Arch. Rational Mech. Anal. 243 (2022) 1773–1810
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-021-01738-0

# *Conducting Flat Drops in a Confining Potential*

Cyrill B. Muratov, Matteo Novaga & Berardo Ruffini

*Communicated by* A. Figalli

**Abstract**

We study a geometric variational problem arising from modeling two-dimensional charged drops of a perfectly conducting liquid in the presence of an external potential. We characterize the semicontinuous envelope of the energy in terms of a parameter measuring the relative strength of the Coulomb interaction. As a consequence, when the potential is confining and the Coulomb repulsion strength is below a critical value, we show existence and regularity estimates for volume-constrained minimizers. We also derive the Euler–Lagrange equation satisfied by regular critical points, expressing the first variation of the Coulombic energy in terms of the normal $\frac{1}{2}$-derivative of the capacitary potential.

## 1. Introduction

This paper is concerned with a geometric variational problem modeling charged liquid drops in two space dimensions, whose study was initiated in [30]. The problem in question arises in the studies of electrified liquids, and one of its main features is that the Coulombic repulsion of charges competes with the cohesive action of surface tension and tends to destabilize the liquid drop [13,33,37], an effect that is used in many concrete applications (see, e.g., [3,6,18]). From a mathematical point of view, this problem is interesting due to the competition between short-range attractive and long-range repulsive forces that produces non-trivial energy minimizing configurations and even nonexistence of minimizers when the total charge is large enough (for an overview, see [7]). The original model in three dimensions was proposed by Lord Rayleigh [33] and later investigated by many authors (see, for example, [4,5,9,15,17,20,21,29,37]; this list is not meant to be exhaustive).

In mathematical terminology, we are interested in the properties of the energy

$$E_\lambda(\Omega) := \mathcal{H}^1(\partial\Omega) + \lambda\, \mathcal{I}_1(\Omega) + \int_\Omega g(x)\,\mathrm{d}x, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a compact set with smooth boundary and prescribed area $|\Omega| = m$,

$$\mathcal{I}_1(\Omega) := \inf_{\mu \in \mathcal{P}(\Omega)} \int_\Omega \int_\Omega \frac{d\mu(x)\, d\mu(y)}{|x-y|}, \tag{1.2}$$

where $\mathcal{P}(\Omega)$ is the space of probability measures supported on $\Omega$, and $g$ is a continuous function. The function $\mathcal{I}_1(\Omega)$ is often referred to as the 1-Riesz capacitary energy of $\Omega$, and the right-hand side of (1.2) admits a unique minimizer $\mu_\Omega$, which is called the *equilibrium measure* of $\Omega$ [24]. Physically, the model describes the energetics of a charged conducting liquid drop sandwiched in a narrow space between the two parallel plates of a Hele-Shaw cell. The terms in (1.1) are, in the order of appearance: the surface energy contribution of the liquid meniscus between the plates, the self-interaction Coulombic energy due to the electric field created in the three-dimensional ambient space by a perfect conductor carrying a fixed charge, and the effect of a confining external potential that may be due to the spatial variations in the liquid-solid interfacial energy, a strong applied magnetic field, or a steady rotation of the cell when the conducting liquid is surrounded by a heavier insulating liquid. The first term in (1.1) acts as a cohesive term. In contrast, the second term is a capacitary term due to the presence of a charge and acts on the drop as a repulsive term. The parameter $\lambda > 0$ measures the relative strength of Coulombic repulsion. We refer to [30] for a more comprehensive derivation of the two-dimensional model, as well as for a deeper physical background.

A minimization problem for (1.1) must take into account the fine balance that exists between the surface and the capacitary term [30]. A rough prediction of the behavior of minimizers, when they exist, is that if $\lambda$ is big enough, then the drop will tend to be unstable, possibly leading to absence of a minimizer at all, while if $\lambda$ is small, the dominant term is the surface one, leading to existence and stability of energy minimizing drops in a suitable class of sets. One of the purposes of this paper is to make the above prediction precise.

We point out that the energy above is a particular case of the more general energy

$$\begin{aligned} E_{\lambda,\alpha,N}(\Omega) &:= \mathcal{H}^{N-1}(\partial\Omega) + \lambda \mathcal{I}_\alpha(\Omega) + \int_\Omega g(x)\,\mathrm{d}x, \\ \mathcal{I}_\alpha(\Omega) &= \inf_{\mu \in \mathcal{P}(\Omega)} \int_\Omega \int_\Omega \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha}, \end{aligned} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a compact set with smooth boundary and with prescribed Lebesgue measure $|\Omega| = m$, and $\alpha \in (0, N)$. For $g \equiv 0$, some mathematical analysis of the minimization problem associated with (1.3) has been carried out in [20,21], where it was shown that the problem is ill-posed for $\alpha < N-1$. Indeed, the nonlocal term $\mathcal{I}_\alpha(\Omega)$ is finite whenever the Hausdorff dimension of a compact set $\Omega$ is greater than $N-\alpha$. On the other hand, the Hausdorff measure $\mathcal{H}^{N-1}$ is trivially null on sets whose Hausdorff dimension is less than $N-1$. Thus, whenever a positive gap between $N-\alpha$ and $N-1$ occurs, it is possible to construct sets with $\mathcal{I}_\alpha$ positive and finite, but arbitrarily small $\mathcal{H}^{N-1}$-measure, ensuing non-existence of minimizers [20]. The existence of a minimizer for (1.3) in the case $g = 0$ and

$\alpha \geq N - 1$ is still open, except in the borderline case $N = 2$ and $\alpha = 1$ (see [30]). In this latter case, we showed that there exists an explicit threshold $\lambda = \lambda_c(m)$, where

$$\lambda_c(m) := \frac{4m}{\pi}, \tag{1.4}$$

such that for $\lambda > \lambda_c(m)$ no minimizer exists, while for $\lambda \leq \lambda_c(m)$ the only minimizer is a ball of measure $m$.

In this paper we address the question of the existence and qualitative properties of minimizers of (1.3) for $N = 2$ and $\alpha = 1$. To this end, in Theorem 1 we characterize the lower semicontinuous envelope of the energy $E_\lambda$ with respect to the $L^1$ topology. As a corollary, we show that the energy $E_\lambda$ is lower semicontinuous as long as $\lambda$ is below the precise threshold $\lambda_c(m)$, which is the same as the one for the case $g \equiv 0$. Then in Theorem 3 we prove, under a suitable coercivity assumption on $g$, the existence of volume-constrained minimizers for $E_\lambda$, as long as $\lambda < \lambda_c(m)$. Furthermore, in Theorem 4 we obtain density estimates and finiteness of the number of connected components of minimizers. Building on these regularity estimates, in Theorem 6 we consider the asymptotic regime $\lambda$, $m \to 0$, with $\limsup(\lambda/\lambda_c(m)) < 1$. In this limit the potential term is of lower order, and we show that minimizers, suitably rescaled, tend to a ball, which is the unique minimizer when $g = 0$ [30]. Finally, in Theorem 7 we compute the first variation of $\mathcal{I}_1$ and, as a consequence, we derive the Euler–Lagrange equation of the functional $E_\lambda$, for sufficiently smooth sets.

Notice that the proof of Theorem 7 is based on recent regularity improvements for nonlocal elliptic equations [34,35]. Such results are valid for more general classes of nonlocal equations than those considered in this paper. We suspect that the technique that we developed to obtain the first variation of the energy may also work for a large class of energies involving nonlocal elliptic operators in dimensions higher than two. Nevertheless, this may require overcoming some additional nontrivial difficulties, which, in particular, is why we decided not to pursue this calculation for $E_{\lambda,\alpha,N}$ in (1.3) with $\alpha \neq 1$ or $N \neq 2$.

Lastly, we point out that to obtain the results of Theorem 1 we crucially exploit the knowledge of the exact threshold, known only in dimension two, for which the ball minimizes the energy for $g \equiv 0$ (see [30]). Furthermore, to get the existence of minimizers in Theorem 3 we need to restrict to the class of compact sets with positive measure and with boundary of finite Hausdorff measure in order to apply first Blaschke Theorem and then Golab Theorem (see [2, Theorems 4.4.15 and 4.4.17]). Notice that the latter result can be applied to our class of compact sets only in dimension two.

## 2. Statement of the Main Results

As was mentioned earlier, the main difficulty in showing existence of minimizers for the variational problems above is that adding a surface term to a nonlocal capacitary term typically leads to an ill-posed problem. The strategy adopted in [30] to study the minimizers of (1.1) with $g = 0$ was to show directly a lower bound on the energy, given by that of a single ball, using some concentration compactness tools and some fine properties of the theory of convex bodies in dimension two. The presence of the bulk energy in (1.1) precludes application of these techniques.

The strategy of this paper is different: we first characterize the lower semicontinuous envelope of the functional $E_\lambda$, in a class of sets which includes compact sets with smooth boundary. Its explicit expression allows us to state that for certain values of $\lambda$, the energy $E_\lambda$ is lower semicontinuous with respect to the $L^1$ convergence. To state the main results of the paper, we introduce some notation. Given $m > 0$, we denote by $\mathcal{A}_m$ the class of all measurable subsets of $\mathbb{R}^2$ of measure $m$:

$$\mathcal{A}_m := \left\{ \Omega \subset \mathbb{R}^2 \; : \; |\Omega| = m \right\}. \tag{2.1}$$

We then introduce the families of sets

$$\mathcal{S}_m := \{ \Omega \in \mathcal{A}_m \; : \; \Omega \text{ compact, } \partial\Omega \text{ smooth} \}, \tag{2.2}$$

$$\mathcal{K}_m := \left\{ \Omega \in \mathcal{A}_m \; : \; \Omega \text{ compact, } \mathcal{H}^1(\partial\Omega) < +\infty \right\}. \tag{2.3}$$

We can extend the functional $E_\lambda$ defined in (1.1) over $\mathcal{S}_m$ to the whole of $\mathcal{K}_m$ by setting

$$E_\lambda(\Omega) := P(\Omega) + \lambda\, \mathcal{I}_1(\Omega) + \int_\Omega g(x)\, \mathrm{d}x, \qquad \Omega \in \mathcal{K}_m, \tag{2.4}$$

where $P(\Omega)$ denotes the De Giorgi perimeter of $\Omega$, defined as

$$P(\Omega) := \sup \left\{ \int_\Omega \operatorname{div} \phi \; \mathrm{d}x \; : \; \phi \in C_c^1(\mathbb{R}^2; \mathbb{R}^2), \; \|\phi\|_{L^\infty(\mathbb{R}^2)} \le 1 \right\}, \tag{2.5}$$

which coincides with $\mathcal{H}^1(\partial\Omega)$ if $\Omega \in \mathcal{S}_m$. Given an open set $A \subset \mathbb{R}^2$, we also define the perimeter of $\Omega$ in $A$ as

$$P(\Omega; A) := \sup \left\{ \int_{\Omega \cap A} \operatorname{div} \phi \; \mathrm{d}x \; : \; \phi \in C_c^1(A; \mathbb{R}^2), \; \|\phi\|_{L^\infty(A)} \le 1 \right\}, \tag{2.6}$$

so that, in particular, $P(\Omega) = P(\Omega; \mathbb{R}^2)$.

