2}}\cdot\frac{d^n_{i_1^n}+\left|\left(b^n_{K_n},c^n_{L_n}\right)\cap III^+_{\delta,n}\right|-b^n_{K_n-2}}{d^n_{i_1^n}+\left|\left(b^n_{K_n},c^n_{L_n}\right)\cap III^+_{\delta,n}\right|-a^n_{K_n-2}}\right)\\
&\quad-\frac{\beta}{\pi}\ln 2\frac{b^n_{K_n-2}}{a^n_1}-\frac{\beta}{\pi}\ln\frac{b^n_{K_n-1}}{a^n_{K_n-1}}-\frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}}-C+\frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right|\\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i_3^n}-a^n_{K_n-2}}{b^n_{K_n-2}}+\frac{\beta}{\pi}\ln\left(\frac{a^n_{K_n-1}-b^n_{K_n-2}}{a^n_{K_n-1}-a^n_{K_n-2}}\cdot\frac{c^n_{i_2^n}-a^n_{K_n-2}}{c^n_{i_2^n}-b^n_{K_n-2}}\right)+\frac{\beta}{\pi}\ln\left(\frac{c^n_{i_1^n}-a^n_{K_n-2}}{c^n_{i_1^n}-b^n_{K_n-2}}\cdot\frac{a^n_{K_n}-b^n_{K_n-2}}{a^n_{K_n}-a^n_{K_n-2}}\right)\\
&\quad-\frac{\beta}{\pi}\ln\left(c^n_{i_3^n}-b^n_{K_n-2}+1\right)-\frac{\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n-2},a^n_{K_n-1}\right)\cap III^+_{\delta,n}\right|\right)-\frac{\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n-1},a^n_{K_n}\right)\cap III^+_{\delta,n}\right|\right)\\
&\quad-\frac{\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n},c^n_{L_n}\right)\cap III^+_{\delta,n}\right|\right)-\frac{\beta}{\pi}\ln\frac{b^n_{K_n-1}}{a^n_{K_n-1}}-\frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}}-C+\frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right|\\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i_3^n}-a^n_{K_n-2}}{b^n_{K_n-2}}+\frac{\beta}{\pi}\ln\left(\frac{a^n_{K_n}-b^n_{K_n-2}}{c^n_{i_1^n}-b^n_{K_n-2}}\cdot\frac{a^n_{K_n-1}-b^n_{K_n-2}}{c^n_{i_2^n}-b^n_{K_n-2}}\right)-\frac{\beta}{\pi}\ln\frac{b^n_{K_n}}{a^n_{K_n}}-\frac{\beta}{\pi}\ln\frac{b^n_{K_n-1}}{a^n_{K_n-1}}-C. \qquad (2.53)
\end{aligned}$$

Taking liminf on both sides, we conclude

$$\liminf\frac{c^n_{i_3^n}}{b^n_{K_n-2}}=\limsup\frac{a^n_{K_n-2}}{b^n_{K_n-2}}=1.$$

Case III-ii-b: no such i_3^n satisfying (2.52) **exists.** Then we must have

$$c^n_{j_3^n}-b^n_{K_n-2}\geq 1 \qquad (2.54)$$

In this case, we can estimate $J_\delta^+(u_n)$ in the same way as case III-i with $c_{i_3^n}^n$ replaced by $c_{j_3^n}^n$ and conclude

$$\liminf \frac{c_{j_3^n}^n}{b_{K_n-2}^n} = \limsup \frac{a_{K_n-2}^n}{b_{K_n-2}^n} = 1.$$

Continuing this way, we conclude that $J_\delta^+(u_n) \to -\infty$ implies

$$\limsup \frac{a_{K_n}^n}{b_{K_n}^n} = \limsup \frac{a_{K_n-1}^n}{b_{K_n-1}^n} = \cdots = \limsup \frac{a_1^n}{b_1^n} = 1.$$

We now pick our subsequence as follows. Pick our first subsequence $\{u_n\}$ such that for its decomposition

$$\ln \frac{a_{K_n}^n}{b_{K_n}^n} < \frac{1}{2} \text{ for all } n.$$

Next we pick a subsequence of the chosen subsequence such that

$$\ln \frac{a_{K_n-1}^n}{b_{K_n-1}^n} < \frac{1}{4} \text{ for all } n.$$

Continuing this way, we pick our final subsequence using a diagonal argument. For simplicity of notations, we use the original sequence. For the final subsequence, we have for all n,

$$\ln \frac{a_{K_n}^n}{b_{K_n}^n} < \frac{1}{2}, \ln \frac{a_{K_n-1}^n}{b_{K_n-1}^n} < \frac{1}{4}, \cdots, \ln \frac{a_{K_n-l}^n}{b_{K_n-l}^n} < \frac{1}{2^l}, \cdots$$

it then follows that

$$\begin{aligned} \int_{I_{\delta,n}^+} \int_{I_{\delta,n}^-} \frac{4}{(x-y)^2} dy dx &\leq 4 \ln \prod_{i=1}^{K_n} \frac{a_i^n + \widetilde{d}_{\widetilde{L}}}{b_i^n + \widetilde{d}_{\widetilde{L}}} \cdot \frac{b_i^n + \widetilde{c}_1}{a_i^n + \widetilde{c}_1} \\ &\leq 4 \sum_{i=1}^{K_n} \ln \frac{b_i^n}{a_i^n} < 4, \end{aligned}$$

a contradiction to the assumption that $J_\delta^+(u_n) \to -\infty$. ■

3 Existence of minimizer of $J(u)$.

3.1 Overview

In this section, we prove the existence of a minimizer of $J(u)$. Observe that in general the boundedness of $J(u_n)$ does not imply the boundedness of $v_n = u_n - \eta$ in $H^1(\mathbb{R})$. A priori it is not clear whether we can obtain a suitable limit function from a minimizing sequence, using the direct method of calculus of variations. On the other hand, for a sequence $\{u_n\}$ satisfying $|u_n(x) + sgn(x)| \geq c_0 > 0$ outside a uniformly bounded interval, the boundedness of $J(u_n)$ implies the boundedness of $\{u_n - \eta\}$ in $H^1(\mathbb{R})$. Our main idea, therefore, is to show that we can replace a minimizing sequence $\{u_n\}$ by another one $\{\overline{u}_n\}$ which satisfies $|\overline{u}_n(x) + sgn(x)| \geq c_0 > 0$ for some constant c_0 outside a uniformly bounded interval. The new sequence $\{\overline{u}_n\}$ has uniformly bounded energy $J(\overline{u}_n)$, and $\{\overline{u}_n - \eta\}$ is bounded in $H^1(\mathbb{R})$. From this, we obtain a subsequence which weakly converges to a limit function $\overline{v} \in \mathcal{A}$ that achieves the minimum energy in $\mathcal{A}$. To construct the replacement sequence $\{\overline{u}_n\}$, we use the interval decompositions of u_n from the previous section. We first construct $\widetilde{u}_n$ by reflecting u_n over suitable regions. Keeping track of the energy contributions from each interval in the decomposition, we show that we can choose our regions of reflection so that the energy difference between $J(\widetilde{u}_n)$ and $J(u_n)$ is approaching zero as $n \to \infty$. Lastly, we define $\overline{u}_n$ by a suitable translation of $\widetilde{u}_n$ so that $|\overline{u}_n(x) + sgn(x)| \geq c_0 > 0$ for some constant c_0 outside a uniformly bounded interval. By periodic translation invariance of J and the above property of $J(\widetilde{u}_n)$, we conclude that $\{\overline{u}_n\}$ is another minimizing sequence.

We first state a translation invariant lemma.

Lemma 3.1 *Given any $c \in \mathbb{Z}$, let $u_c(x) = u(x+c)$, then $J(u_c(x)) = J(u(x))$.*

Proof. Since the first two terms are translation invariant for $c \in \mathbb{Z}$, $J(u_c) = J(u) + D(\eta_c, \eta)$, where

$$D(\eta_c, \eta) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{(\eta_c(x) - \eta_c(y))^2}{(x-y)^2} - \frac{(\eta(x) - \eta(y))^2}{(x-y)^2} \right) dydx.$$

By Lemma 2.1 in [27], we have $D(\eta_c, \eta) = 0$ for any constant c. The conclusion of the Lemma then follows. ■

3.2 Existence of a minimizer

Let $\{u_n\}$ be a minimizing sequence. By Lemma 2.2, we may assume that $u_n - \eta$ is compactly supported in $\mathbb{R}$. By Lemma 3.1 and our assumption on the behavior of $\{u_n\}$ at infinity, after a suitable translation by an integer there exists $c_n \in [0,1)$ such that $u_n(1+c_n) = 0$ and

$$u_n(x) \leq 0 \text{ for } x \leq 1 + c_n.$$

Throughout this section, we assume that on $[1+c_n, \infty)$, u_n has a decomposition

$$I^+_{\delta,n} = \bigcup_{i=1}^{K_n} \left[a_i^n,\ b_i^n\right] \tag{3.1}$$

and

$$II^+_{\delta,n} = \bigcup_{j=1}^{L_n} \left[c_j^n, d_j^n\right]. \tag{3.2}$$

Here we understand $d^n_{L_n} = \infty$. Throughout this section, we fix $\delta \ll 1$ such that $W''(u) \geq \frac{1}{4}W''(1)$ when $(1-u) \leq \delta$ and $W''(u) \geq \frac{1}{4}W''(-1)$ when $1+u \geq \delta$. Since $W(u) > 0$ for $u \in (-1,1)$, there exists $C_\delta > 0$ such that

$$W(u) \geq C_\delta (1-u)^2 \quad \text{when } 1+u \geq \delta \tag{3.3}$$

and

$$W(u) \geq C_\delta (1+u)^2 \quad \text{when } 1-u \geq \delta. \tag{3.4}$$

3.2.1 Case I: limsup $b^n_{K_n} < \infty$.

In this case, we prove the following proposition.