For $\Omega \in \mathcal{A}_m$ we introduce the $L^1$-relaxed energy for $E_\lambda$ restricted to $\mathcal{S}_m$:

$$\overline{E}_\lambda(\Omega) := \inf_{\Omega_n \in \mathcal{S}_m,\, |\Omega_n \Delta \Omega| \to 0} \; \liminf_{n \to \infty} E_\lambda(\Omega_n). \tag{2.7}$$

We observe that, as a consequence of Proposition 14 and Corollary 15 in the following section, we can equivalently define $\overline{E}_\lambda$ starting from sets $\Omega_n \in \mathcal{K}_m$ in (2.7), that is, there holds

$$\overline{E}_\lambda(\Omega) = \inf_{\Omega_n \in \mathcal{K}_m,\, |\Omega_n \Delta \Omega| \to 0} \; \liminf_{n \to \infty} E_\lambda(\Omega_n). \tag{2.8}$$

Our first result, below, provides an explicit characterization of the relaxed energy for sets in $\mathcal{K}_m$.

**Theorem 1.** *Let $g$ be a continuous function bounded from below and let $m > 0$. Then for any $\Omega \in \mathcal{K}_m$ we have $\overline{E}_\lambda(\Omega) = \mathcal{E}_\lambda(\Omega)$, where*

$$\mathcal{E}_\lambda(\Omega) := \begin{cases} E_\lambda(\Omega) & \text{if } \lambda \leq \lambda_\Omega, \\ E_{\lambda_\Omega}(\Omega) + 2\pi\left(\sqrt{\lambda} - \sqrt{\lambda_\Omega}\right) & \text{if } \lambda > \lambda_\Omega, \end{cases} \tag{2.9}$$

*and*

$$\lambda_\Omega := \left(\frac{\pi}{\mathcal{I}_1(\Omega)}\right)^2. \tag{2.10}$$

*Moreover, $E_\lambda$ is lower semicontinuous on $\mathcal{K}_m$ with respect to the $L^1$−convergence if and only if $\lambda \leq \lambda_c(m)$.*

We recall that the quantity $\lambda_c(m)$ is defined in (1.4).

Note that as can be easily seen from the definition of $\lambda_\Omega$, we have

$$E_\lambda(\Omega) \geq E_{\lambda_\Omega}(\Omega) + 2\pi(\sqrt{\lambda} - \sqrt{\lambda_\Omega}) \qquad \forall \lambda > 0; \tag{2.11}$$

see Lemma 16. Therefore, the result of Theorem 1 may be interpreted as follows: either it is energetically convenient to distribute all the charges over the set $\Omega$ or it is favorable to send some excess charge off to infinity. More precisely, for a given set $\Omega$ such that $\lambda > \lambda_\Omega$ it is possible to find a sequence of sets converging to $\Omega$ in the $L^1$ sense that contain vanishing parts with positive capacitary energy. In particular, the vanishing parts contribute a finite amount of energy to the limit, which is a non-trivial property of the considered problem.

The above result implies existence of minimizers for $\overline{E}_\lambda$ in $\mathcal{A}_m$, as long as we require the coercivity and the local Lipschitz continuity of the function $g$.

**Definition 2.** We say that a function $g : \mathbb{R}^2 \to \mathbb{R}$ is coercive if

$$\lim_{|x| \to +\infty} g(x) = +\infty, \tag{2.12}$$

Furthermore, we define the class of functions $\mathcal{G}$ as follows:

$$\mathcal{G} := \{g : \mathbb{R}^2 \to [0, +\infty) \ : \ g \text{ is locally Lipschitz continuous and coercive}\}. \tag{2.13}$$

Note that the assumption of positivity of $g$ in (2.13) is not essential and may be replaced by boundedness of $g$ from below. For this class of functions, which represent the effect of confinement by an external potential $g$, we have the following existence result.

**Theorem 3.** *Let $m > 0$, let $\lambda < \lambda_c(m)$ and let $g \in \mathcal{G}$. Then there exists a minimizer $\Omega_\lambda$ for $E_\lambda$ over all sets in $\mathcal{K}_m$.*

We stress that the existence result stated in Theorem 3 is not a direct consequence of Theorem 1, the reason being that the class $\mathcal{K}_m$ is not closed under $L^1$−convergence, and is in fact one of the main results of this paper. Furthermore, we notice that Theorem 3 does not include existence for $\lambda = \lambda_c(m)$, in contrast with the case when $g \equiv 0$. We believe it to be a technical limitation of our proof, and we conjecture that existence should hold also in this limiting case.

Given $\Omega \in \mathcal{K}_m$, we let $\Omega^+$ be defined as

$$\Omega^+ := \{x \in \Omega : \ |\Omega \cap B_r(x)| > 0 \text{ for all } r > 0\}. \tag{2.14}$$

Notice that $\Omega^+$ is a closed set. Indeed, recalling that $\Omega$ is closed we have that $x \in (\Omega^+)^c$ if and only if there exists $r > 0$ such that $|\Omega \cap B_r(x)| = 0$. Then, for every $y \in B_r(x)$ there holds $|\Omega \cap B_{r-|y-x|}(y)| = 0$, hence $y \in (\Omega^+)^c$, that is, $(\Omega^+)^c$ is open and $\Omega^+$ is closed. Furthermore, if $\Omega \in \mathcal{K}_m$, we have $\Omega^+ \subset \Omega$ and $\Omega \setminus \Omega^+ = \{x \in \Omega : \ |\Omega \cap B_r(x)| = 0 \text{ for some } r > 0\} \subset \partial\Omega$, so that

$$\mathcal{H}^1(\Omega \setminus \Omega^+) \le \mathcal{H}^1(\partial\Omega) < +\infty. \tag{2.15}$$

As a consequence, we get $|\Omega| = |\Omega^+|$, $P(\Omega) = P(\Omega^+)$. Moreover, since the Hausdorff dimension of $\Omega \setminus \Omega^+$ is at most 1, then $\mathcal{I}_1(\Omega) = \mathcal{I}_1(\Omega^+)$ (see Lemma 10 below). Therefore $E_\lambda(\Omega) = E_\lambda(\Omega^+)$, and $\Omega$ is a minimizer of $E_\lambda$ if and only if $\Omega^+$ is a minimizer. We observe that $\Omega^+$ is a representative of $\Omega$ which is in general more regular, and for which we can show density estimates which do not necessarily hold for $\Omega$ itself.

We now state a regularity estimate for the minimizers given in Theorem 3.

**Theorem 4.** *Let $m > 0$, $\lambda < \lambda_c(m)$ and $g \in \mathcal{G}$. Let also $\Omega_\lambda$ be a minimizer of $E_\lambda$ over $\mathcal{K}_m$. Then there exist $c > 0$ universal and $r_0 > 0$ depending only on $m$, $\lambda$ and $g$ such that for every $0 < r \le r_0$ and every $x \in \partial\Omega_\lambda^+$ there holds*

$$|\Omega_\lambda \cap B_r(x)| \ge c\left(1 - \frac{\lambda}{\lambda_c(m)}\right)^2 r^2 \quad \text{and} \quad |\Omega_\lambda^c \cap B_r(x)| \ge c\left(1 - \frac{\lambda}{\lambda_c(m)}\right)^2 r^2. \tag{2.16}$$

*Furthermore, both $\Omega_\lambda$ and $\Omega_\lambda^c$ have a finite number of indecomposable components in the sense of* [1, Section 4].

*Remark 5.* From Theorem 4 and [27, Theorem II.5.14] it follows that

$$\mathcal{H}^1(\partial\Omega_\lambda^+) = P(\Omega_\lambda^+) = P(\Omega_\lambda). \tag{2.17}$$

Therefore, the set $\Omega_\lambda^+$ also minimizes the energy $E_\lambda$ as defined in (1.1), among all sets in $\mathcal{K}_m$.

The semicontinuity of $E_\lambda$ allows us to get existence of minimizers for $\lambda < \lambda_c(m)$, but we cannot say much about their qualitative shape, besides the regularity estimate given in Theorem 4. On the other hand, for $m$ sufficiently small and $\lambda$ small relative to $\lambda_c(m)$ we can show that the minimizers become close to a single ball of mass $m$ located at a minimum of $g$.

**Theorem 6.** *Let $g \in \mathcal{G}$ and let $m_k$, $\lambda_k > 0$, $k \in \mathbb{N}$, be two sequences such that*

$$\lim_{k\to+\infty} m_k = 0 \quad \textit{and} \quad \limsup_{k\to+\infty} \frac{\lambda_k}{\lambda_c(m_k)} < 1. \tag{2.18}$$

*Then the following assertions are true:*

*(1) For every k large enough there exists a minimizer $\Omega_k$ of $E_{\lambda_k}$ over $\mathcal{K}_{m_k}$.*
*(2) As $k \to \infty$, there exists a bounded sequence $(x_k) \in \mathbb{R}^2$ such that the boundaries of the translated and rescaled minimizers $(\pi/m_k)^{\frac{1}{2}}(\Omega_k - x_k)$ converge to $\partial B_1(0)$ in the Hausdorff distance.*
*(3) If $x_0$ is a cluster point of $(x_k)$, then $x_0 \in \operatorname{argmin} g$.*

We note that in the local setting, i.e., when $\lambda = 0$, the result in Theorem 6 was obtained by Figalli and Maggi in [14], who in fact also obtained strong quantitative estimates of the rate of convergence of these minimizers to balls in this perimeter-dominated regime. This is made possible in the context of local isoperimetric problems with confining potentials by an extensive use of the regularity theory available for such problems [27]. In contrast, minimizers of our problem fail to be quasi-minimizers of the perimeter and, therefore, their $C^{1,\alpha}$-regularity is a difficult open question. The proof of Theorem 6, which extends some results of [14] to the nonlocal setting involving capacitary energies relies on the arguments used to obtain some regularity estimates of the minimizers in the subcritical regime in Theorem 4. These estimates are also the first step towards the full regularity theory of the minimizers of $E_\lambda$.

Finally, we derive the Euler–Lagrange equation for the energy $E_\lambda$ under some smoothness assumptions on the shape of the minimizer. The main issue here is to compute the first variation of the functional $\mathcal{I}_1(\Omega)$ with respect to the deformations of the set $\Omega$. To that end, given a compact set $\Omega$ with a sufficiently smooth boundary, we introduce the potential function

$$v_\Omega(x) := \int_\Omega \frac{d\mu_\Omega(y)}{|x-y|}, \tag{2.19}$$

where $\mu_\Omega$ is the equilibrium measure of $\Omega$ minimizing $\mathcal{I}_1$. The normal $\frac{1}{2}$-derivative of the potential of $v_\Omega$ at the boundary of $\Omega$ is then defined as

$$\partial_\nu^{1/2} v_\Omega(x) := \lim_{s\to 0^+} \frac{v_\Omega(x+s\nu(x)) - v_\Omega(x)}{s^{1/2}}, \tag{2.20}$$

where $x \in \partial\Omega$ and $\nu(x)$ is the outward normal vector to $\partial\Omega$ at $x$.

**Theorem 7.** *Let $\Omega$ be a compact set with boundary of class $C^2$, let $\zeta \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, and let $(\Phi_t)_{t\in\mathbb{R}}$ be a smooth family of diffeomorphisms of the plane satisfying $\Phi_0 = \mathrm{Id}$ and $\frac{d}{dt}\Phi_t\big|_{t=0} = \zeta$. Then the normal $\frac{1}{2}$-derivative $\partial_\nu^{1/2} v_\Omega$ of the potential $v_\Omega$ from (2.19) is well-defined and belongs to $C^\beta(\partial\Omega)$ for any $\beta \in (0, 1/2)$. Moreover, we have*

$$\frac{d}{dt}\mathcal{I}_1(\Phi_t(\Omega))\bigg|_{t=0} = -\frac{1}{8}\int_{\partial\Omega} (\partial_\nu^{1/2} v_\Omega(x))^2 \zeta(x)\cdot\nu(x)\, d\mathcal{H}^1(x). \tag{2.21}$$

*As a consequence, the Euler–Lagrange equation for a critical point $\Omega \in \mathcal{A}_m$ of $E_\lambda$ satisfying the above smoothness conditions is*

$$\kappa - \frac{\lambda}{8}(\partial_\nu^{1/2} v_\Omega)^2 + g = p \quad \text{on } \partial\Omega, \tag{2.22}$$

*where $\kappa$ is the curvature of $\partial\Omega$ (positive if $\Omega$ is convex) and $p \in \mathbb{R}$ is a Lagrange multiplier due to the mass constraint.*

We note that the result in Theorem 7 relies on recent regularity estimates for fractional elliptic PDEs obtained in [10,34,35]. It is also closely related to the result of Dalibard and Gérard-Varet [8] on the shape derivative of a fractional shape optimization problem.

## 3. Preliminaries: Capacitary Estimates, Perimeters and Connected Components

In this section we give some preliminary definitions and results about the functionals $\mathcal{I}_1$ and $P$ that define $\mathcal{E}_\lambda$. We begin with an important remark about the necessity of introducing the classes $\mathcal{K}_m$ and $\mathcal{S}_m$.

*Remark 8.* As mentioned in the Introduction, we have to choose carefully the admissible class for the minimization of $E_\lambda$. A natural choice would be minimizing $E_\lambda$ in the class of finite perimeter sets. However, in this class the functional $E_\lambda$ is never lower semicontinuous. Indeed, given a set $\Omega \subset \mathbb{R}^2$ and $\varepsilon > 0$ it is possible to find another set $\Omega_\varepsilon$, with $|\Omega \Delta \Omega_\varepsilon| = 0$ and $P(\Omega) = P(\Omega_\varepsilon)$, but with $\mathcal{I}_1(\Omega_\varepsilon) < \varepsilon$ (see the Introduction in [30]). Such a construction cannot be accomplished in $\mathcal{K}_m$. In this sense, $\mathcal{K}_m$ is the largest class in which it is meaningful to consider the minimization of $E_\lambda$.

In [30, Theorems 1 and 2] uniform bounds on $E_\lambda(\Omega)$ were proved for $g = 0$, which are attained on balls. These estimates will play a crucial role in the proof of Theorem 1, and we recall them in the following lemma.

**Lemma 9.** *For any $\Omega \in \mathcal{K}_m$ there holds*

$$\mathcal{H}^1(\partial\Omega) + \lambda\mathcal{I}_1(\Omega) \geq 2\pi\sqrt{\lambda}. \tag{3.1}$$

*Moreover, if $\lambda \leq \lambda_c(m)$ there also holds*

$$\mathcal{H}^1(\partial\Omega) + \lambda\mathcal{I}_1(\Omega) \geq \mathcal{H}^1(\partial B_r(x_0)) + \lambda\mathcal{I}_1(B_r(x_0)), \tag{3.2}$$

*where $r = \sqrt{m/\pi}$ and $x_0 \in \mathbb{R}^2$, i.e., $B_r(x_0)$ is a ball of measure $m$, and the equality holds if and only of $\Omega = \overline{B_r(x_0)}$ for some $x_0 \in \mathbb{R}^2$.*

We now recall some basic facts about the functional $\mathcal{I}_1$.

**Lemma 10.** [30, Lemma 1] *Let $\Omega \subset \mathbb{R}^2$ be a compact set such that $|\Omega| > 0$ and $\mathcal{H}^1(\partial\Omega) < +\infty$. Then there exists a unique probability measure $\mu$ over $\mathbb{R}^2$ supported on $\Omega$ such that*

$$\mathcal{I}_1(\Omega) = \int_\Omega \int_\Omega \frac{d\mu(x)d\mu(y)}{|x-y|}. \tag{3.3}$$

*Furthermore, $\mu(\partial\Omega) = 0$, and we have $d\mu(x) = \rho(x)dx$ for some $\rho \in L^1(\Omega)$ satisfying $0 < \rho(x) \leq C/\mathrm{dist}\,(x, \partial\Omega)$ for some constant $C > 0$ and all $x \in \mathrm{int}(\Omega)$.*

Another useful estimate is the following:

**Lemma 11.** [30, Lemma 2] *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be compact sets with positive measure such that $\mathcal{H}^1(\partial\Omega_i) < +\infty$ for $i \in \{1, 2\}$, and $|\Omega_1 \cap \Omega_2| = 0$. Then, for all $t \in [0, 1]$ there holds*

$$\mathcal{I}_1(\Omega_1 \cup \Omega_2) \le t^2 \mathcal{I}_1(\Omega_1) + (1-t)^2 \mathcal{I}_1(\Omega_2) + \frac{2t(1-t)}{\operatorname{dist}(\Omega_1, \Omega_2)}, \tag{3.4}$$

*and there exists $\bar{t} \in (0, 1)$ such that*

$$\mathcal{I}_1(\Omega_1 \cup \Omega_2) > \bar{t}^2 \mathcal{I}_1(\Omega_1) + (1-\bar{t})^2 \mathcal{I}_1(\Omega_2). \tag{3.5}$$

From [20, Section 2] (see also [24]) we have that

$$\mathcal{I}_1(\Omega) = \frac{2\pi}{\operatorname{cap}_1(\Omega)}, \tag{3.6}$$

whenever $\Omega$ is a compact set, where $\operatorname{cap}_1(\Omega)$ is the $\frac{1}{2}$-capacity of $\Omega$ defined as

$$\operatorname{cap}_1(\Omega) := \inf\left\{ \|u\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} : u \in C_c^\infty(\mathbb{R}^2), \quad u \ge \chi_\Omega \right\}, \tag{3.7}$$

and

$$\|u\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} := \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^3} \,\mathrm{d}x\mathrm{d}y \tag{3.8}$$

is the Gagliardo norm of the homogeneous fractional Sobolev space obtained via completion of $C_c^\infty(\mathbb{R}^2)$ with respect to that norm [11,26]. For the sake of completeness, we provide a short justification of this fact: Let $v_\Omega := \mu_\Omega * |\cdot|^{-1}$ be the potential of $\Omega$, where $\mu_\Omega$ is the equilibrium measure for $\Omega$. Then $v_\Omega$ satisfies

$$(-\Delta)^{\frac{1}{2}} v_\Omega = 2\pi \mu_\Omega \tag{3.9}$$

distributionally in $\mathbb{R}^2$, and

$$\begin{cases} (-\Delta)^{\frac{1}{2}} v_\Omega = 0 & \text{in } \Omega^c, \\ v_\Omega = \mathcal{I}_1(\Omega) & \text{in } \Omega \backslash \Omega_0, \\ \lim_{|x| \to +\infty} v_\Omega(x) = 0, & \end{cases} \tag{3.10}$$

for some $\Omega_0 \subset \Omega$ with $\operatorname{cap}_1(\Omega_0) = 0$ (see, for instance, [20, Lemma 2.11] and [24, p. 137]). Furthermore, arguing as in the proof of [25, Theorem 11.16] one can see that

$$u_\Omega := \mathcal{I}_1^{-1}(\Omega)\, v_\Omega \tag{3.11}$$

is the $\frac{1}{2}$-capacitary potential of $\Omega$, which satisfies $\operatorname{cap}_1(\Omega) = \|u_\Omega\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}$, and

$$\mathcal{I}_1(\Omega) = \int_{\mathbb{R}^2} v_\Omega \, d\mu_\Omega = \frac{1}{2\pi} \int_{\mathbb{R}^2} v_\Omega (-\Delta)^{1/2} v_\Omega \,\mathrm{d}x$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left| (-\Delta)^{1/4} v_\Omega \right|^2 \, \mathrm{d}x = \frac{1}{2\pi} \| v_\Omega \|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}, \tag{3.12}$$

so that

$$\mathrm{cap}_1(\Omega) = \| u_\Omega \|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} = \frac{\| v_\Omega \|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}}{\mathcal{I}_1(\Omega)^2} = \frac{2\pi}{\mathcal{I}_1(\Omega)}. \tag{3.13}$$

The link with the classical Newtonian capacity, defined for $\Omega \subset \mathbb{R}^3$ as

$$\mathrm{cap}(\Omega) := \inf \left\{ \| \nabla u \|^2_{L^2(\mathbb{R}^3)} \, : \, u \in C^1_c(\mathbb{R}^3), \quad u \geq \chi_\Omega \right\}, \tag{3.14}$$

is given by the equality

$$\mathrm{cap}_1(\Omega) = 2 \, \mathrm{cap}(\Omega \times \{0\}) \tag{3.15}$$

for any compact set $\Omega \subset \mathbb{R}^2$ [25, Theorem 11.16]. Finally, we recall that $\mathrm{cap}_1(\Omega) = 0$ if $\mathcal{H}^1(\Omega) < \infty$ (see [24, Theorem 3.14]).

We note that, a priori, the functional $\mathcal{I}_1$ is not lower semicontinuous with respect to $L^1$ convergence. However, given a compact set $\Omega$, $\mathcal{I}_1$ is semicontinuous along a specific family of sets, namely sets of the form

$$\Omega^\delta := \{ x \in \mathbb{R}^2 : \, \mathrm{dist}(x, \Omega) \leq \delta \} \tag{3.16}$$

for $\delta \to 0$. This is formalized in the next lemma, and then exploited in Proposition 17.

**Lemma 12.** *Let $\Omega$ be a compact subset of $\mathbb{R}^2$ and let $(\delta_n)_{n \in \mathbb{N}} \subset [0, +\infty)$ and $\bar{\delta} \in [0, +\infty)$ be such that $\delta_n \to \bar{\delta}$ as $n \to \infty$. Then*

$$\mathcal{I}_1(\Omega^{\bar{\delta}}) \leq \liminf_{n \to +\infty} \mathcal{I}_1(\Omega^{\delta_n}). \tag{3.17}$$

*Moreover, if $\delta_n \searrow \bar{\delta}$ it holds that*

$$\mathcal{I}_1(\Omega^{\bar{\delta}}) = \lim_{n \to +\infty} \mathcal{I}_1(\Omega^{\delta_n}). \tag{3.18}$$

*Proof.* We can suppose that $(\delta_n)_{n \in \mathbb{N}}$ is a monotone sequence. We have two cases: If $\delta_n \nearrow \bar{\delta}$, then $\delta_n \leq \bar{\delta}$ for any $n$ and thus by the monotonicity of $\mathcal{I}_1$ with respect to set inclusions, we have that $\mathcal{I}_1(\Omega^{\delta_n}) \geq \mathcal{I}_1(\Omega^{\bar{\delta}})$ and the lower semicontinuity is proven.