Proposition 3.2 *Let $\delta > 0$ be such that $W(u)$ satisfies* (3.3) *and* (3.4)*. Let u_n be a minimizing sequence for $J(u)$ in $\mathcal{A}_0$ with decompositions (3.1) and (3.2). If there exists a constant $M > 1$ such that $b^n_{K_n} < M$ for all n, then a subsequence of $\{u_n\}$ converges weakly to a minimizer u_0 of $J(u)$ in $\mathcal{A}$.*

Proof. Without loss of generality, we can assume $|u_n| \leq 1$. Consider $v_n := u_n - \eta$. Decomposition (3.1) and our assumption imply

$$u_n + 1 \geq \delta \text{ for } x \geq M \tag{3.5}$$

and

$$u_n - 1 \leq -1 \text{ for } x \leq 1 + c_n,\ \ c_n \in [0,1) \tag{3.6}$$

We write $J(u_n)$ in terms of v_n as follows:

$$\begin{aligned} J(u_n) &= \int_{\mathbb{R}} \left[\frac{\alpha}{2}|u_n'|^2 + g(x)W(u_n)\right]dx + \frac{\beta}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{(u_n(x)-u_n(y))^2}{(x-y)^2} - \frac{(\eta(x)-\eta(y))^2}{(x-y)^2}\right)dydx \\ &= \int_{\mathbb{R}} \left[\frac{\alpha}{2}|u_n'|^2 + g(x)W(u_n)\right]dx + \frac{\beta}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(v_n(x)-v_n(y))^2}{(x-y)^2}dydx \\ &\quad -\frac{\beta}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(v_n(x)-v_n(y))(\eta(x)-\eta(y))}{(x-y)^2}dydx. \end{aligned} \tag{3.7}$$

We have

$$\begin{aligned} &-\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(v_n(x)-v_n(y))(\eta(x)-\eta(y))}{(x-y)^2}dydx \\ =\ & -2\int_1^\infty\int_{-1}^1\frac{(v_n(x)-v_n(y))(\eta(x)-\eta(y))}{(x-y)^2}dydx - 2\int_1^\infty\int_{-\infty}^{-1}\frac{(v_n(x)-v_n(y))(\eta(x)-\eta(y))}{(x-y)^2}dydx \\ & -2\int_{-\infty}^{-1}\int_{-1}^1\frac{(v_n(x)-v_n(y))(\eta(x)-\eta(y))}{(x-y)^2}dydx - \int_{-1}^1\int_{-1}^1\frac{(v_n(x)-v_n(y))(\eta(x)-\eta(y))}{(x-y)^2}dydx \\ \geq\ & -4\int_1^\infty\frac{v_n(x)}{x+1}dx + 4\int_{-\infty}^{-1}\frac{v_n(y)}{1-y}dy - \frac{1}{4}\int_1^\infty\int_{-1}^1\frac{(v_n(x)-v_n(y))^2}{(x-y)^2}dydx \\ & -4\int_1^\infty\int_{-1}^1\frac{(\eta(x)-\eta(y))^2}{(x-y)^2}dydx - \frac{1}{4}\int_{-\infty}^{-1}\int_{-1}^1\frac{(v_n(x)-v_n(y))^2}{(x-y)^2}dydx \\ & -4\int_{-\infty}^{-1}\int_{-1}^1\frac{(\eta(x)-\eta(y))^2}{(x-y)^2}dydx - \frac{1}{4}\int_{-1}^1\int_{-1}^1\frac{(v_n(x)-v_n(y))^2}{(x-y)^2} - \int_{-1}^1\int_{-1}^1\frac{(\eta(x)-\eta(y))^2}{(x-y)^2} \\ \geq\ & -\varepsilon\int_1^\infty v_n^2(x)\,dx - \varepsilon\int_{-\infty}^{-1}v_n^2(y)\,dy - \frac{C}{\varepsilon} - C(\|\eta'\|_{L^\infty}) - \frac{1}{4}\int_1^\infty\int_{-1}^1\frac{(v_n(x)-v_n(y))^2}{(x-y)^2}dydx \\ & -\frac{1}{4}\int_{-\infty}^{-1}\int_{-1}^1\frac{(v_n(x)-v_n(y))^2}{(x-y)^2}dydx - \frac{1}{4}\int_{-1}^1\int_{-1}^1\frac{(v_n(x)-v_n(y))^2}{(x-y)^2}dydx. \end{aligned} \tag{3.8}$$

The last inequality in (3.8) follows from Hölder inequality and bounds on

$$\int_{-\infty}^{-1}\int_{-1}^1\frac{(\eta(x)-\eta(y))^2}{(x-y)^2}dydx, \qquad \int_1^\infty\int_{-1}^1\frac{(\eta(x)-\eta(y))^2}{(x-y)^2}dydx.$$

By $(3.3), (3.4), (3.7)$ and (3.8), we have

$$\begin{aligned}
C &\geq J(u_n) \geq \int_{\mathbb{R}} \frac{\alpha}{2} |u_n'|^2 dx + \int_1^{\infty} \gamma W(u_n) dx + \int_{-\infty}^{-1} \gamma W(u_n) dx \\
&\quad - \frac{\beta}{2\pi}\varepsilon \int_1^{\infty} v_n^2(x) dx - \frac{\beta}{2\pi}\varepsilon \int_{-\infty}^{-1} v_n^2(y) dy - \frac{C\beta}{2\pi\varepsilon} - C(\|\eta'\|_{L^\infty}) \\
&\quad + \frac{3\beta}{8\pi} \int_{-\infty}^{-1} \int_{-1}^{1} \frac{(v_n(x) - v_n(y))^2}{(x-y)^2} dydx + \frac{\beta}{4\pi} \int_{-\infty}^{-1} \int_{-\infty}^{-1} \frac{(v_n(x) - v_n(y))^2}{(x-y)^2} dydx \\
&\quad + \frac{\beta}{4\pi} \int_1^{\infty} \int_1^{\infty} \frac{(v_n(x) - v_n(y))^2}{(x-y)^2} dydx + \frac{\beta}{8\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(v_n(x) - v_n(y))^2}{(x-y)^2} dydx \\
&\quad + \frac{3\beta}{8\pi} \int_1^{\infty} \int_{-1}^{1} \frac{(v_n(x) - v_n(y))^2}{(x-y)^2} dydx + \frac{\beta}{2\pi} \int_{-\infty}^{-1} \int_1^{\infty} \frac{(v_n(x) - v_n(y))^2}{(x-y)^2} dydx \\
&\geq \frac{\alpha}{2} \int_{\mathbb{R}} |u_n'|^2 dx + C_{\delta,\gamma} \int_{\mathbb{R}} v_n^2 dx + \frac{\beta}{8\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v_n(x) - v_n(y))^2}{(x-y)^2} dydx - C(M, \|\eta'\|_{L^\infty}).
\end{aligned}$$

From this we conclude that v_n is bounded in $H^1(\mathbb{R})$, and, hence, there exists a subsequence $v_{n_k} \rightharpoonup v \in H^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v_{n_k}(x) - v_{n_k}(y))(\eta(x) - \eta(y))}{(x-y)^2} dydx \to \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{(x-y)^2} dydx.$$

Here the convergence above follows from the identity

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v_{n_k}(x) - v_{n_k}(y))(\eta(x) - \eta(y))}{(x-y)^2} dydx = 2 \int_{\mathbb{R}} v_{n_k}(x) \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{\eta(x) - \eta(y)}{|x-y|^2} dydx,$$

the fact that

$$\lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{\eta(x) - \eta(y)}{|x-y|^2} dy$$

lies in $L^2(\mathbb{R})$ (see the discussion around (4.5) in section 4) and the weak convergence of v_{n_k} in $L^2(\mathbb{R})$.

Let $u_0 = v + \eta$, then $u_0 \in \mathcal{A}$ and

$$\begin{aligned}
&\liminf J(u_n) \\
= &\liminf \left\{ \int_{\mathbb{R}} \left[\frac{\alpha}{2} |u_n'|^2 + g(x) W(u_n) \right] dx \right. \\
&\left. + \frac{\beta}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{(v_n(x) - v_n(y))^2}{(x-y)^2} - 2 \frac{(v_n(x) - v_n(y))(\eta(x) - \eta(y))}{(x-y)^2} \right] dydx \right\} \\
\geq &\int_{\mathbb{R}} \left[\frac{\alpha}{2} |u_0'|^2 + g(x) W(u_0) \right] dx + \frac{\beta}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2} dydx \\
&- \frac{\beta}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{(x-y)^2} dydx \\
= &J(u_0).
\end{aligned}$$

■

3.2.2 Case II: $\limsup b_{K_n}^n = \infty$.

We prove the following proposition.

Proposition 3.3 *If* $\limsup b^n_{K_n} = \infty$, *we can find a new minimizing sequence* $\{\overline{u}_n\}$ *and* $\overline{c}_n \in [0,1)$ *such that*

$$\overline{u}_n \leq 0 \quad \text{for} \quad x \leq 1 + \overline{c}_n$$

and

$$[1+\overline{c}_n,\ \infty) \cap I^+_{\delta,n} = \cup_{i=1}^{\overline{K}_n} [a^n_i, b^n_i]$$

with $\limsup \overline{b}^n_{K_n} < \infty$

Proof. Let $\rho^n_i < a^n_i$ be the biggest zero point of u_n that lies to the left of a^n_i and $\sigma^n_i > b^n_i$ be the smallest zero of u_n lying to the right of b^n_i. Set

$$A^+_n = \{x \geq 1 + c_n,\ u_n(x) \geq 0\},$$

and

$$A^-_n = \{x \geq 1 + c_n,\ u_n(x) \leq 0\},$$

We construct our replacement minimizing sequence $\{\widetilde{u}_n\}$ in two cases.

Case I: There exists $0 \leq l \leq K_n$, **such that** $\limsup_n \frac{a^n_{K_n-l}}{b^n_{K_n-l}} = \nu_l < 1$ and $\limsup_n \frac{a^n_{K_n-i}}{b^n_{K_n-i}} = 1$ **for all** $i = 0, 1, \cdots, l-1$.

The main idea to construct a replacement minimizing sequence in this case is to show that we must have $\limsup_n \left(a^n_{K_n-l} - b^n_1\right) < \infty$ and $\limsup_n \frac{a^n_1}{b^n_{K_n-l}} = 0$. We then reflect the positive part of u_n defined on $\left[1+c_n, \rho^n_{K_n-l}\right]$ to $-u_n$. It can be shown that the energy of the resulting function differs from initial minimizing sequence by a small amount, a suitable translation of this reflected minimizing sequence satisfies the assumption in Proposition 3.2 and we can obtain a limit function from this replacement minimizing sequence. To illustrate our main idea, we first assume $l = 0$.

Case I-i: $\limsup_n \frac{a^n_{K_n}}{b^n_{K_n}} = 0$.

By definition of ρ^n_i, σ^n_i, we have $\limsup_n \frac{\rho^n_{K_n}}{\sigma^n_{K_n}} = 0$. In this case, we consider the sequence $\{\widetilde{u}_n\}$ defined by

$$\widetilde{u}_n(x) = \begin{cases} -u_n(x) & x \in \left[1+c_n, \rho^n_{K_n}\right] \cap A^+_n \\ u_n(x) & \text{otherwise} \end{cases}. \tag{3.9}$$

Then

$$\begin{aligned} \frac{4\pi}{\beta}\left[J(u_n) - J(\widetilde{u}_n)\right] &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{(u_n(x) - u_n(y))^2}{(x-y)^2} - \frac{(\widetilde{u}_n(x) - \widetilde{u}_n(y))^2}{(x-y)^2}\right] dydx \\ &= -8 \int_{\left[1+c_n, \rho^n_{K_n}\right]\cap A^+_n} \int_{\mathbb{R}\setminus\left(\left[1+c_n,\rho^n_{K_n}\right]\cap A^+_n\right)} \frac{u_n(x)\, u_n(y)}{(x-y)^2} dydx \\ &\geq -8 \int_{1+c_n}^{\rho^n_{K_n}} \int_{\sigma^n_{K_n}}^{\infty} \frac{1}{(x-y)^2} dydx \\ &= -8 \ln \frac{\sigma^n_{K_n} - 1 - c_n}{\sigma^n_{K_n} - \rho^n_{K_n}}. \end{aligned} \tag{3.10}$$

Since

$$\limsup_n \frac{\rho^n_{K_n}}{\sigma^n_{K_n}} = 0,$$

(3.10) implies the subsequence of $\{\widetilde{u}_n\}$ is also a minimizing sequence. Let $s_n = \left[\sigma^n_{K_n}\right]$ be the largest integer smaller than $\sigma^n_{K_n}$. By periodic translation invariance of the energy, we define $\overline{u}_n(x) = \widetilde{u}_n(x + s_n - 1)$. Then $J(\overline{u}_n) = J(\widetilde{u}_n)$ and $\overline{u}_n$ satisfies

$$\overline{u}_n(x) \leq 0 \text{ for } x \leq 1 + \sigma^n_{K_n} - s_n,$$

and

$$\overline{u}_n(x) \geq -1+\delta \text{ for } x \geq 1+\sigma^n_{K_n} - s_n.$$

We conclude from Proposition 3.2 that $\overline{u}_n$ is bounded in $H^1(\mathbb{R})$, and $\overline{u}_n$ converges weakly to a minimizer in $\mathcal{A}$.

Case I-ii: $\limsup_n \frac{a^n_{K_n}}{b^n_{K_n}} = \nu < 1$. Then we must have $\limsup_n \left(a^n_{K_n} - b^n_1\right) < \infty$ and $\limsup_n \frac{a^n_1}{b^n_{K_n}} = 0$. In this case, we prove that we essentially get back to the same situation as case I-i, with $a^n_{K_n}$ replaced by a^n_1.

By estimates in section 2 (we use the same notations in this section)

$$\begin{aligned}
J^+_\delta(u_n) &= \frac{\beta}{4\pi}\int_{I^+_{\delta,n}}\int_{II^+_{\delta,n}} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx - \int_{I^+_{\delta,n}}\int_{I^-_{\delta,n}} \frac{\beta}{\pi(x-y)^2}dydx + \frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{4\pi}\int_{a^n_{K_n}}^{b^n_{K_n}}\int_{c^n_{i^n_1}}^{\infty} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx - \frac{\beta}{4\pi}\int_{a^n_{K_n}}^{b^n_{K_n}}\int_{d^n_{i^n_1}}^{d^n_{i^n_1}+\left|\left(b^n_{K_n},\infty\right)\cap III^+_{\delta,n}\right|} \frac{(u_n(x)-u_n(y))^2}{(x-y)^2}dydx \\
&\quad -\int_{I^+_{\delta,n}}\int_{I^-_{\delta,n}} \frac{\beta}{\pi(x-y)^2}dydx + \frac{\gamma}{4}\int_1^\infty W(u_n)\,dx \\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n}}{c^n_{i^n_1}-b^n_{K_n}} - \frac{\beta}{\pi}\ln\left(2\frac{b^n_{K_n}}{a^n_1}\right) + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right| - \frac{\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n},\infty\right)\cap III^+_{\delta,n}\right|\right) - C \\
&\geq \frac{\beta}{\pi}\ln\frac{c^n_{i^n_1}-a^n_{K_n}}{b^n_{K_n}} - C + \frac{\gamma\min_{|u|\leq 1-\delta}W(u)}{4}\left|III^+_{\delta,n}\right| - \frac{\beta}{\pi}\ln\left(c^n_{i^n_1}-b^n_{K_n}\right) \qquad (3.11) \\
&\quad -\frac{\beta}{\pi}\ln\left(1+\left|\left(b^n_{K_n},\infty\right)\cap III^+_{\delta,n}\right|\right).
\end{aligned}$$

Assuming $\limsup_n \frac{a^n_{K_n}}{b^n_{K_n}} = \nu$, (3.11) and the boundedness of $J^+_\delta(u_n)$ imply $\left|III^+_{\delta,n}\right| \leq C$. We construct $\widetilde{u}_n$ as follows.

$$\widetilde{u}_n = \begin{cases} -u_n(x) & x\in\left[1+c_n,\rho^n_{K_n}\right]\cap A^+_n \\ u_n(x) & \text{elsewhere} \end{cases}. \qquad (3.12)$$

We first show that there does not exist an s such that $\limsup_n \left(a^n_{K_n} - a^n_{s+1}\right) = A_s < \infty$, $\limsup_n \left(a^n_{K_n} - b_s\right) = \infty$.

Otherwise, letting $t_n = \left|\left[1+c_n, \rho^n_{K_n}\right] \cap A^+_n\right|$, $t^n_s = \left|\left[b^n_s, a^n_{s+1}\right] \cap II^+_{\delta,n}\right|$, we can write

$$\begin{aligned}
& \frac{4\pi}{\beta}\left[J(u_n) - J(\widetilde{u}_n)\right] \\
= \; & \int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{(u_n(x)-u_n(y))^2}{(x-y)^2} - \frac{(\widetilde{u}_n(x)-\widetilde{u}_n(y))^2}{(x-y)^2}\right]dydx \\
= \; & -8\int_{\left[1+c_n,\,\rho^n_{K_n}\right]\cap A^+_n}\int_{\mathbb{R}\setminus\left(\left[1+c_n,\rho^n_{K_n}\right]\cap A^+_n\right)}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
= \; & -8\int_{\left[1+c_n,\,\rho^n_{K_n}\right]\cap A^+_n}\int_{\sigma^n_{K_n}}^{\infty}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx - 8\int_{\left[1+c_n,\,\rho^n_{K_n}\right]\cap A^+_n}\int_{-\infty}^{1+c_n}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
& -8\int_{\left[1+c_n,\,\rho^n_{K_n}\right]\cap A^+_n}\int_{\left[1+c_n,\,\rho^n_{K_n}\right]\cap A^-_n}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx - 8\int_{\left[1+c_n,\,\rho^n_{K_n}\right]\cap A^+_n}\int_{\rho^n_{K_n}}^{\sigma^n_{K_n}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
\geq \; & -8\int_{\rho^n_{K_n}-t_n}^{\rho^n_{K_n}}\int_{\sigma^n_{K_n}}^{\infty}\frac{1}{(x-y)^2}dydx - 8\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap A^+_n}\int_{\rho^n_{K_n}}^{\sigma^n_{K_n}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
\geq \; & -8\int_{\rho^n_{K_n}-t_n}^{\rho^n_{K_n}}\int_{\sigma^n_{K_n}}^{\infty}\frac{1}{(x-y)^2}dydx - 8\int_{\left[b^n_s,a^n_{s+1}\right]\cap II^+_{\delta,n}}\int_{a^n_{K_n}}^{b^n_{K_n}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
\geq \; & -8\ln\frac{\sigma^n_{K_n}-\rho^n_{K_n}+t_n}{\sigma^n_{K_n}-\rho^n_{K_n}} + 8\int_{b^n_s}^{b^n_s+t^n_s}\int_{a^n_{K_n}}^{b^n_{K_n}}\frac{(1-\delta)^2}{(x-y)^2}dydx \\
= \; & -8\ln\frac{\sigma^n_{K_n}-\rho^n_{K_n}+t_n}{\sigma^n_{K_n}-\rho^n_{K_n}} \\
& +8(1-\delta)^2\ln\left(\frac{b^n_{K_n}-b^n_s-t^n_s}{b^n_{K_n}-b^n_s}\cdot\frac{a^n_{K_n}-b^n_s}{a^n_{K_n}-b^n_s-t_s}\right).
\end{aligned} \tag{3.13}$$

Recall that

$$\frac{t_n}{\sigma^n_{K_n}-\rho^n_{K_n}} \leq \frac{\rho^n_{K_n}}{\sigma^n_{K_n}-\rho^n_{K_n}}, \tag{3.14}$$

$$\frac{b^n_{K_n}-b^n_s-t^n_s}{b^n_{K_n}-b^n_s} \geq \frac{b^n_{K_n}-\rho^n_{K_n}}{b^n_{K_n}-b^n_s} \geq 1-\frac{\rho^n_{K_n}}{b^n_{K_n}}, \tag{3.15}$$

and

$$\begin{aligned}
\limsup_n\left(a^n_{K_n}-b^n_s-t^n_s\right) &= \limsup_n\left(a^n_{K_n}-a^n_{s+1}+a^n_{s+1}-b^n_s-t^n_s\right) \\
&\leq A_s + \left|\left[b^n_s, a^n_{s+1}\right]\cap III^+_{\delta,n}\right| \leq C.
\end{aligned} \tag{3.16}$$

Taking liminf on both sides of (3.13), we have

$$\liminf_n\left[J(u_n) - J(\widetilde{u}_n)\right] = \infty,$$

contradicting the assumption that $\{u_n\}$ is a minimizing sequence and the fact that J is bounded from below. Therefore we must have $\limsup_n\left(a^n_{K_n}-b^n_1\right) = a_0 < \infty$.