We deal now with the case $\delta_n \searrow \bar{\delta}$. Let us fix $\varepsilon > 0$ and let $\varphi \in C^1_c(\mathbb{R}^2)$ be such that $\varphi > \chi_{\Omega^{\bar{\delta}}}$, and $\| \varphi \|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq \mathrm{cap}_1(\Omega^{\bar{\delta}}) + \varepsilon$. Then, since $\{ \varphi > 1 \}$ is an open set which contains $\Omega^{\bar{\delta}}$, for $n$ big enough (depending on $\varepsilon$) $\varphi$ is also a test function for $\mathrm{cap}_1(\Omega^{\delta_n})$, and we get

$$\mathrm{cap}_1(\Omega^{\bar{\delta}}) \leq \mathrm{cap}_1(\Omega^{\delta_n}) \leq \| \varphi \|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq \mathrm{cap}_1(\Omega^{\bar{\delta}}) + \varepsilon. \tag{3.19}$$

Letting $\varepsilon \searrow 0$, we get the continuity of $\mathrm{cap}_1$ and hence of $\mathcal{I}_1$ by (3.13). □

We prove now a result which turns out to be very useful in the proof of the semicontinuity result in Theorem 1, as well as of the existence and regularity results in Theorems 3 and 4.

**Lemma 13.** *Let $\Omega = U \cup V$ with $U$ and $V$ compact sets of finite positive measure and such that $|U \cap V| = 0$. Then we have*

$$\mathcal{I}_1(\Omega) \geq \mathcal{I}_1(U) - \frac{\pi}{4|U|} P(V). \tag{3.20}$$

*Proof.* By Lemma 11 we have

$$\mathcal{I}_1(\Omega) \geq \min_{t\in[0,1]} \left\{ t^2 \mathcal{I}_1(U) + (1-t)^2 \mathcal{I}_1(V) \right\}. \tag{3.21}$$

By computing the minimum on the right-hand side, we get

$$\mathcal{I}_1(\Omega) \geq \mathcal{I}_1(U) - \frac{\mathcal{I}_1^2(U)}{\mathcal{I}_1(U) + \mathcal{I}_1(V)} \geq \mathcal{I}_1(U) - \frac{\mathcal{I}_1^2(U)}{\mathcal{I}_1(V)}. \tag{3.22}$$

We recall that $\mathcal{I}_1$ is maximized by the ball among sets of fixed volume. Letting $B := B_{\sqrt{|U|/\pi}}(0)$, we then get that $|B| = |U|$ and [30, Lemma 3.3]

$$\mathcal{I}_1(U) \leq \mathcal{I}_1(B) = \frac{\pi^{\frac{3}{2}}}{2\sqrt{|U|}}. \tag{3.23}$$

Moreover, by [31, Corollary 3.2] the dilation invariant functional $\mathcal{F} := \mathcal{I}_1(\cdot)P(\cdot)$ is minimized by balls, and on a ball $B_r$ it takes the value $\mathcal{I}_1(B_r)P(B_r) = \pi^2$, so that

$$\mathcal{I}_1(V) \geq \frac{\pi^2}{P(V)}. \tag{3.24}$$

We plug these two estimates into (3.22) to get

$$\mathcal{I}_1(\Omega) \geq \mathcal{I}_1(U) - \left( \frac{\pi^{\frac{3}{2}}}{2\sqrt{|U|}} \right)^2 \frac{P(V)}{\pi^2} = \mathcal{I}_1(U) - \frac{\pi}{4|U|} P(V), \tag{3.25}$$

which is the desired estimate. □

In the proof of Theorem 3 we shall use some topological features of sets of finite perimeter in dimension two. Since these sets are defined in the $L^1-$sense (as equivalence classes), it is not a priori immediate how to define what a connected component for a set of finite perimeter is. A suitable notion of connected components for sets of finite perimeter was introduced in [1]. Below we recall some of their main features that we shall use in the sequel.

Given $\Omega \in \mathcal{K}_m$, let $\mathring{\Omega}^M$ be its measure theoretic interior, namely:

$$\mathring{\Omega}^M := \left\{ x \in \mathbb{R}^2 \ : \ \lim_{r\to 0} \frac{|\Omega \cap B_r(x)|}{\pi r^2} = 1 \right\}. \tag{3.26}$$

Since $P(\mathring{\Omega}^M) = P(\Omega) = \mathcal{H}^1(\partial^M\Omega) \le \mathcal{H}^1(\partial\Omega) < +\infty$, where $\partial^M\Omega$ is the essential boundary of $\Omega$ [27], the set $\mathring{\Omega}^M$ is a set of finite perimeter. Therefore, following [1], there exists an at most countable family of sets of finite perimeter $\Omega_i$ such that $\mathring{\Omega}^M = \left(\bigcup_i \mathring{\Omega}_i^M\right) \cup \Sigma$, with $\mathcal{H}^1(\Sigma) = 0$, where the sets $\mathring{\Omega}_i^M$ are the so-called *indecomposable components* of $\mathring{\Omega}^M$. In particular, the sets $\Omega_i$ admit unique representatives that are connected and satisfy the following properties:

(i) $\mathcal{H}^1(\Omega_i \cap \Omega_j) = 0$ for $i \ne j$,
(ii) $|\Omega| = \sum_i |\Omega_i|$,
(iii) $P(\Omega) = \sum_i P(\Omega_i)$,
(iv) $\Omega_i = \overline{\mathring{\Omega}_i^M}$.

Moreover, each set $\Omega_i$ is indecomposable in the sense that it cannot be further decomposed as above. We refer to these representatives of $\Omega_i$ as the *connected components* of $\Omega$. We point out that this notion coincides with the standard notion of connected components in the following sense: if $\Omega$ has a regular boundary (Lipschitz continuous being enough) then the components $\Omega_i$ are the closures of the usual connected components of the interior of $\Omega$.

Such a representation of $\Omega$ as a union of connected components allows us to convexify the components in order to decrease the energy. Indeed, for every $i \in \mathbb{N}$ there holds $\mathcal{I}_1(co(\Omega_i)) \le \mathcal{I}_1(\Omega_i)$ and $\mathcal{H}^1(\partial\, co(\Omega_i)) \le \mathcal{H}^1(\partial\Omega_i)$, where $co(\Omega_i)$ denotes the convex envelope of the component $\Omega_i$. This follows from the fact that $\Omega_i \subseteq co(\Omega_i)$, and that the outer boundary of a connected component can be parametrized by a Jordan curve of finite length (see [1, Section 8]). In addition, since $\partial\Omega$ is negligible with respect to the equilibrium measure for $\mathcal{I}_1(\Omega)$ by Lemma 10, we have $\mathcal{I}_1(\Omega) = \mathcal{I}_1(\mathring{\Omega}^M)$.

The next result shows that the relaxations of $E_\lambda$ in $\mathcal{S}_m$ and in $\mathcal{K}_m$ coincide.

**Proposition 14.** *Given $\Omega \in \mathcal{K}_m$, there exists a sequence of sets $\Omega_n \in \mathcal{S}_m$ such that*

$$\lim_{n\to\infty} |\Omega_n \Delta \Omega| = 0 \qquad \textit{and} \qquad \limsup_{n\to\infty} E_\lambda(\Omega_n) \le E_\lambda(\Omega). \tag{3.27}$$

*Proof.* Assume first that $P(\Omega) = \mathcal{H}^1(\partial\Omega)$. Then by [36, Theorem 1.1] applied to $B_R(0) \setminus \Omega$, for $R > 0$ big enough there exists a sequence of compact sets $\widetilde{\Omega}_n$ with smooth boundaries such that $\widetilde{\Omega}_n \supset \Omega$, $|\widetilde{\Omega}_n \Delta \Omega| \to 0$ and $P(\widetilde{\Omega}_n) \to P(\Omega)$ as $n \to \infty$. Furthermore, by monotonicity of $\mathcal{I}_1$ with respect to set inclusions we have $\mathcal{I}_1(\widetilde{\Omega}_n) \le \mathcal{I}_1(\Omega)$. Now, we define $\Omega_n := (m/|\widetilde{\Omega}_n|)^{1/2}\widetilde{\Omega}_n \in \mathcal{S}_m$, and in view of the fact that $|\widetilde{\Omega}_n| \to m$ as $n \to \infty$ we obtain the result.

Let us now consider the general case. By [1, Corollary 1], there exists a sequence of sets $\Omega_n \in \mathcal{K}_m$ such that $\partial\Omega_n$ is a finite union of Jordan curves, and as $n \to \infty$ we have

$$|\Omega_n \Delta \Omega| \to 0, \qquad P(\Omega_n) \to P(\Omega), \qquad P(\Omega \setminus \Omega_n) \to 0. \tag{3.28}$$

In particular, $P(\Omega_n) = \mathcal{H}^1(\partial\Omega_n)$ for every $n \in \mathbb{N}$. Then by Lemma 13 it follows that

$$\mathcal{I}_1(\Omega_n) \le \mathcal{I}_1(\Omega_n \cap \Omega) \le \mathcal{I}_1(\Omega) + \omega_n, \tag{3.29}$$

with

$$\omega_n := \frac{\pi}{4\,|\Omega_n \cap \Omega|}\, P(\Omega \setminus \Omega_n) \to 0 \qquad \text{as } n \to \infty, \tag{3.30}$$

so that

$$\limsup_{n\to+\infty} E_\lambda(\Omega_n) \le E_\lambda(\Omega). \tag{3.31}$$

Applying now the approximation with regular sets to each set $\Omega_n$, we conclude by a diagonal argument. □

Proposition 14 yields the following characterization of the relaxed energy $\overline{E}_\lambda$.

**Corollary 15.** *For every $\Omega \in \mathcal{A}_m$ there holds*

$$\overline{E}_\lambda(\Omega) = \inf_{\Omega_n \in \mathcal{K}_m,\, |\Omega_n \Delta \Omega| \to 0} \liminf_{n\to\infty} E_\lambda(\Omega_n). \tag{3.32}$$

We finish this section with the following elementary lemma establishing (2.11):

**Lemma 16.** *Let $m > 0$ and $\Omega \in \mathcal{K}_m$. Then (2.11) holds, where $\lambda_\Omega$ is defined in (2.10).*

*Proof.* Using the definition of $\lambda_\Omega$, we have

$$E_\lambda(\Omega) - E_{\lambda_\Omega}(\Omega) - 2\pi(\sqrt{\lambda} - \sqrt{\lambda_\Omega}) = \left(\sqrt{\lambda \mathcal{I}_1(\Omega)} - \frac{\pi}{\sqrt{\mathcal{I}_1(\Omega)}}\right)^2 \ge 0, \tag{3.33}$$

which yields the claim. □

## 4. The Relaxed Energy: Proof of Theorem 1

In this section we prove Theorem 1. We divide the proof into first characterizing the relaxation of $E_\lambda$ in Proposition 17 and then showing the semicontinuity of $E_\lambda$ for $\lambda \le \lambda_c(m)$ in Proposition 18.

**Proposition 17.** *For any $\Omega \in \mathcal{K}_m$, it holds that*

$$\overline{E}_\lambda(\Omega) = \begin{cases} E_\lambda(\Omega) & \text{if } \lambda \le \lambda_\Omega, \\ E_{\lambda_\Omega}(\Omega) + 2\pi\left(\sqrt{\lambda} - \sqrt{\lambda_\Omega}\right) & \text{if } \lambda > \lambda_\Omega, \end{cases} \tag{4.1}$$

*where $\lambda_\Omega$ is defined in (2.10).*

*Proof.* Let $\Omega_n$ be a sequence of sets in $\mathcal{S}_m$ such that $|\Omega_n \Delta \Omega| \to 0$ as $n \to \infty$. For any $\delta > 0$ we let $\Omega^\delta$ as in (3.16). Notice that there exists $\delta_0 > 0$ such that $\Omega^\delta \in \mathcal{K}_{m+\omega(\delta)}$, for any $\delta \le \delta_0$, where $\omega(\delta) \to 0$ as $\delta \to 0$ by the monotone convergence theorem.