Next we show that $\limsup_n \frac{a^n_1}{b^n_{K_n}} = 0$. If $\limsup_n \frac{a^n_1}{b^n_{K_n}} > 0$ and $\kappa_n := \left|\left[1+c_n, \rho^n_{K_n}\right]\cap II^+_{\delta,n}\right|$, then

$$t_n - \kappa_n \leq \left|III^+_{\delta,n}\right| \leq C,$$

and

$$\begin{aligned}\kappa_n &= \rho^n_{K_n} - (1+c_n) - \left|\left[1+c_n, \rho^n_{K_n}\right] \cap I^+_{\delta,n}\right| \\ &\quad - \left|\left[1+c_n, \rho^n_{K_n}\right] \cap III^+_{\delta,n}\right| \\ &\leq a^n_{K_n} - (1+c_n) - (b^n_1 - a^n_1).\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{4\pi}{\beta}\left[J(u_n) - J(\widetilde{u}_n)\right] \\
= &\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{(u_n(x)-u_n(y))^2}{(x-y)^2} - \frac{(\widetilde{u}_n(x)-\widetilde{u}_n(y))^2}{(x-y)^2}\right]dydx \\
= &-8\int_{\left[1+c_n,\ \rho^n_{K_n}\right]\cap A^+_n}\int_{\mathbb{R}\setminus\left(\left[1,\ \rho^n_{K_n}\right]\cap A^+_n\right)}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
= &-8\int_{\left[1+c_n,\ \rho^n_{K_n}\right]\cap A^+_n}\int_{\sigma^n_{K_n}}^{\infty}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx - 8\int_{\left[1+c_n,\ \rho^n_{K_n}\right]\cap A^+_n}\int_{-\infty}^{1+c_n}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
&-8\int_{\left[1+c_n,\ \rho^n_{K_n}\right]\cap A^+_n}\int_{\left[1+c_n,\ \rho^n_{K_n}\right]\cap A^-_n}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx - 8\int_{\left[1+c_n,\ \rho^n_{K_n}\right]\cap A^+_n}\int_{\rho^n_{K_n}}^{\sigma^n_{K_n}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
\geq &-8\int_{\rho^n_{K_n}-t_n}^{\rho^n_{K_n}}\int_{\sigma^n_{K_n}}^{\infty}\frac{1}{(x-y)^2}dydx - 8\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap II^+_{\delta,n}}\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap I^+_{\delta,n}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx \\
&-8\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap II^+_{\delta,n}}\int_{a^n_{K_n}}^{b^n_{K_n}}\frac{u_n(x)\,u_n(y)}{(x-y)^2} \\
\geq &-8\ln\frac{\sigma^n_{K_n}-\rho^n_{K_n}+t_n}{\sigma^n_{K_n}-\rho^n_{K_n}} + 8(1-\delta)^2\int_{1+c_n}^{1+c_n+\kappa_n}\int_{a^n_{K_n}-b^n_1+a^n_1}^{b^n_{K_n}}\frac{1}{(x-y)^2} \\
= &-8\ln\frac{\sigma^n_{K_n}-\rho^n_{K_n}+t_n}{\sigma^n_{K_n}-\rho^n_{K_n}} \\
&+8(1-\delta)^2\ln\frac{b^n_{K_n}-1-c_n-\kappa_n}{b^n_{K_n}-1-c_n}\cdot\frac{a^n_{K_n}-b^n_1+a^n_1-1-c_n}{a^n_{K_n}-b^n_1+a^n_1-1-c_n-\kappa_n}.
\end{aligned} \tag{3.17}$$

Since

$$\limsup_n\frac{\sigma^n_{K_n}-\rho^n_{K_n}+t_n}{\sigma^n_{K_n}-\rho^n_{K_n}} \leq \limsup_n\frac{\sigma^n_{K_n}}{\sigma^n_{K_n}-\rho^n_{K_n}} < \infty, \tag{3.18}$$

$$\liminf_n\frac{b^n_{K_n}-1-c_n-\kappa_n}{b^n_{K_n}-1-c_n} \geq 1 - \limsup_n\frac{a^n_1-1-c_n+a^n_{K_n}-b^n_1}{b^n_{K_n}-1} > 1-\nu > 0, \tag{3.19}$$

and

$$a^n_{K_n} - b^n_1 + a^n_1 - 1 - c_n - \kappa_n \leq a_0 + \left|III^+_{\delta,n}\right| \leq C, \tag{3.20}$$

taking liminf on both sides of (3.17), it follows from (3.18), (3.19) and (3.20) that

$$\liminf_n\left(J(u_n) - J(\widetilde{u}_n)\right) = \infty,$$

a contradiction.

Lastly we estimate the energy difference between u_n and $\widetilde{u}_n$. We have

$$\begin{aligned}
&\frac{4\pi}{\beta}\left[J\left(u_n\right)-J\left(\widetilde{u}_n\right)\right]\\
=\ &\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{\left(u_n\left(x\right)-u_n\left(y\right)\right)^2}{\left(x-y\right)^2}-\frac{\left(\widetilde{u}_n\left(x\right)-\widetilde{u}_n\left(y\right)\right)^2}{\left(x-y\right)^2}\right]dydx\\
=\ &-8\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap A_n^+}\int_{\mathbb{R}\setminus\left(\left[1+c_n,\rho^n_{K_n}\right]\cap A_n^+\right)}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx\\
=\ &-8\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap A_n^+}\int_{\sigma^n_{K_n}}^{\infty}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx-8\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap A_n^+}\int_{-\infty}^{1+c_n}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx\\
&-8\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap A_n^+}\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap A_n^-}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx-8\int_{\left[1+c_n,\rho^n_{K_n}\right]\cap A_n^+}\int_{\rho^n_{K_n}}^{\sigma^n_{K_n}}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx\\
\geq\ &-8\int_{1+c_n}^{\rho_1^n}\int_{\sigma^n_{K_n}}^{\infty}\frac{1}{\left(x-y\right)^2}dydx-8\int_{\sigma_1^n}^{\rho^n_{K_n}}\int_{\sigma^n_{K_n}}^{\infty}\frac{1}{\left(x-y\right)^2}dydx\\
\geq\ &-8\ln\frac{\sigma^n_{K_n}-1-c_n}{\sigma^n_{K_n}-\rho_1^n}-8\ln\frac{\sigma^n_{K_n}-\rho^n_{K_n}+\rho^n_{K_n}-\sigma_1^n}{\sigma^n_{K_n}-\rho^n_{K_n}}. \qquad (3.21)
\end{aligned}$$

Recalling that

$$\limsup\frac{\rho^n_{K_n}}{\sigma^n_{K_n}}\leq\limsup\frac{a^n_{K_n}}{b^n_{K_n}}=\nu<1,\ \ \limsup\frac{\rho_1^n}{\sigma^n_{K_n}}\leq\limsup\frac{a_1^n}{b^n_{K_n}}=0,$$

and

$$\limsup\left(\rho^n_{K_n}-\sigma_1^n\right)\leq\limsup\left(a^n_{K_n}-b_1^n\right)=a_0<\infty,$$

we conclude that

$$\begin{aligned}
\limsup_n\frac{\sigma^n_{K_n}-\rho^n_{K_n}+\rho^n_{K_n}-\sigma_1^n}{\sigma^n_{K_n}-\rho^n_{K_n}} &= 1,\\
\liminf_n\frac{\sigma^n_{K_n}-1-c_n}{\sigma^n_{K_n}-\rho_1^n} &= 1.
\end{aligned}$$

Thus by (3.21) we have that $\{\widetilde{u}_n\}$ is also a minimizing sequence. Defining

$$\overline{u}_n=\widetilde{u}_n\left(x+s_n-1\right),$$

where $s_n=\left[\sigma^n_{K_n}\right]$ is the largest integer smaller than $\sigma^n_{K_n}$, then $\{\overline{u}_n\}$ is a minimizing sequence satisfying

$$\overline{u}_n\left(x\right)\leq 0\ \text{ for } x\leq 1+\sigma^n_{K_n}-s_n \text{ and } \overline{u}_n\left(x\right)\geq -1+\delta\ \text{ for } x>1+\sigma^n_{K_n}-s_n.$$

Proposition 3.2 applies to $\overline{u}_n$, from which we can extract a converging subsequence to a minimizer $u_0\in\mathcal{A}_0$.

Case I-iii: There exists $l>1$ **such that** $\limsup_n\frac{a^n_{K_n-j}}{b^n_{K_n-j}}=1$ for $j=0,1,\cdots,l-1$ **and** $\limsup_n\frac{a^n_{K_n-l}}{b^n_{K_n-l}}<1$. Then we must have $\limsup_n\left(a^n_{K_n-l}-b_1^n\right)<\infty$ and $\limsup_n\frac{a_1^n}{b^n_{K_n-l}}=0$.

We construct $\widetilde{u}_n$ as follows.

$$\widetilde{u}_n=\begin{cases}-u_n\left(x\right) & x\in\left[1+c_n,\rho_l^n\right]\cap A_n^+\\ u_n\left(x\right) & \text{elsewhere}\end{cases}. \qquad (3.22)$$

We also need

$$\int_{I_{\delta,n}^{+}}\int_{I_{\delta,n}^{-}}\frac{4}{(x-y)^{2}}dydx\leq 4\ln\left(\prod_{j=0}^{l-1}\frac{b_{K_n-j}^{n}}{a_{K_n-j}^{n}}\right)+4\ln\left(2\frac{b_{K_n-l}^{n}}{a_{1}^{n}}\right).$$

We can follow a similar argument in **Case I-i and Case I-ii** by replacing any estimates on $\left[a_{K_n}^{n},b_{K_n}^{n}\right]$ in (3.13) , (3.21) and (3.17) by estimates on $\left[a_{K_n-l}^{n},b_{K_n-l}^{n}\right]$, using $\overline{u}_n=\widetilde{u}_n\left(x+s_{n,l}-1\right)$, where $s_{n,l}=\left[\sigma_l^n\right]$ is the largest integer less than or equal to σ_l^n.