For any $n \in \mathbb{N}$ we let $\Omega_n(\delta) := \Omega_n \cap \Omega^\delta$ and $\widetilde{\Omega}_n(\delta) := \overline{\Omega_n \setminus \Omega^\delta}$. By [27, Section II.7.1], we have

$$P(\Omega_n) \geq P(\Omega_n; \mathrm{int}(\Omega^\delta)) + P(\Omega_n; \mathbb{R}^2 \setminus \Omega^\delta) = P(\Omega_n(\delta)) + P(\widetilde{\Omega}_n(\delta)) - 2\mathcal{H}^1(\Omega_n \cap \partial\Omega^\delta). \quad (4.2)$$

Notice that for any fixed $\delta \in (0, \delta_0)$, by Coarea Formula [27, Theorem 18.1] we also have

$$\int_0^\delta \mathcal{H}^1(\Omega_n \cap \partial\Omega^t)\, dt = |\Omega_n \cap (\Omega^\delta \setminus \Omega)| \leq |\Omega_n \Delta \Omega|, \quad (4.3)$$

Therefore we can choose $\delta_n \in (\delta/2, \delta)$ such that

$$\mathcal{H}^1(\Omega_n \cap \partial\Omega^{\delta_n}) \leq \frac{2\,|\Omega_n \Delta \Omega|}{\delta}. \quad (4.4)$$

Recalling (4.2) this gives

$$P(\Omega_n) \geq P(\Omega_n(\delta_n)) + P(\widetilde{\Omega}_n(\delta_n)) - \omega_n^\delta, \quad (4.5)$$

where $\omega_n^\delta \leq \frac{4}{\delta}|\Omega_n \Delta \Omega|$. Up to a subsequence, we can assume that $\delta_n \to \bar{\delta}$ as $n \to \infty$ for some $\bar{\delta} \in [\delta/2, \delta]$. Moreover, we can choose $\delta_n$ such that $P(\widetilde{\Omega}_n(\delta_n)) = \mathcal{H}^1(\partial\widetilde{\Omega}_n(\delta_n))$ (see [28, Equation (68)]).

We now estimate the nonlocal term. Since $\Omega_n \subset \Omega_n(\delta_n) \cup \widetilde{\Omega}_n(\delta_n)$, we have

$$\begin{aligned} \mathcal{I}_1(\Omega_n) &\geq \mathcal{I}_1(\Omega_n(\delta_n) \cup \widetilde{\Omega}_n(\delta_n)) \\ &\geq \min_{t\in[0,1]} \left( t^2 \mathcal{I}_1(\Omega_n(\delta_n)) + (1-t)^2 \mathcal{I}_1(\widetilde{\Omega}_n(\delta_n)) \right) \\ &\geq \min_{t\in[0,1]} \left( t^2 \mathcal{I}_1(\Omega^{\delta_n}) + (1-t)^2 \mathcal{I}_1(\widetilde{\Omega}_n(\delta_n)) \right), \end{aligned} \quad (4.6)$$

where the second inequality follows from Lemma 11, while the third is due to the fact that $\Omega_n(\delta_n)$ is contained in $\Omega^{\delta_n}$ and that $\mathcal{I}_1$ is decreasing with respect to set inclusions.

By Lemma 9 we have that

$$\mathcal{H}^1(\partial\widetilde{\Omega}_n(\delta_n)) + \lambda(1-t)^2 \mathcal{I}_1(\widetilde{\Omega}_n(\delta_n)) \geq 2\pi(1-t)\sqrt{\lambda}. \quad (4.7)$$

Thus, by combining (4.6) with (4.5), recalling that $P(\widetilde{\Omega}_n(\delta_n)) = \mathcal{H}^1(\partial\widetilde{\Omega}_n(\delta_n))$, we obtain

$$\begin{aligned} E_\lambda(\Omega_n) \geq\ & P(\Omega_n(\delta_n)) + \int_{\Omega_n} g\, \mathrm{d}x - \omega_n^\delta + \mathcal{H}^1(\partial\widetilde{\Omega}_n(\delta_n)) \\ & + \lambda \min_{t\in[0,1]} \left( t^2 \mathcal{I}_1(\Omega^{\delta_n}) + (1-t)^2 \mathcal{I}_1(\widetilde{\Omega}_n(\delta_n)) \right) \\ \geq\ & P(\Omega_n(\delta_n)) + \int_{\Omega_n} g\, \mathrm{d}x - \omega_n^\delta + \min_{t\in[0,1]} \left( \lambda t^2 \mathcal{I}_1(\Omega^{\delta_n}) + 2\pi(1-t)\sqrt{\lambda} \right) \\ =\ & P(\Omega_n(\delta_n)) + \int_{\Omega_n} g\, \mathrm{d}x - \omega_n^\delta + \begin{cases} 2\pi\sqrt{\lambda} - \dfrac{\pi^2}{\mathcal{I}_1(\Omega^{\delta_n})} & \text{if } \mathcal{I}_1(\Omega^{\delta_n}) > \frac{\pi}{\sqrt{\lambda}} \\ \lambda\, \mathcal{I}_1(\Omega^{\delta_n}) & \text{if } \mathcal{I}_1(\Omega^{\delta_n}) \leq \frac{\pi}{\sqrt{\lambda}} \end{cases}. \end{aligned}$$

$$(4.8)$$

Therefore, thanks to the lower semicontinuity of the perimeter with respect to the $L^1$ convergence (notice that $|\Omega_n(\delta_n)\Delta\Omega^{\bar{\delta}}| \to 0$ as $n \to +\infty$) and thanks to the semicontinuity of $\mathcal{I}_1$ in Lemma 12, in the limit as $n \to \infty$ we obtain

$$\liminf_{n\to\infty} E_\lambda(\Omega_n) \geq P(\Omega^{\bar{\delta}}) + \int_\Omega g\,\mathrm{d}x + \begin{cases} 2\pi\sqrt{\lambda} - \dfrac{\pi^2}{\mathcal{I}_1(\Omega^{\bar{\delta}})} & \text{if } \mathcal{I}_1(\Omega^{\bar{\delta}}) > \frac{\pi}{\sqrt{\lambda}} \\ \lambda\,\mathcal{I}_1(\Omega^{\bar{\delta}}) & \text{if } \mathcal{I}_1(\Omega^{\bar{\delta}}) \leq \frac{\pi}{\sqrt{\lambda}} \end{cases}, \tag{4.9}$$

Letting now $\delta \to 0$ in (4.9), and again using that $\mathcal{I}_1(\Omega^{\bar{\delta}}) \to \mathcal{I}_1(\Omega)$ by Lemma 12, we finally get

$$\begin{aligned}\liminf_{n\to\infty} E_\lambda(\Omega_n) &\geq P(\Omega) + \int_\Omega g\,\mathrm{d}x + \begin{cases} 2\pi\sqrt{\lambda} - \dfrac{\pi^2}{\mathcal{I}_1(\Omega)} & \text{if } \lambda > \lambda_\Omega \\ \lambda\,\mathcal{I}_1(\Omega) & \text{if } \lambda \leq \lambda_\Omega \end{cases} \\ &= \mathcal{E}_\lambda(\Omega),\end{aligned} \tag{4.10}$$

where $\mathcal{E}_\lambda(\Omega)$ is defined in (2.9).

We now have to show that there exists a sequence $\Omega_n$ in $\mathcal{S}_m$ such that $|\Omega_n\Delta\Omega| \to 0$ as $n \to +\infty$ and

$$\limsup_{n\to\infty} E_\lambda(\Omega_n) \leq \mathcal{E}_\lambda(\Omega). \tag{4.11}$$

Recalling Corollary 15, it is enough to find a sequence $\Omega_n$ in $\mathcal{K}_m$ with the desired properties.

If $\lambda \leq \lambda_\Omega$ we can take $\Omega_n := \Omega$ and there is nothing to prove. If $\lambda > \lambda_\Omega$ we let $R > 0$ such that $\Omega \subset B_{R/2}(0)$. Notice that, for all $n$ large enough (depending on $R$) there exist $n$ points $x_1, \ldots, x_n$ in $B_{2R}(0) \setminus B_R(0)$ such that $|x_i - x_j| \geq R/\sqrt{n}$ for all $i \neq j$. We then take $\Omega_n := \rho_n\,\Omega \cup \left(\cup_{i=1}^n \overline{B_{r/n}(x_i)}\right)$, where

$$r := \frac{\sqrt{\lambda} - \sqrt{\lambda_\Omega}}{2} \qquad \text{and} \qquad \rho_n := \sqrt{1 - \frac{\pi r^2}{m\,n}}. \tag{4.12}$$

Notice that with these choices of $r$ and $\rho_n$ we have that the sets $\rho_n\Omega$ and $\overline{B_{r/n}(x_i)}$ are disjoint, $|\Omega_n| = m$ and

$$\mathrm{dist}(\rho_n\Omega, \cup_{i=1}^n \overline{B_{r/n}(x_i)}) \geq \frac{R}{2} - \frac{r}{n} \geq \frac{R}{4} \tag{4.13}$$

for all $n$ large enough. Letting $t = \sqrt{\lambda_\Omega/\lambda}$, by Lemma 11 we estimate

$$\begin{aligned} E_\lambda(\Omega_n) &\leq E_{\lambda t^2}(\rho_n\Omega) + E_{\lambda(1-t)^2}\left(\cup_{i=1}^n \overline{B_{r/n}(x_i)}\right) + \frac{2t(1-t)\lambda}{\mathrm{dist}(\rho_n\Omega, \cup_{i=1}^n \overline{B_{r/n}(x_i)})} \\ &\leq E_{\lambda t^2}(\rho_n\Omega) + E_{\lambda(1-t)^2}\left(\cup_{i=1}^n \overline{B_{r/n}(x_i)}\right) + \frac{2\lambda}{R} \end{aligned}$$

$$= E_{\lambda_\Omega}(\rho_n\Omega) + E_{(\sqrt{\lambda}-\sqrt{\lambda_\Omega})^2}\Big(\cup_{i=1}^n \overline{B_{r/n}(x_i)}\Big) + \frac{2\lambda}{R}. \tag{4.14}$$

Let now $\mu_i$ be the equilibrium measure for $\overline{B_{r/n}(x_i)}$. Then $\frac{1}{n}\sum_{i=1}^n \mu_i$ is an admissible measure in the definition of $\mathcal{I}_1(\cup_i \overline{B_{r/n}(x_i)})$, so that

$$\begin{aligned}
\mathcal{I}_1(\cup_{i=1}^n \overline{B_{r/n}(x_i)}) &\le n\cdot\frac{1}{n^2}\mathcal{I}_1(\overline{B_{r/n}(x_1)}) + \frac{1}{n^2}\sum_{i\neq j}\int_{B_{r/n}(x_i)}\int_{B_{r/n}(x_j)}\frac{d\mu_i(x)\,d\mu_j(y)}{|x-y|}\\
&\le \frac{1}{n}\mathcal{I}_1(\overline{B_{r/n}(x_1)}) + \frac{1}{n^2}\sum_{i\neq j}\frac{1}{|x_i-x_j|-\frac{2r}{n}} \qquad (4.15)\\
&\le \frac{1}{n}\mathcal{I}_1(\overline{B_{r/n}(x_1)}) + \frac{2}{n^2}\sum_{i\neq j}\frac{1}{|x_i-x_j|},
\end{aligned}$$

for $n$ large enough. Since for any $i = 1,\dots,n$ we have

$$\int_{B_{r/n}(x_i)} g(y)\,\mathrm{d}y \le \frac{\pi r^2}{n^2}\|g\|_{L^\infty(B_{2R}(0))}, \tag{4.16}$$

from (4.14) we obtain

$$\begin{aligned}
E_\lambda(\Omega_n) &\le E_{\lambda_\Omega}(\rho_n\Omega) + nP(\overline{B_{r/n}(x_1)}) + \frac{(\sqrt{\lambda}-\sqrt{\lambda_\Omega})^2}{n}\mathcal{I}_1(\overline{B_{r/n}(x_i)}) + \frac{\pi r^2}{n}\|g\|_{L^\infty(B_{2R}(0))}\\
&\quad + \frac{2(\sqrt{\lambda}-\sqrt{\lambda_\Omega})^2}{n^2}\sum_{i\neq j}\frac{1}{|x_i-x_j|} + \frac{2\lambda}{R}\\
&= E_{\lambda_\Omega}(\rho_n\Omega) + 2\pi r + \frac{\pi}{2r}(\sqrt{\lambda}-\sqrt{\lambda_\Omega})^2 + \frac{\pi r^2}{n}\|g\|_{L^\infty(B_{2R}(0))}\\
&\quad + \frac{2(\sqrt{\lambda}-\sqrt{\lambda_\Omega})^2}{n^2}\sum_{i\neq j}\frac{1}{|x_i-x_j|} + \frac{2\lambda}{R}\\
&= E_{\lambda_\Omega}(\rho_n\Omega) + 2\pi(\sqrt{\lambda}-\sqrt{\lambda_\Omega}) + \frac{\pi(\sqrt{\lambda}-\sqrt{\lambda_\Omega})^2}{4n}\|g\|_{L^\infty(B_{2R}(0))}\\
&\quad + \frac{2(\sqrt{\lambda}-\sqrt{\lambda_\Omega})^2}{n^2}\sum_{i\neq j}\frac{1}{|x_i-x_j|} + \frac{2\lambda}{R},
\end{aligned} \tag{4.17}$$

where we used (4.12) and the fact that $\mathcal{I}_1(\overline{B_r}) = \frac{\pi}{2r}$ (see [30, Equation (2.5)]). Notice that, since $|x_i - x_j| \ge R/\sqrt{n}$, we have

$$\sum_{i\neq j}\frac{1}{|x_i-x_j|} \le \frac{Cn^2}{R},$$

for some universal constant $C > 0$ and $n$ large enough depending only on $R$. Notice also that $\rho_n \to 1$ as $n \to \infty$, so that

$$\lim_{n\to\infty} E_{\lambda_\Omega}(\rho_n\Omega) = \lim_{n\to\infty}\left(\rho_n P(\Omega) + \lambda_\Omega\rho_n^{-1}\mathcal{I}_1(\Omega) + \int_{\rho_n\Omega} g(x)\mathrm{d}x\right) = E_{\lambda_\Omega}(\Omega), \tag{4.18}$$

where in the last term we passed to the limit using $g \in \mathcal{G}$ and the Dominated Convergence Theorem. Sending $n \to \infty$ in (4.17) we then get

$$\limsup_{n\to\infty} E_\lambda(\Omega_n) \le E_{\lambda_\Omega}(\Omega) + 2\pi(\sqrt{\lambda} - \sqrt{\lambda_\Omega}) + \frac{2\lambda + C(\sqrt{\lambda} - \sqrt{\lambda_\Omega})^2}{R}.$$

Sending now $R \to +\infty$, we eventually obtain (4.11) and this concludes the proof. □

From Proposition 17 we get the following result:

**Proposition 18.** *The functional $E_\lambda$ is lower semicontinuous in $\mathcal{K}_m$ if and only if $\lambda \le \lambda_c(m)$.*

*Proof.* Since $\mathcal{I}_1(\Omega) \le \mathcal{I}_1(\overline{B_m})$ for any $\Omega \in \mathcal{K}_m$ (see [32, VII.7.3, p.157]), where $\overline{B}_m$ is a closed ball of measure $m$, we have that $\lambda_\Omega \ge \lambda_{\overline{B_m}} = 4m/\pi$, with equality if and only if $\Omega = \overline{B_m}$. Thus, if $\lambda \le 4m/\pi$, the energy $E_\lambda$ coincides with its lower semicontinuous envelope $\overline{E_\lambda}$ by Proposition 17. On the other hand, if $\lambda > \lambda_{\overline{B_m}}$ then $\overline{E_\lambda}(\overline{B}_m) < E_\lambda(\overline{B}_m)$. Indeed, recalling (2.10) we have

$$\begin{aligned}
\frac{E_\lambda(\overline{B_m}) - \overline{E_\lambda}(\overline{B_m})}{\sqrt{\lambda} - \sqrt{\lambda_{\overline{B_m}}}} &= \left(\sqrt{\lambda} + \sqrt{\lambda_{\overline{B_m}}}\right)\mathcal{I}_1(\overline{B_m}) - 2\pi \\
&= \left(\sqrt{\lambda} + \frac{\pi}{\mathcal{I}_1(\overline{B_m})}\right)\mathcal{I}_1(\overline{B_m}) - 2\pi \\
&= \sqrt{\lambda}\,\mathcal{I}_1(\overline{B_m}) - \pi = \left(\sqrt{\lambda} - \sqrt{\lambda_{\overline{B_m}}}\right)\mathcal{I}_1(\overline{B_m}) > 0.
\end{aligned}$$

In particular, $E_\lambda$ is not lower semicontinuous for $\lambda > \lambda_{\overline{B_m}}$. □

Lastly, Theorem 1 directly follows from Propositions 17 and 18.

## 5. Existence of Minimizers: Proof of Theorem 3

In this section we show existence of minimizers of $E_\lambda$ under suitable assumptions on $\lambda$ and on the function $g$. We start with a simple existence result for minimizers of the relaxed energy $\overline{E}_\lambda$.

**Proposition 19.** *Let $g \in \mathcal{G}$. Then $\overline{E}_\lambda$ admits a minimizer $\Omega_\lambda$ over $\mathcal{A}_m$ for every $\lambda > 0$.*

*Proof.* Let $\Omega_k$ be a minimizing sequence for $\overline{E}_\lambda(\Omega)$. Notice that $P(\Omega_k) < c$ for some positive constant $c$ independent of $k$. Letting $\Omega_k^R := \Omega_k \cap \overline{B_R(0)}$, we have $P(\Omega_k^R) \le P(\Omega_k) + P(B_R(0)) \le c + 2\pi R$. Thus, by the compactness of the immersion of $BV(B_R(0))$ into $L^1(B_R(0))$, applied to the sequence $\chi_{\Omega_k^R}$, we get that there exists a set $\Omega^R \subset B_R(0)$ such that $\chi_{\Omega_k^R} \to \chi_{\Omega^R}$ in $L^1$, up to a (not relabeled) subsequence, as $k \to +\infty$. Sending $R \to +\infty$, by a diagonal argument we get that there exists $\Omega_\lambda \subset \mathbb{R}^2$ such that, up to extracting a further subsequence, the functions $\chi_{\Omega_k}$ converge to $\chi_{\Omega_\lambda}$ in $L^1_{\text{loc}}(\mathbb{R}^2)$.

Now we observe that, since $\Omega_k$ is a minimizing sequence for $\overline{E}_\lambda$, there exists $C > 0$ such that, for all $R > 0$ large enough, we have

$$|\Omega_k \setminus B_R(0)| \inf_{x \in B_R^c(0)} g(x) \le \int_{\Omega_k \setminus B_R(0)} g(x)\,\mathrm{d}x \le \int_{\Omega_k} g(x)\,\mathrm{d}x \le C, \tag{5.1}$$

so that

$$|\Omega_k \setminus B_R(0)| \le \frac{C}{\inf_{x \in B_R^c(0)} g(x)}. \tag{5.2}$$

In particular, by (2.12) for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that $|\Omega_k \setminus B_{R_\varepsilon}(0)| \le \varepsilon$ for all $k$. Thus, recalling the convergence of $\chi_{\Omega_k}$ to $\chi_{\Omega_\lambda}$ in $L^1_{\text{loc}}(\mathbb{R}^2)$ as $k \to \infty$, there also exists $k_\varepsilon \in \mathbb{N}$ such that

$$|\Omega_k \Delta \Omega_\lambda| = |(\Omega_k \Delta \Omega_\lambda) \cap B_{R_\varepsilon}(0)| + |(\Omega_k \Delta \Omega_\lambda) \setminus B_{R_\varepsilon}(0)| \le 2\varepsilon, \tag{5.3}$$

for all $k \ge k_\varepsilon$, that is, the sequence $\chi_{\Omega_k}$ converges to $\chi_{\Omega_\lambda}$ in $L^1(\mathbb{R}^2)$ as $k \to \infty$.

Since, by definition, $\overline{E}_\lambda$ is lower semicontinuous in $L^1(\mathbb{R}^2)$, we eventually get that $\Omega_\lambda$ is a minimizer of $\overline{E}_\lambda$. □

The main difficulty in proving Theorem 3 is to show that the minimizer $\Omega_\lambda$ is indeed an element of $\mathcal{K}_m$, so that it is also a minimizer of $E_\lambda$ by Proposition 17.

We first show that $\text{cap}_1(\Omega)$ depends continuously on smooth perturbations of $\Omega$, where $\Omega \subset \mathbb{R}^2$ is a compact set with Lipschitz boundary.

**Lemma 20.** *Let $\Omega \subset \mathbb{R}^2$ be a compact set with positive measure and Lipschitz boundary. Let $\eta \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$, let $\Phi_t(x) := x + t\eta(x)$ be the corresponding family of (Lipschitz) diffeomorphisms, defined for $t \in (-t_0, t_0)$ and $t_0$ sufficiently small, and let $\Omega_t := \Phi_t(\Omega)$.*

*Then, for $t \in (-t_0, t_0)$ it holds that*

$$\text{cap}_1(\Omega) \le \text{cap}_1(\Omega_t)(1 + Ct), \tag{5.4}$$

*where the constant $C > 0$ depends only on the $W^{1,\infty}$-norm of $\eta$.*

*Proof.* Let $u_t$ be the $\frac{1}{2}$-capacitary potential of $\Omega_t$ minimizing (3.7) with $\Omega$ replaced by $\Omega_t$, and let $u := u_t \circ \Phi_t$. Notice that $u$ is an admissible function for the minimum problem (3.7). In particular, we have

$$\text{cap}_1(\Omega) \le \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^3}\,\mathrm{d}x\mathrm{d}y. \tag{5.5}$$

We now compute

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^3}\,\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u_t(\Phi_t(x)) - u_t(\Phi_t(y))|^2}{|x - y|^3}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u_t(X)-u_t(Y)|^2}{|\Phi_t^{-1}(X)-\Phi_t^{-1}(Y)|^3} \left|\det\nabla\Phi_t^{-1}(X)\right|\left|\det\nabla\Phi_t^{-1}(Y)\right| \,\mathrm{d}X\mathrm{d}Y, \tag{5.6}$$

where we performed the change of variables $X = \Phi_t(x)$, $Y = \Phi_t(y)$. Observing that

$$|\det\nabla\Phi_t^{-1}(X) - 1| \le Ct \qquad \text{and} \qquad |\Phi_t^{-1}(X) - \Phi_t^{-1}(Y)| \ge (1 - Ct)|X - Y|, \tag{5.7}$$

where $C > 0$ depends only on the $W^{1,\infty}$-norm of $\eta$, from (5.5) and (5.6) we readily obtain (5.4). □

From Lemma 20 and (3.6), we immediately get the following result:

**Corollary 21.** *Under the assumptions of Lemma 20, there holds*

$$\mathcal{I}_1(\Omega_t) \le \mathcal{I}_1(\Omega)(1 + Ct), \tag{5.8}$$

*where the constant $C > 0$ depends only on the $W^{1,\infty}$-norm of $\eta$.*

We now show that, if $\lambda < \lambda_c(m)$, we can decrease the energy of a set $\Omega \in \mathcal{K}_m$ by reducing the number of its connected components and holes.