Case II: No such l exists, i.e., $\limsup_n \frac{a_j^n}{b_j^n}=1$ **for all** j **where** $\limsup_n b_j^n=\infty$. Let l be such that $\limsup_n b_{K_n-j}^n=\infty$ for all $j<l$ and $\limsup_n b_{K_n-l}^n<\infty$. In this case, we will reflect the negative part of u_n to $-u_n$ outside a big portion of $\left(b_{K_n-l}^{n},a_{K_n-l+1}^{n}\right)\cap II_{\delta,n}^{+}$. Following notations in section 2, we write

$$\left(b_{K_n-l}^{n},a_{K_n-l+1}^{n}\right)\cap II_{\delta,n}^{+}=\cup_{j=j_{l+1}^{n}}^{j_l^n-1}\left[c_j^n,\ d_j^n\right].$$

First by

$$\begin{aligned} C &> J_\delta^{+}\left(u_n\right)\geq-\int_{I_{\delta,n}^{+}}\int_{I_{\delta,n}^{-}}\frac{\beta}{\pi\left(x-y\right)^{2}}dydx+\frac{\gamma}{4}\int_{1}^{\infty}W\left(u_n\right)dx \\ &\geq -\frac{\beta}{\pi}\ln\left(2\frac{b_{K_n-l}^{n}}{a_1^n}\right)-\sum_{i=K_n-l+1}^{K_n}\frac{\beta}{\pi}\ln\frac{b_i^n}{a_i^n}+\frac{\gamma\min_{|u|\leq 1-\delta}}{4}\left|III_{\delta,n}^{+}\right| \end{aligned} \tag{3.23}$$

we conclude that

$$\left|III_{\delta,n}^{+}\right|\leq C.$$

Case II-i: There exists $j\left(l\right)\in\left\{j_{l+1}^{n},\cdots,j_l^n-1\right\}$ **such that**

$$\limsup_n\left(d_{j(l)}^{n}-c_{j(l)}^{n}\right)=\infty. \tag{3.24}$$

Let $T_{n,l}=\left[d_{j(l)}^{n},c_{L_n}^{n}\right]\cap A_n^{-}$, $M_{n,l}=\left|T_{n,l}\cap III_{\delta,n}^{+}\right|$. We define

$$\overline{u}_n\left(x\right)=\begin{cases}-u_n\left(x\right) & x\in\left[d_{j(l)}^{n},c_{L_n}^{n}\right]\cap A_n^{-}\\ u_n\left(x\right) & \text{otherwise}\end{cases}.$$

Since

$$\frac{u_n\left(y_1\right)}{\left(x-y_1\right)^{2}}\geq-\frac{u_n\left(y_2\right)}{\left(x-y_2\right)^{2}}$$

for $x\in T_{n,l}$, $y_1\in\left[c_{j(l)}^{n},d_{j(l)}^{n}\right]$ and $y_2\in\left[b_{K_n-l}^{n},\ c_{j(l)}^{n}\right]\cap A_n^{-}$, together with (3.24) and the fact that

$$\left|\left[b_{K_n-l}^{n},\ c_{j(l)}^{n}\right]\cap A_n^{-}\right|\leq\left|III_{\delta,n}^{+}\right|\leq C,$$

we conclude that

$$\int_{T_{n,l}}\int_{\left[b_{K_n-l}^{n},\ c_{j(l)}^{n}\right]\cap A_n^{-}}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^{2}}dydx+\int_{T_{n,l}}\int_{c_{j(l)}^{n}}^{d_{j(l)}^{n}}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^{2}}dydx\leq 0.$$

It then follows that

$$
\begin{aligned}
&\frac{4\pi}{\beta}\left[J\left(\overline{u}_n\right)-J\left(u_n\right)\right]\\
=\;& 8\int_{T_{n,l}}\int_{\mathbb{R}\setminus T_{n,l}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx\\
\leq\;& 8\int_{T_{n,l}}\int_{-\infty}^{c^n_{j(l)}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx+8\int_{T_{n,l}}\int_{\left[d^n_{j(l)},\,c^n_{L_n}\right]\cap A_n^+}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx\\
&+8\int_{T_{n,l}}\int_{c^n_{j(l)}}^{d^n_{j(l)}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx+8\int_{T_{n,l}}\int_{c^n_{L_n}}^{\infty}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx\\
\leq\;& 8\int_{T_{n,l}}\int_{-\infty}^{b^n_{K_n-l}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx+8\int_{T_{n,l}}\int_{\left[b^n_{K_n-l},\,c^n_{j(l)}\right]\cap A_n^-}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx\\
&+8\int_{T_{n.,l}}\int_{c^n_{j(l)}}^{d^n_{j(l)}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx\\
\leq\;& \int_{-\infty}^{c^n_{j(l)}}\int_{d^n_{j(l)}}^{M_{n,l}+d^n_{j(l)}}\frac{8}{(x-y)^2}dydx+\sum_{i=0}^{l-1}\int_{-\infty}^{b^n_{K_n-l}}\int_{\rho^n_{K_n-i}}^{\sigma^n_{K_n-i}}\frac{8}{(x-y)^2}dydx\\
&+8\int_{T_{n,l}}\int_{\left[b^n_{K_n-l},\,c^n_{j(l)}\right]\cap A_n^-}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx+8\int_{T_{n,l}}\int_{c^n_{j(l)}}^{d^n_{j(l)}}\frac{u_n(x)\,u_n(y)}{(x-y)^2}dydx\\
\leq\;& 8\ln\frac{M_{n,l}+d^n_{j(l)}-c^n_{j(l)}}{d^n_{j(l)}-c^n_{j(l)}}+\sum_{i=0}^{l-1}\ln\frac{\sigma^n_{K_n-i}-b^n_{K_n-l}}{\rho^n_{K_n-i}-b^n_{K_n-l}}\to 0,
\end{aligned}
$$

i.e., $\{\overline{u}_n\}$ is also a minimizing sequence with $\overline{b}^n_{K_n}=b^n_{K_n-l}$ satisfying $\limsup\overline{b}^n_{K_n}<\infty$.

Case II-ii:

$$\limsup_n\left[\max_{j^n_{l+1}\leq j\leq j^n_l-1}\left(d^n_j-c^n_j\right)\right]<\infty.$$

Let $S_0=\left\{\,j^n_{l+1}+1,\ \cdots,\ j^n_l-1\right\}$. We define S_0^+ as follows.

$$S_0^+=\left\{j\in S_0:d^n_j-c^n_j>c^n_j-d^n_{j-1}\right\}$$

There exists $k=k(n)$ and indices $p(1),\cdots,\ p(k)\in S_0$, $q(1),\cdots,q(k)\in S_0$ such that $i\in S_0^+$ if $p(s)\leq i\leq q(s)$ and $i\notin S_0^+$ if $q(s)\leq i\leq p(s+1)$ for $s=1,\cdots,k$. We write $S_1=\{1,\cdots,k\}$.

Case II-ii-1: There exists $s\leq k$ such that

$$\limsup_n\sum_{i=p(s)}^{q(s)}\left(d^n_i-c^n_i\right)\to\infty.$$

Let

$$
\begin{aligned}
T_{n,l,s} &= \left[d^n_{q(s)},c^n_{L_n}\right]\cap A_n^-\\
M_{n,l,s} &= \left|III^+_{\delta,n}\cap T_{n,l,s}\right|.
\end{aligned}
$$

We define

$$\overline{u}_n(x)=\begin{cases}-u_n(x) & x\in\left[d^n_{q(s)},c^n_{L_n}\right]\cap A_n^-\\ u_n(x) & \text{otherwise}\end{cases}.$$

Since

$$\left|\left(c_j^n, d_j^n\right)\right| > \left|\left(d_{j-1}^n, c_j^n\right)\right| \text{ for } p(s) \le j \le q(s), \ \cup_{j=p(s)}^{q(s)} \left(d_{j-1}^n, c_j^n\right) \subset III_{\delta,n}^+$$

and

$$\limsup_n \sum_{i=p(s)}^{q(s)} \left(d_i^n - c_i^n\right) \to \infty, \ \left|III_{\delta,n}^+\right| \le C,$$

together with the observation that

$$\frac{u_n(y_1)}{(x-y_1)^2} > -\frac{u_n(y_2)}{(x-y_2)^2}$$

for $x \in T_{n,l,s}$, $y_1 \in \left(c_j^n, d_j^n\right)$, $y_2 \in \left(d_{j-1}^n, c_j^n\right)$, $p(s) \le j \le q(s)$, and

$$\frac{u_n(y_1)}{(x-y_1)^2} > -\frac{u_n(y_2)}{(x-y_2)^2}$$

for $x \in T_{n,l,s}$, $y_1 \in \cup_{j=p(s)}^{q(s)} \left(c_j^n, d_j^n\right)$, $y_2 \in III_{\delta,n}^+ \cap \left(b_{K_n-l}^n, c_{p(s)}^n\right)$, we conclude that

$$\begin{aligned} & \int_{T_{n,l,s}} \int_{\left[b_{K_n-l}^n,\, c_{p(s)}^n\right] \cap A_n^-} \frac{u_n(x)\, u_n(y)}{(x-y)^2} dy dx + \sum_{j=p(s)}^{q(s)} \int_{T_{n,l,s}} \int_{c_j^n}^{d_j^n} \frac{u_n(x)\, u_n(y)}{(x-y)^2} dy dx \\ & + \sum_{j=p(s)}^{q(s)} \int_{T_{n,l,s}} \int_{d_{j-1}^n}^{c_j^n} \frac{u_n(x)\, u_n(y)}{(x-y)^2} dy dx \\ \le \ & 0. \end{aligned}$$

Therefore

$$
\begin{aligned}
& \frac{4\pi}{\beta}\left[J\left(\overline{u}_n\right)-J\left(u_n\right)\right] \\
= \quad & 8\int_{T_{n,l,s}}\int_{\mathbb{R}\setminus T_{n,l,s}}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx \\
= \quad & 8\int_{T_{n,l,s}}\int_{-\infty}^{d_{q(s)}^n}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx+8\int_{T_{n,l,s}}\int_{\left[d_{q(s)}^n,\ c_{L_n}^n\right]\cap A_n^+}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx \\
& +8\int_{T_{n,l,s}}\int_{c_{L_n}^n}^{\infty}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx \\
\leq \quad & 8\int_{T_{n,l,s}}\int_{-\infty}^{c_{p(s)}^n}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx+8\int_{T_{n,l,s}}\int_{c_{p(s)}^n}^{d_{q(s)}^n}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx \\
\leq \quad & 8\int_{T_{n,l,s}}\int_{-\infty}^{b_{K_n-l}^n}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx+8\int_{T_{n,l,s}}\int_{\left[b_{K_n-l}^n,\ c_{p(s)}^n\right]\cap A_n^-}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx \\
& +8\int_{T_{n,l,s}}\int_{c_{p(s)}^n}^{d_{q(s)}^n}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx \\
\leq \quad & \int_{-\infty}^{c_{p(s)}^n}\int_{d_{q(s)}^n}^{M_{n,l,s}+d_{q(s)}^n}\frac{8}{\left(x-y\right)^2}dydx+\sum_{i=0}^{l-1}\int_{-\infty}^{b_{K_n-l}^n}\int_{\rho_{K_n-i}^n}^{\sigma_{K_n-i}^n}\frac{8}{\left(x-y\right)^2}dydx \\
& +8\int_{T_{n,l,s}}\int_{\left[b_{K_n-l}^n,\ c_{p(s)}^n\right]\cap A_n^-}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx+8\sum_{j=p(s)}^{q(s)}\int_{T_{n,l,s}}\int_{c_j^n}^{d_j^n}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx \\
& +8\sum_{j=p(s)}^{q(s)}\int_{T_{n,l,s}}\int_{d_{j-1}^n}^{c_j^n}\frac{u_n\left(x\right)u_n\left(y\right)}{\left(x-y\right)^2}dydx \\
\leq \quad & \int_{-\infty}^{c_{p(s)}^n}\int_{d_{q(s)}^n}^{M_{n,l,s}+d_{q(s)}^n}\frac{8}{\left(x-y\right)^2}dydx+\sum_{i=0}^{l-1}\int_{-\infty}^{b_{K_n-l}^n}\int_{\rho_{K_n-i}^n}^{\sigma_{K_n-i}^n}\frac{8}{\left(x-y\right)^2}dydx \\
\leq \quad & 8\ln\frac{M_{n,l,s}+d_{q(s)}^n-c_{p(s)}^n}{d_{q(s)}^n-c_{p(s)}^n}+8\sum_{i=0}^{l-1}\ln\frac{\sigma_{K_n-i}^n-b_{K_n-l}^n}{\rho_{K_n-i}^n-b_{K_n-l}^n} \\
\to \quad & 0
\end{aligned}
$$

Case II-ii-2:

$$\sup_s\left(\limsup_n\sum_{i=p(s)}^{q(s)}\left(d_i^n-c_i^n\right)\right)<\infty.$$

We consider

$$S_1^+=\left\{\alpha\in S_1:\sum_{i=p(\alpha)}^{q(\alpha)}\left(d_i^n-c_i^n\right)>\sum_{i=1+q(\alpha-1)}^{q(\alpha)}\left(c_i^n-d_{i-1}^n\right)\right\}.$$

There exists $m=m\left(n\right)\leq k$ and $p_1\left(\tau\right),q_1\left(\tau\right)\in S_1$ for each $\tau\leq m$ and $\tau\in\mathbb{N}$ such that $\gamma\in S_1^+$ if $p_1\left(\tau\right)\leq\gamma\leq q_1\left(\tau\right)$ and $\gamma\notin S_1^+$ if $q_1\left(\tau\right)\leq\gamma\leq p_1\left(\tau+1\right)$. Let $S_2=\{1,\cdots,m\}$

Case II-ii-2-a: There exists $\tau=\tau\left(n\right)$ such that

$$\limsup_n\sum_{\gamma=p_1(\tau)}^{q_1(\tau)}\sum_{i=p(\gamma)}^{q(\gamma)}\left(d_i^n-c_i^n\right)\to\infty. \tag{3.25}$$

Then we consider

$$\overline{u}_n(x) = \begin{cases} -u_n(x) & x \in \left[d^n_{q(q_1(\tau))}, c^n_{L_n}\right] \cap A^- \\ u_n(x) & \text{otherwise} \end{cases}$$

Let $T_{n,l,q_1(\tau)} = \left[d^n_{q(q_1(\tau))}, \sigma^n_{K_n}\right] \cap A^-$. $M_{n,l,q_1(\tau)} = \left|\left[d^n_{q(q_1(\tau))}, \sigma^n_{K_n}\right] \cap A^- \cap III^+_{\delta,n}\right|$. Observe

$$\frac{u_n(y_1)}{(x-y_1)^2} \geq -\frac{u_n(y_2)}{(x-y_2)^2}$$

for $x \in T_{n,l,q_1(\tau)}$, $y_1 \in \left(c^n_j, d^n_j\right)$, $y_2 \in \left(d^n_{j-1}, c^n_j\right)$ when $p(p_1(\tau)) \leq j \leq q(q_1(\tau))$. The same inequality also holds for $x \in T_{n,l,q_1(\tau)}$, $y_1 \in \cup_{j=p(p_1(\tau))}^{q(q_1(\tau))} \left(c^n_j, d^n_j\right)$, $y_2 \in \left[b^n_{K_n-l},\ c^n_{p(p_1(\tau))}\right] \cap A^-_n$. Moreover, by (3.25),

$$\left|\left[b^n_{K_n-l},\ c^n_{p(p_1(\tau))}\right] \cap A^-_n\right| \leq \left|III^+_{\delta,n}\right| \leq C,$$

and

$$\cup_{j=p(p_1(\tau))}^{q(q_1(\tau))} \left(d^n_{j-1}, c^n_j\right) \subset III^+_{\delta,n},$$

we conclude that

$$\begin{aligned}
&\int_{T_{n,l,q_1(\tau)}} \int_{\left[b^n_{K_n-l},\ c^n_{p(p_1(\tau))}\right] \cap A^-_n} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx + \sum_{j=p(p_1(\tau))}^{q(q_1(\tau))} \int_{T_{n,l,q_1(\tau)}} \int_{c^n_j}^{d^n_j} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx \\
&+ \sum_{j=p(p_1(\tau))}^{q(q_1(\tau))} \int_{T_{n,l,q_1(\tau)}} \int_{d^n_{j-1}}^{c^n_j} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx \leq 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{4\pi}{\beta}\left[J(\overline{u}_n) - J(u_n)\right] \\
= \ & 8\int_{T_{n,l,q_1(\tau)}} \int_{\mathbb{R} \setminus T_{n,l,q_1(\tau)}} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx \\
= \ & 8\int_{A^-_2} \int_{-\infty}^{d^n_{q(q_1(\tau))}} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx + 8\int_{T_{n,l,q_1(\tau)}} \int_{\left[d^n_{q(q_1(\tau))},\ c^n_{L_n}\right] \cap A^+_n} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx \\
& +8\int_{T_{n,l,q_1(\tau)}} \int_{c^n_{L_n}}^{\infty} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx \\
\leq \ & 8\int_{T_{n,l,q_1(\tau)}} \int_{-\infty}^{b^n_{K_n-l}} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx + 8\int_{T_{n,l,q_1(\tau)}} \int_{b^n_{K_n-l}}^{d^n_{q(q_1(\tau))}} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx \\
\leq \ & \int_{-\infty}^{c^n_{p(p_1(\tau))}} \int_{d^n_{q(q_1(\tau))}}^{M_{n,l,q_1(\tau)}+d^n_{q(q_1(\tau))}} \frac{8}{(x-y)^2} dydx + \sum_{i=0}^{l-1} \int_{-\infty}^{b^n_{K_n-l}} \int_{\rho^n_{K_n-i}}^{\sigma^n_{K_n-i}} \frac{8}{(x-y)^2} dydx \\
& +8\int_{T_{n,l,q_1(\tau)}} \int_{\left[b^n_{K_n-l},\ c^n_{p(p_1(\tau))}\right] \cap A^-_n} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx + 8\sum_{j=p(p_1(\tau))}^{q(q_1(\tau))} \int_{T_{n,l,q_1(\tau)}} \int_{c^n_j}^{d^n_j} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx \\
& +8\sum_{j=p(p_1(\tau))}^{q(q_1(\tau))} \int_{T_{n,l,q_1(\tau)}} \int_{d^n_{j-1}}^{c^n_j} \frac{u_n(x)\,u_n(y)}{(x-y)^2} dydx \\
\leq \ & 8\ln \frac{M_{n,l,q_1(\tau)} + d^n_{q(q_1(\tau))} - c^n_{p(p_1(\tau))}}{d^n_{q(q_1(\tau))} - c^n_{p(p_1(\tau))}} + 8\sum_{i=0}^{l-1} \ln \frac{\sigma^n_{K_n-i} - b^n_{K_n-l}}{\rho^n_{K_n-i} - b^n_{K_n-l}} \\
\to \ & 0.
\end{aligned}$$

Continuing this way if necessary, we can define the set S_i inductively by each $m \in S_i$, $p_i(m), q_i(m) \in S_{i-1}$ such that any $p_i(m) \le p \le q_i(m)$, $p \in S_{i-1}^+$, if $q_i(m) \le p \le p_i(m+1)$, $p \in S_{i-1} \backslash S_{i-1}^+$. Here

$$S_i^+ := \left\{ m \in S_i : \sum_{l_i=p_i(m)}^{q_i(m)} \sum_{l_{i-1}=p_{i-1}(l_i)}^{q_{i-1}(l_i)} \cdots \sum_{l_1=p(l_2)}^{q(l_2)} \left(d_{l_1}^n - c_{l_1}^n\right) > \sum_{l_i=p_i(m)}^{q_i(m)} \sum_{l_{i-1}=p_{i-1}(l_i)}^{q_{i-1}(l_i)} \cdots \sum_{l_1=p(l_2)}^{q(l_2)} \left(c_{l_1}^n - d_{l_1-1}^n\right) \right\}.$$

By the definition of S_i, we have $|S_i| \le |S_{i-1}| \le \cdots \le |S_0|$. Since $\left|III_{\delta,n}^+\right|$ is uniformly bounded and $\limsup \left(a_{K_n-l+1}^n - b_{K_n-l}^n\right) = \infty$, we would be able to find r_n such that $S_{r_n} = S_{r_n}^+$. Let $\mu = |S_{r_n}|$ and define

$$\overline{u}_n(x) = \begin{cases} -u_n(x) & x \in \left[d_{q(q_1(\cdots q_{r_n}(\mu)))}^n, c_{L_n}^n\right] \cap A_n^- \\ u_n(x) & \text{otherwise} \end{cases}.$$

By a similar argument, we can show that $\{\overline{u}_n\}$ is a minimizing sequence which is close to ± 1 away from a uniformly bounded interval. ■

Proof of the first half of Theorem 1.1. Given a minimizing sequence $\{u_n\}$, if $\limsup_n b_{K_n}^n < \infty$ we obtain a minimizer by Proposition 3.2. If $\limsup b_{K_n}^n = \infty$, we obtain a new minimizing sequence $\{\overline{u}_n\}$ which satisfies $\limsup \overline{b}_{K_n}^n < \infty$ by Proposition 3.3, existence then follows from Proposition 3.2.