**Proposition 22.** *Let $\lambda < \lambda_c(m)$ and $g \in \mathcal{G}$. Then, for any $\Omega \in \mathcal{K}_m$ we can find $\widetilde{\Omega} \in \mathcal{K}_m$ such that $E_\lambda(\widetilde{\Omega}) \le E_\lambda(\Omega)$, $\widetilde{\Omega} \subset B_R(0)$, and the numbers of connected components of both $\widetilde{\Omega}$ and of $\widetilde{\Omega}^c$ are bounded above by $N$, where the numbers $R$, $N$ depend only on $\lambda, m, g$ and $E_\lambda(\Omega)$.*

*Proof.* We divide the proof into two steps.
*Step 1: Construction of a uniformly bounded set with a uniformly bounded number of connected components.*
Let $\Omega_i$ be the connected components of $\Omega$, and up to a relabeling we can suppose that if $m_i := |\Omega_i|$, then $m_i \ge m_{i+1}$. Let $\varepsilon \in (0, m/2)$. We claim that there exists $N_\varepsilon \in \mathbb{N}$ depending only on $\varepsilon$ and $m$ such that

$$\left|\Omega \setminus \bigcup_{i>N_\varepsilon} \Omega_i\right| \ge m - \frac{\varepsilon}{2} > \frac{3}{4}m. \tag{5.9}$$

Indeed, we have $\sum_{i=1}^{\infty} m_i = m$, and by the isoperimetric inequality we get

$$\sum_{i=1}^{\infty}\sqrt{4\pi m_i} \le \sum_{i=1}^{\infty} P(\Omega_i) \le E_\lambda(\Omega). \tag{5.10}$$

Recalling that the sequence $i \mapsto m_i$ is decreasing, it follows that

$$m_i \le \frac{E_\lambda^2(\Omega)}{4\pi i^2}. \tag{5.11}$$

Hence there exists $C > 0$ depending only on $m$ and $E_\lambda(\Omega)$ such that

$$\sum_{i \geq k} m_i \leq \frac{C}{k}, \tag{5.12}$$

so that (5.9) holds for $N_\varepsilon \geq 2C/\varepsilon$.

Let us set

$$U_\varepsilon := \bigcup_{i=1}^{N_\varepsilon} \Omega_i. \tag{5.13}$$

We claim that there exists $\bar{R} \geq 1$, depending only on $m$, $g$ and $E_\lambda(\Omega)$ such that

$$|U_\varepsilon \cap B_{\bar{R}}(0)| \geq \frac{2}{3} m. \tag{5.14}$$

Notice that the previous equation implies in particular that

$$|U_\varepsilon \setminus B_{\bar{R}}(0)| \leq \frac{1}{3} m. \tag{5.15}$$

Indeed, reasoning as in the proof of Proposition 19, for any $R > 0$ we can write

$$E_\lambda(\Omega) \geq \int_{\Omega \setminus B_R(0)} g \, \mathrm{d}x \geq |\Omega \setminus B_R(0)| \inf_{x \in B_R^c(0)} g(x), \tag{5.16}$$

so that

$$|\Omega \setminus B_R(0)| \leq \frac{E_\lambda(\Omega)}{\inf_{x \in B_R^c(0)} g(x)}. \tag{5.17}$$

Take now $\bar{R} \geq 1$ such that $\frac{E_\lambda(\Omega)}{\inf_{x \in B_{\bar{R}}^c(0)} g(x)} \leq \frac{m}{12}$. Such a radius exists in view of the coercivity of $g$. Then we have

$$|U_\varepsilon \cap B_{\bar{R}}(0)| \geq |U_\varepsilon| - \frac{m}{12} > \frac{3}{4} m - \frac{m}{12} = \frac{2}{3} m, \tag{5.18}$$

which gives (5.14).

By the same argument, there exists $R_\varepsilon \geq 2\bar{R}$ such that $|U_\varepsilon \cap B_{R_\varepsilon}(0)| \geq m - \varepsilon$. Moreover, since $P(U_\varepsilon) \leq E_\lambda(\Omega)$, we can also find a radius $R_\varepsilon^n \in [R_\varepsilon, R_\varepsilon']$, with $R_\varepsilon' := R_\varepsilon + E_\lambda(\Omega)$, such that

$$\mathcal{H}^1(U_\varepsilon \cap \partial B_{R_\varepsilon^n}(0)) = 0. \tag{5.19}$$

Indeed, since we are working in dimension two we can write

$$\mathcal{H}^1\left(\left\{t \in \mathbb{R} : \mathcal{H}^1(\partial B_t(0) \cap U_\varepsilon) > 0\right\}\right) \leq \sum_{i=1}^{N_\varepsilon} \operatorname{diam}(\Omega_i) \leq \frac{1}{2} P(\Omega) \leq \frac{1}{2} E_\lambda(\Omega), \tag{5.20}$$

which ensures the existence of $R_\varepsilon^n$ satisfying (5.19).

Let now $\varphi : \mathbb{R} \to \mathbb{R}$ be a cutoff function defined as

$$\varphi(s) := \begin{cases} 1 & \text{if } |s| \le \bar{R} \\ 2 - \dfrac{|s|}{\bar{R}} & \text{if } \bar{R} < |s| \le 2\bar{R} \\ 0 & \text{if } |s| \ge 2\bar{R}. \end{cases} \tag{5.21}$$

For $t \ge 0$, we introduce the localized dilation $\Phi_t(x) := (1 + t\varphi(|x|))x$, and we observe that

$$\det \nabla\Phi_t(x) = (1 + t\varphi(|x|))^2 + t|x|\varphi'(|x|) + t^2|x|\varphi(|x|)\varphi'(|x|). \tag{5.22}$$

In particular, the map $t \mapsto |\Phi_t(A)| = \int_A \det \nabla\Phi_t(x)\,\mathrm{d}x$ is continuous for every set $A \subset \mathbb{R}^2$ of finite measure. We notice that a similar construction of a localized dilation can be found in the proof of [12, Theorem 1].

Recalling (5.14), (5.15) and letting $\widetilde{U}_\varepsilon := U_\varepsilon \cap B_{R^n_\varepsilon}(0)$, we have

$$\begin{aligned} |\Phi_t(\widetilde{U}_\varepsilon)| &= \int_{\widetilde{U}_\varepsilon} \det \nabla\Phi_t(x)\,\mathrm{d}x \\ &= \int_{\widetilde{U}_\varepsilon} \Big[(1 + t\varphi(|x|))^2 + t|x|\varphi'(|x|) + t^2|x|\varphi(|x|)\varphi'(|x|)\Big]\,\mathrm{d}x \\ &\ge |\widetilde{U}_\varepsilon| + (2t + t^2)|U_\varepsilon \cap B_{\bar{R}}(0)| - \frac{t + t^2}{\bar{R}} \int_{U_\varepsilon \cap B_{2\bar{R}}(0)\setminus B_{\bar{R}}(0)} |x|\,\mathrm{d}x \\ &\ge |\widetilde{U}_\varepsilon| + \frac{2}{3}m(2t + t^2) - \frac{2}{3}m(t + t^2) \;=\; |\widetilde{U}_\varepsilon| + \frac{2}{3}mt\,, \end{aligned}$$

which implies that

$$\left|\Phi_{\frac{3(m - |\widetilde{U}_\varepsilon|)}{2m}}(\widetilde{U}_\varepsilon)\right| \ge m. \tag{5.23}$$

Noting that $|\Phi_0(\widetilde{U}_\varepsilon)| = |\widetilde{U}_\varepsilon| = |U_\varepsilon \cap B_{R^n_\varepsilon}(0)| \le m$, we obtain that there exists $t_\varepsilon \ge 0$ such that $|\Phi_{t_\varepsilon}(\widetilde{U}_\varepsilon)| = m$ and

$$t_\varepsilon \le \frac{3(m - |\widetilde{U}_\varepsilon|)}{2m} \le \frac{3\varepsilon}{2m}. \tag{5.24}$$

Let now $W_\varepsilon := \Phi_{t_\varepsilon}(\widetilde{U}_\varepsilon)$. Recalling Corollary 21 and [22, Proposition 3.1] (see also [27, Proposition 17.1]), the following properties hold:

$(i)$ $W_\varepsilon \subset B_{R'_\varepsilon}(0)$ and $W_\varepsilon$ has at most $N_\varepsilon$ connected components;
$(ii)$ $|W_\varepsilon| = m$;
$(iii)$ $P(W_\varepsilon) \le \mathrm{Lip}(\Phi_{t_\varepsilon})\, P(\widetilde{U}_\varepsilon) \le (1 + t_\varepsilon)P(\widetilde{U}_\varepsilon) \le P(\widetilde{U}_\varepsilon) + Ct_\varepsilon$;
$(iv)$ $\mathcal{I}_1(W_\varepsilon) \le \mathcal{I}_1(\widetilde{U}_\varepsilon) + Ct_\varepsilon$;
$(v)$ $\int_{W_\varepsilon} g(x) \le \int_{\widetilde{U}_\varepsilon} g(x)\,\mathrm{d}x + Ct_\varepsilon$;

where the constant $C > 0$ depends only on $g$, $m$ and $E_\lambda(\Omega)$. Indeed, the first two assertions follow by construction. Assertion $(iii)$ holds true since $\|\varphi\|_{L^\infty(\mathbb{R})} \le 1$. Assertion $(iv)$ follows by Corollary 21, while $(v)$ holds true since

$$\begin{aligned}\int_{W_\varepsilon} g(x)\,\mathrm{d}x &= \int_{\widetilde{U}_\varepsilon} g(\Phi_{t_\varepsilon}(x))\det\nabla\Phi_{t_\varepsilon}(x)\,\mathrm{d}x\\ &\le (1+t_\varepsilon)^2\left(\int_{\widetilde{U}_\varepsilon} g(x)\,\mathrm{d}x + C\|\nabla g\|_{L^\infty(B_{2\bar{R}}(0))} m t_\varepsilon\right)\\ &\le \int_{\widetilde{U}_\varepsilon} g(x)\,\mathrm{d}x + C' t_\varepsilon\,,\end{aligned}$$

for some $C, C' > 0$ depending only on $g$, $m$ and $E_\lambda(\Omega)$.