4 Regularity of the minimizers

Proof of the second half of Theorem 1.1.

Proposition 4.1 *Any minimizer u_0 of J over $\mathcal{A}$ is a $C^{2,\frac{1}{2}}(\mathbb{R})$ solution of*

$$-\alpha u_0'' + g(x) W'(u_0) + \beta \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_0 = 0,$$

where we understand the fractional operator in the sense of (1.4)*.*

Proof. Let $v_0 = u_0 - \eta$. We write $J(u_0)$ in terms of v_0 as

$$\begin{aligned} J(v_0+\eta) &= \frac{\alpha}{2}\int_{\mathbb{R}} |v_0' + \eta'|^2 dx + \int_{\mathbb{R}} g(x) W(v_0+\eta) dx + \frac{\beta}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v_0(x)-v_0(y))^2}{(x-y)^2} dydx \\ &\quad + \frac{\beta}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v_0(x)-v_0(y))(\eta(x)-\eta(y))}{(x-y)^2} dydx. \end{aligned}$$

Consider now variations $v_\varepsilon = v_0 + \varepsilon\varphi$, where φ is any smooth compactly supported function. Since u_0 is a minimizer, we must have

$$\begin{aligned} 0 &= \left.\frac{d}{d\varepsilon} J(v_\varepsilon + \eta)\right|_{\varepsilon=0} = \int_{\mathbb{R}} \left(\alpha u_0'\varphi' + g(x) W'(v_0+\eta)\varphi\right) dx \\ &\quad + \frac{\beta}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v_0(x)-v_0(y))(\varphi(x)-\varphi(y))}{(x-y)^2} dydx \\ &\quad + \frac{\beta}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\eta(x)-\eta(y))(\varphi(x)-\varphi(y))}{(x-y)^2} dydx. \end{aligned} \tag{4.1}$$

Since $v_0 \in H^1(\mathbb{R})$, we can define $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} v_0$ via Fourier transform as (see e.g. [29] Proposition 3.3)

$$\widehat{\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} v_0}(\xi) = |\xi| \widehat{v_0}(\xi),$$

and write the second term in (4.1) (see [29] Remark 3.7) as

$$\frac{\beta}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\left(v_0\left(x\right)-v_0\left(y\right)\right)\left(\varphi\left(x\right)-\varphi\left(y\right)\right)}{\left(x-y\right)^2}dydx=\beta\int_{\mathbb{R}}\varphi\left(x\right)\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}}v_0\left(x\right).$$

Since $\eta\in C^{\infty}\left(\mathbb{R}\right),$ for $x>1$ take $\varepsilon\ll 1$ such that

$$\int_{|x-y|\geq\varepsilon}\frac{\eta\left(x\right)-\eta\left(y\right)}{\left(x-y\right)^2}dy=\int_{-\infty}^{1}\frac{\eta\left(x\right)-\eta\left(y\right)}{\left(x-y\right)^2}dy\leq\frac{2}{x-1}, \tag{4.2}$$

and for $x<-1$ take $\varepsilon\ll 1$ such that

$$\int_{|x-y|\geq\varepsilon}\frac{\eta\left(x\right)-\eta\left(y\right)}{\left(x-y\right)^2}dy=\int_{-1}^{\infty}\frac{\eta\left(x\right)-\eta\left(y\right)}{\left(x-y\right)^2}dy\leq\frac{2}{x+1}. \tag{4.3}$$

For $-1\leq x\leq 1$, we can write

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{\eta\left(x\right)-\eta\left(y\right)}{\left(x-y\right)^2}dy &= \frac{1}{2}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\frac{\eta\left(x+y\right)+\eta\left(x-y\right)-2\eta\left(x\right)}{y^2}dy\\ &= \frac{1}{2}\int_{\mathbb{R}}\frac{\eta\left(x+y\right)+\eta\left(x-y\right)-2\eta\left(x\right)}{y^2}dy,\end{aligned}$$

where the last step follows from the fact that $\eta\in C^{\infty}\left(\mathbb{R}\right).$

For each $x\in\mathbb{R}$ we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\frac{\eta\left(x+y\right)+\eta\left(x-y\right)-2\eta\left(x\right)}{y^2}dy\right| &\leq \int_{1}^{\infty}\frac{4}{y^2}+\int_{-\infty}^{-1}\frac{4}{y^2}+\left|\int_{-1}^{1}\frac{\eta\left(x+y\right)+\eta\left(x-y\right)-2\eta\left(x\right)}{y^2}dy\right|\\ &\leq 8+2\left\|D^2\eta\right\|_{L^{\infty}}.\end{aligned} \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we conclude that the function

$$\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{\eta\left(x\right)-\eta\left(y\right)}{\left(x-y\right)^2}dy \tag{4.5}$$

belongs to $L^2(\mathbb{R})$. Thus the third term in (4.1) can be written as

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\left(\eta\left(x\right)-\eta\left(y\right)\right)\left(\varphi\left(x\right)-\varphi\left(y\right)\right)}{\left(x-y\right)^2}dydx &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x-y|\geq\varepsilon}\frac{\left(\eta\left(x\right)-\eta\left(y\right)\right)\left(\varphi\left(x\right)-\varphi\left(y\right)\right)}{\left(x-y\right)^2}dydx\\ &= 2\int_{\mathbb{R}}\varphi\left(x\right)\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{\eta\left(x\right)-\eta\left(y\right)}{\left(x-y\right)^2}dydx.\end{aligned}$$

We now introduce the notation

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}}u_0:=\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}}v_0+\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon}\frac{\eta\left(x\right)-\eta\left(y\right)}{\left(x-y\right)^2}dy,$$

where the fractional operator in the right-hand side is understood via Fourier transform. Since φ is arbitrary, we conclude from (4.1) that u_0 satisfies the following equation in the distributional sense:

$$-\alpha u_0''+g\left(x\right)W'\left(u_0\right)+\beta\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}}u_0=0. \tag{4.6}$$

Since $|u_0| \leq 1$ and $v_0 = u_0 - \eta \in H^1(\mathbb{R})$, we have $W'(u_0) \in L^2(\mathbb{R})$ and $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} v_0 \in L^2(\mathbb{R})$. Thus (4.5) implies that $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_0 \in L^2(\mathbb{R})$. By elliptic estimates, we then conclude that $u_0 \in W^{2,2}(\mathbb{R})$.

Weakly differentiating (4.6) yields

$$-\alpha u_{0x}'' + g'(x) W'(u_0) + g(x) W''(u_0) u_{0x} + \beta \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_{0x} = 0$$

in the sense of distributions. Here we used the facts that

$$\frac{d}{dx}\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} v_0 = \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} v_{0x} \tag{4.7}$$

and

$$\frac{d}{dx}\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \frac{\eta(x)-\eta(y)}{(x-y)^2}dy = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \frac{\eta'(x)-\eta'(y)}{(x-y)^2}dy, \tag{4.8}$$

which follow from the properties of Fourier transform of Sobolev functions and the following calculation:

$$\begin{aligned}
&\frac{d}{dx}\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \frac{\eta(x)-\eta(y)}{(x-y)^2}dy \\
=\; & \lim_{h\to 0}\frac{1}{h}\left[\lim_{\varepsilon\to 0}\int_{|x+h-y|\geq\varepsilon} \frac{\eta(x+h)-\eta(y)}{(x+h-y)^2}dy - \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \frac{\eta(x)-\eta(y)}{(x-y)^2}dy\right] \\
=\; & \lim_{h\to 0}\frac{1}{h}\left[\lim_{\varepsilon\to 0}\int_{|x-z|\geq\varepsilon} \frac{\eta(x+h)-\eta(z+h)}{(x-z)^2}dz - \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \frac{\eta(x)-\eta(y)}{(x-y)^2}dy\right] \\
=\; & \lim_{h\to 0}\frac{1}{h}\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \frac{\eta(x+h)-\eta(x)-\eta(y+h)+\eta(y)}{(x-y)^2}dy \\
=\; & \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \lim_{h\to 0}\frac{1}{h}\left[\frac{\eta(x+h)-\eta(x)-\eta(y+h)+\eta(y)}{(x-y)^2}\right]dy \\
=\; & \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon} \frac{\eta'(x)-\eta'(y)}{(x-y)^2}dy.
\end{aligned}$$

The same arguments as in the case of (4.5) can be used to prove that the function

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon} \frac{(\eta'(x)-\eta'(y))}{(x-y)^2}dy$$

belongs to $L^2(\mathbb{R})$ as well. Define

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_{0x} = \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} v_{0x} + \frac{d}{dx}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon} \frac{\eta'(x)-\eta'(y)}{(x-y)^2}dy.$$

Since $W \in C^{2,1}(\mathbb{R})$, we have $g'W'(u_0) + gW''(u_0)u_{0x} \in L^2(\mathbb{R})$, $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} v_{0x} \in L^2(\mathbb{R})$, we have $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_{0x} \in L^2(\mathbb{R})$. Thus elliptic estimates imply $u_{0x} \in W^{2,2}(\mathbb{R})$, i.e., $u_0 \in W^{3,2}(\mathbb{R}) \subset C^{2,\frac{1}{2}}(\mathbb{R})$. Thus u_0 is a classical solution of (4.6). Moreover, since $u_0 \in C^{2,\frac{1}{2}}(\mathbb{R})$, we can write

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_0(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon} \frac{u_0(x)-u_0(y)}{(x-y)^2}dy.$$

The second half of Theorem 1.1 follows. ■

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References

[1] S.B. Angenent, J. Mallet-Paret and L. A. Peletier: Stable transition layers in a semilinear boundary value problem. *J. Differential Equations.* **67** (1987), 212-242.

[2] G. Alberti, L. Ambrosio and X. Cabré: On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property. Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday. *Acta Appl. Math.* **65** (2001), no. 1-3, 9-33.

[3] F. Alessio, C. Gui and P Montecchiari: Saddle solutions to Allen-Cahn equations in doubly periodic media. *Indiana U. Math. J.* **65** (2016), no. 1, 199-221.

[4] F. Alessio, L. Jeanjean and P. Montecchiari: Stationary layered solutions in $\mathbb{R}^2$ for a class of non autonomous Allen-Cahn equations. *Calc. Var. Partial Differ. Equ.* **11** (2000), no. 11, 177-202.

[5] F. Alessio, L. Jeanjean and P. Montecchiari: Existence of infinitely many stationary layered solutions in $\mathbb{R}^2$ for a class of periodic Allen-Cahn equations. *Commun. Partial Differ. Equ.* **27** (2002), No. 7-8, 1537-1574.

[6] F. Alessio and P. Montecchiari: Entire solutions in $\mathbb{R}^2$ for a class of Allen-Cahn equations. *ESIAM: COCV* **11** (2005), 633-672.

[7] F. Alessio and P. Montecchiari: Brake orbits type solutions to some class of semiliear elliptic equations. *Calc. Var. Partial Differ. Equ.* **30** (2007), 51-83.

[8] F. Alessio and P. Montecchiari: Layered solutions with multiple asymptotes for non autonomous Allen-Cahn equations in $\mathbb{R}^3$. *Calc. Var. Partial Differ. Equ.* **46** (2013), 591-622.

[9] L. Ambrosio and X. Cabré: Entire solutions of semilinear elliptic equations in three dimensions and a conjecture of De Giorgi. *J. Amer. Math. Soc.* **13** (2000), no. 4, 725-739.

[10] V. Bangert: On minimal laminations on the torus. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **6** (1989), 95-138.

[11] M. Barlow, R. Bass and C. Gui: The Liouville property and a conjecture of De Giorgi. *Comm. Pure Appl. Math.* **53** (2000), no. 8, 1007-1038.

[12] H. Berestycki, F. Hamel and R. Monneau: One-dimensional symmetry for some bounded entire solution of some elliptic equations. *Duke Math. J.* **103** (2000), no. 3, 375-396.

[13] U. Bessi: Many solutions of elliptic problems on $\mathbb{R}^n$ of irrational slope. *Commun. Partial Differ. Equ.* **30** (2005) no. 12, 1773-1804.

[14] U. Bessi: Slope-changing solutions of elliptic problems on $\mathbb{R}^n$. *Nonlinear Anal. Theory Methods Appl.* **68** (2008), no. 12, 3923-3947.

[15] C. Bucur and E. Valdinoci: Nonlocal diffusion and applications. Lecture notes of the Unione Matematica Italiana, 20, Springer, 2016.

[16] J. Byeon and P. Rabinowitz: On a phase transition model. *Calc. Var. Partial Differ. Equ.* **47** (2013), 1-23.

[17] J. Byeon and P. Rabinowitz: Asymptotic behavior of minima and mountain pass solutions for a class of Allen-Cahn models. *Commun. Inf. Sysr.* **13** (2013), 79-95.

[18] J. Byeon and R. Rabinowitz: Solutions of higer topological type for an Allen-Cahn model equation. *J. Fixed Point Theory Appl.* **15** (2014), 379-404.

[19] J. Byeon and R. Rabinowitz: Unbounded solutions for a periodic phase transition model. *J. Differential Equations* **260** (2016), 1126-1153.

[20] X. Cabré and E. Cinti: Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian. *Discrete Contin. Dyn. Syst.* **28** (2010), no.3, 1179-1206.

[21] X. Cabré and E. Cinti: Sharp energy estimates for nonlinear fractional diffusion equations. *Cal. Var. Partial Diff. Equ.* **49** (2014), no. 1-2, 233-269.

[22] X. Cabré and J. Serra: An extension problem for sums of fractional Laplacians and 1-D symmetry of phase transitions. *Nonlinear Analysis.* **137** (2016), 246-265.

[23] X.Cabré and Y. Sire: Nonlinear equations for fractional Laplacians I: Regularity, Maximum Principles, and Hamiltonian estimates. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31** (2014), no.1, 23-53.

[24] X. Cabré and Y. Sire: Nonlinear equations for fractional Laplacians II; Existence, uniqueness, and qualitative properties of solutions. *Trans. Amer. Math. Soc.* **367** (2015), no. 2, 911-941.

[25] X. Cabré and J. Solà-Morales: Layer solutions in a half-space for boundary reactions. *Comm. Pure Appl. Math* **58** (2005), no. 12, 1678-1732.

[26] L. Caffarelli and L. Silvestre: An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* **32** (2007), no. 7-9, 1245-1260.

[27] K. Chen, C. Muratov and X. Yan: Layer solutions of a one-dimensional nonlocal model of Ginzburg-Landau type. *Math. Model. nat. Phenom.*, **12** (2017), no. 6, 68-90.

[28] M. del Pino, M. Kowalczyk and J. Wei: On De Giorgi's conjecture in dimension $N \geq 9$. Ann. of Math. **174** (2011), 1485-1569.

[29] E. Di Nezza, G. Palatucci and E. Valdinoci: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), no. 5, 521-573.

[30] W. Ding, F. Hamel and X. Zhao: Bistable pulsating fronts for reaction-diffusion equations in a periodic habitat. *Indiana Univ. Math. J.* **66** (2017), no. 4, 1189-1265.

[31] S. Dipierro, A. Farina and E. Valdinoci: A three-dimensional symmetry result for a phase transition equation in the genuinely nonlocal regime. *Calc. Var. Partial Differential Equations* **57** (2018), no. 1, Art 15, 21pp.

[32] S. Dipierro, J. Serra and E. Valdinoci: Improvement of flatness for nonlocal phase transitions. *Amer. J. Math.* **142** (2020), no. 4, 1083–1160.

[33] N. Dirr and N. K. Yip: Pinning and de-pinning phenomena in front propagation in heterogeneous media. *Interfaces and Free Boundaries* **8** (2006), 79-109.

[34] Z. Du, C. Gui, Y. Sire and J. Wei: Layered solutions for a fractional inhomogeneous Allen-Cahn equation. *Nonlinear Differ. Equ. Appl.* (2016) 23-29.

[35] A. Farina: Symmetry for solutions of semilinear elliptic equations in $\mathbb{R}^N$ and related conjecturs. *Ricerche di Matematica (in Memory of Ennio De Giorgi)* **48** (1999), 129-154.

[36] A. Figalli and J. Serra: On stable solution for boundary reactions: a De Giorgi type result in dimension 4+1. *Invent. Math.* **219** (2020), no. 1, 153–177.

[37] N. Ghoussoub and C. Gui: On a conjecture of De Giorgi and some related problems. *Math. Ann.* **311** (1998), no. 3, 481-491.

[38] N. Ghoussoub and C. Gui: On De Giorgi's conjecture in dimensions 4 and 5. *Ann. of Math.* (2) **157** (2003), no. 1, 313-334.

[39] J. Hale: Existence and stability of transition layers. *Japan J. Appl. Math.* **5** (1988), 367-405.

[40] Y. Hu: Layer solutions for a class of semilinear elliptic equations involving fractional Laplacians. *Boundary Value Probl.* 2014, 2014:41, 20 pp.

[41] J. P. Keener: Propagation of waves in an excitable medium with discrete release sites. *SIAM. J. Appl. Math.* **61** (2000), no. 1, 317-334.

[42] J. Lu, V. Moroz and C. B. Muratov: Orbital-free density functional theory of out-of-plane charge screening in graphene. *J. Nonlin. Sci.* **25** (2015), 1391-1430.

[43] J. Moser: Minimal solutions of variational problems on a torus. *AIHP Analyse Nonlinéaire* **3** (1986), 229-272.

[44] C. B. Muratov and X. Yan: Uniqueness of one-dimensional Néel wall profiles. *Proc. R. Soc. Lond. Ser. A* **472** (2016), 20150762.

[45] K. Nakashima: Stable transition layers in a balanced bistable equation. *Diff. Int. Equations* **13** (2000), 1025-1038.

[46] K. Nakashima: Multi-layered solutions for a spatially inhomogeneous Allen-Cahn equations. *J. Differential Equations* **191** (2003), 234-276.

[47] K. Nakashima and K. Tanaka: Clustering layers and boundary layers in spatially inhomogeneous phase transition problems. *Ann. Inst. H. Poincaré Anal. Nonlinéaire* **20** (2003), 107-143.

[48] G. Palatucci, O. Savin and E. Valdinoci: Local and global minimizers for a variational energy involving a fractional norm. *Ann. Mat. Pura Appl. (4)* **192** (2013), no. 4, 673-718.

[49] P. Rabinowitz: Single and multitransition solutions for a family of semilinear elliptic PDE's. *Milan J. Math.* **79** (2011), 113-127.

[50] P. Rabinowitz and E. Stredulinsky: Mixed states for an Allen-Cahn type equation. *Commun. Pure Appl. Math.* **56** (2003), no. 8, 1078-1134.

[51] P. Rabinowitz and E. Stredulinsky: Mixed states for an Allen-Cahn type equation II. *Calc. Var. Partial Differ. Equ.* **21** (2004), no. 2, 157-207.

[52] P. Rabinowitz and E. Stredulinsky: On some results of Moser and Bangert. *Ann. Inst. H. Poincaré Anal. Nonlinéaire* **21** (2004), 673-688.

[53] P. Rabinowitz and E. Stredulinsky: On some results of Moser and Bangert II. *Adv. Nonlinear Stud.* **4** (2004), 377-396.

[54] P. Rabinowitz and E. Stredulinsky: Extensions of Moser-Bangert Theory: Locally Minimal Solutions. Progress in Nonlinear Differential Equations and Their Applications, vol. 81, Birkhäuser, Boston (2011).

[55] O. Savin: Regularity of flat level sets in phase transitions. *Ann. of Math.* (2) **169** (2009), no. 1, 41-78.

[56] O. Savin: Some remarks on the classification of global solutions with asymptotically flat level sets. *Calc. Var. Partial Differential Equations* **56** (2017), no. 5, Art 141, 21pp.

[57] O. Savin: Rigidity of minimizers in nonlocal phase transitions. *Anal. PDE* **11** (2018), no. 8, 1881–1900. **11** (2018), no. 8, 1881–1900.

[58] Y. Sire and E. Valdinoci: Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result. *J. Funct. Anal* **256** (2009), no. 6, 1842-1864.

[59] P. R. Stinga: Fractional powers of second order partial differential operators: extension problem and regularity theory. *Ph. D. Thesis-Universidad Autónoma de Madrid.*

[60] J. Xin: Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media. *J. Sta. Phys.* **73** (1993), no. 5/6, 893-926.

[61] Z. Zhou: Existence of infinitely many solutions for a class of Allen-Cahn Equations in $\mathbb{R}^2$. *Osaka J. Math.* **48** (2011), 51-67.