We claim that $E_\lambda(W_\varepsilon) \le E_\lambda(\Omega)$ for $\varepsilon$ small enough. Letting $V_\varepsilon := \Omega\setminus\widetilde{U}_\varepsilon$, we compute

$$\begin{aligned}\delta_\varepsilon := E_\lambda(\Omega) - E_\lambda(W_\varepsilon) &= P(\widetilde{U}_\varepsilon) + P(V_\varepsilon) + \lambda\mathcal{I}_1(\widetilde{U}_\varepsilon\cup V_\varepsilon) + \int_{\widetilde{U}_\varepsilon} g\,\mathrm{d}x + \int_{V_\varepsilon} g\,\mathrm{d}x\\ &\quad - P(W_\varepsilon) - \lambda\mathcal{I}_1(W_\varepsilon) - \int_{W_\varepsilon} g\,\mathrm{d}x\\ &\ge P(V_\varepsilon) + \lambda\mathcal{I}_1(\widetilde{U}_\varepsilon\cup V_\varepsilon) - \lambda\mathcal{I}_1(\widetilde{U}_\varepsilon) - Ct_\varepsilon\,,\end{aligned}\tag{5.25}$$

where the constant $C > 0$ depends only on $g$, $m$, and $E_\lambda(\Omega)$, and we took advantage of (5.19). We also have $|V_\varepsilon| < \varepsilon$ by construction. Using Lemma 13 with $U = \widetilde{U}_\varepsilon$ and $V = V_\varepsilon$, we obtain

$$P(V_\varepsilon) + \lambda\mathcal{I}_1(\widetilde{U}_\varepsilon\cup V_\varepsilon) - \lambda I_1(\widetilde{U}_\varepsilon) \ge P(V_\varepsilon)\left(1 - \frac{\lambda\pi}{4|\widetilde{U}_\varepsilon|}\right) \ge P(V_\varepsilon)\left(1 - \frac{\lambda\pi}{4(m-\varepsilon)}\right).\tag{5.26}$$

Recalling that $\lambda < 4m/\pi$, we can choose $\varepsilon$ small enough so that

$$\left(1 - \frac{\lambda\pi}{4(m-\varepsilon)}\right) \ge \frac{1}{2}\left(1 - \frac{\lambda\pi}{4m}\right).\tag{5.27}$$

Recalling (5.24) as well, with the help of the isoperimetric inequality we then get

$$\delta_\varepsilon \ge \frac{1}{2}\left(1 - \frac{\lambda\pi}{4m}\right)P(V_\varepsilon) - \frac{3C}{2m}|V_\varepsilon| \ge \sqrt{\pi}\left(1 - \frac{\lambda\pi}{4m}\right)|V_\varepsilon|^{\frac{1}{2}} - \frac{3C}{2m}|V_\varepsilon| \ge 0,\tag{5.28}$$

provided we choose $\varepsilon$ small enough depending only on $g$, $m$ and $E_\lambda(\Omega)$. We thus proved that $\delta_\varepsilon \ge 0$, that is, $E_\lambda(W_\varepsilon) \le E_\lambda(\Omega)$.

*Step 2: Construction of a set with a uniformly bounded number of holes.*

In Step 1 we built a set $W \in \mathcal{K}_m$, with a uniformly bounded number of connected components and such that $E_\lambda(W) \le E_\lambda(\Omega)$. In particular, there exists a uniform radius $R > 0$ such that $W \subset B_R(0)$. Starting from this, we construct another set with a uniformly bounded number of holes, where a hole is a bounded connected component of the complement set.

Let us denote by $\{H_i\}_{i\in\mathbb{N}}$ the connected components of $W^c$ which are bounded. As in Step 1, for $\varepsilon \in (0, m/2)$ we can find $N_\varepsilon$ such that $\sum_{i>N_\varepsilon} |H_i| \le \varepsilon$. Let us set $H_\varepsilon := \bigcup_{i>N_\varepsilon} H_i$ and

$$\Omega_\varepsilon := \sqrt{\frac{m}{m+|H_\varepsilon|}}\,(W \cup H_\varepsilon) \in \mathcal{K}_m. \tag{5.29}$$

Notice that

$$P(\Omega_\varepsilon) \le P(W \cup H_\varepsilon) = P(W) - P(H_\varepsilon), \tag{5.30}$$

$$\mathcal{I}_1(\Omega_\varepsilon) = \sqrt{\frac{m+|H_\varepsilon|}{m}}\,\mathcal{I}_1(W \cup H_\varepsilon) \le \left(1 + \frac{|H_\varepsilon|}{2m}\right)\mathcal{I}_1(W), \tag{5.31}$$

$$\begin{aligned}\int_{\Omega_\varepsilon} g(x)\,\mathrm{d}x &= \frac{m}{m+|H_\varepsilon|}\int_W g\left(\sqrt{\frac{m}{m+|H_\varepsilon|}}\,x\right)\mathrm{d}x\\ &\le \int_W g(x)\,\mathrm{d}x + C|H_\varepsilon|,\end{aligned} \tag{5.32}$$

where the constant $C > 0$ depends on $m$, $g$ and $R$, and in obtaining (5.31), we used concavity of the square root and monotonicity of the capacitary term with respect to filling the holes. Putting together (5.30), (5.31) and (5.32), we then get

$$E_\lambda(\Omega_\varepsilon) \le E_\lambda(W) - P(H_\varepsilon) + \left(\frac{E_\lambda(\Omega)}{2m} + C\right)|H_\varepsilon|, \tag{5.33}$$

which yields the claim for $\varepsilon$ small enough by the isoperimetric inequality. □

We now prove Theorem 3.

*Proof of Theorem 3.* Let $\Omega_n \in \mathcal{K}_m$ be a minimizing sequence for $E_\lambda$. In particular $E_\lambda(\Omega_n) \le c$, for some $c = c(\lambda, g, m) > 0$ depending only on $g$ and $m$.

Thanks to Proposition 22, we can assume that the sets $\Omega_n$ are uniformly bounded and the number of connected components both of $\Omega_n$ and of $(\Omega_n)^c$ is uniformly bounded. In particular, the number of connected components of $\partial\Omega_n$ is also uniformly bounded.

Since $\mathcal{H}^1(\partial\Omega_n) \le c$, it follows by Blaschke Theorem (see [2, Theorem 4.4.15]) that $\partial\Omega_n \to \Gamma$ in the Hausdorff distance, as $n \to +\infty$ up to passing to a subsequence, for some compact set $\Gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(\Gamma) < +\infty$.

Up to passing to a further subsequence, we also have that the sets $\Omega_n$ converge to some compact set $\Omega$, again in the Hausdorff distance. We notice that

$$\partial\Omega \subset \Gamma. \tag{5.34}$$

Indeed if $x \in \partial\Omega \setminus \Gamma$, then there exists $x_n \in \Omega_n$ such that $x_n \to x$. On the other hand, there exists $N \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that $d_H(x_n, \partial\Omega_n) \ge \varepsilon_0$ for $n \ge N$. Otherwise there would exists $y_n \in \partial\Omega_n$ such that $|y_n - x_n| = d(x_n, \partial\Omega_n) \to 0$ and thus

$$|y_n - x| \le |y_n - x_n| + |x_n - x| \to 0, \tag{5.35}$$

which is impossible, since $x \notin \Gamma$. But then the ball $B_{\varepsilon_0}(x_n)$ is contained, for $n \geq N$, in $\Omega_n$ and converges in Hausdorff distance to $B_{\varepsilon_0}(x) \subset \Omega$. In particular we get $x \notin \partial\Omega$, which gives a contradiction. Thanks to Golab Theorem [2, Theorem 4.4.17], we then obtain

$$\mathcal{H}^1(\partial\Omega) \leq \mathcal{H}^1(\Gamma) \leq \liminf_{n\to+\infty} \mathcal{H}^1(\partial\Omega_n) \leq c. \tag{5.36}$$

Let now $x \in \mathbb{R}^N \setminus \Gamma$. Then there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for $n \geq N$, we have that $B_\varepsilon(x) \subset \Omega_n$ or $B_\varepsilon(x) \subset (\mathbb{R}^N \setminus \Omega_n)$. Thus $\chi_{\Omega_n}(x) = 1$ or $\chi_{\Omega_n}(x) = 0$ for $n$ large enough. In particular $\chi_{\Omega_n} \to \chi_\Omega$ almost everywhere and, by the Dominated Convergence Theorem, we obtain that $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(\mathbb{R}^2)$.

We can now conclude that $\Omega$ is a minimizer in $\mathcal{K}_m$. Indeed, the minimality is granted by the lower semicontinuity of $E_\lambda$ w.r.t. the $L^1$-convergence, since $\lambda < \lambda_c(m)$. Moreover, $\Omega \in \mathcal{K}_m$ since it is compact, it has measure $m$ and $\mathcal{H}^1(\partial\Omega) < +\infty$ by (5.36). The proof is concluded. □

## 6. Partial Regularity of Minimizers: Proof of Theorem 4

In this section we show that all minimizers of our problem are bounded, contain finitely many connected components and holes, and satisfy suitable density estimates. For the latter, the argument essentially follows the classical proof of the density estimates for quasi-minimizers of the perimeter (see [27]). Note, however, that the standard regularity theory of quasi-minimizers of the perimeter cannot be applied directly, as the nonlocal term $\mathcal{I}_1$ presents a crititcal perturbation to the perimeter. This additional complication may be overcome with the help of Lemma 13 for subcritical values of $\lambda < \lambda_c(m)$, yielding some mild regularity of the minimizers.

*Proof of Theorem 4.* Throughout the proof, we identify the minimizer $\Omega_\lambda$ with its regular representative $\Omega_\lambda^+$, and drop the superscript "+" for ease of notation.

First of all, the assertion about the number of connected components of $\Omega_\lambda$ and $\Omega_\lambda^c$ follows from the argument in the proof of Proposition 22, observing that the inequality in that proposition becomes strict otherwise, contradicting the minimality of $\Omega_\lambda$. Therefore, the rest of the proof focuses on the density estimates (2.16), whose proof uses the estimates similar to those in the proof of Proposition 22. It is enough to show the first assertion, since the second one can be proved analogously, taking into account that $\Omega_\lambda^+$ is a closed set.

For $r \in (0, \sqrt{m/(2\pi)})$, so that $|\Omega_\lambda \setminus B_r(x)| \geq m/2$, we set

$$v(0) := 0, \qquad v(r) := |\Omega_\lambda \cap B_r(x)|, \qquad \Omega_{\lambda,r} := \sqrt{\frac{m}{m - v(r)}}\,\big(\Omega_\lambda \setminus B_r(x)\big) \in \mathcal{K}_m. \tag{6.1}$$

Since $x \in \partial\Omega_\lambda^+$, we have that $v(r) > 0$ for all $r > 0$. Moreover, since $\Omega_\lambda$ has finite perimeter, for almost every $r > 0$ it holds (see [27]) that

$$P(\Omega_\lambda) = P(\Omega_\lambda; B_r(x)) + P(\Omega_\lambda; B_r^c(x)) \qquad \text{and} \qquad \frac{dv}{dr}(r) = \mathcal{H}^1(\partial B_r(x) \cap \Omega_\lambda). \tag{6.2}$$

Recalling the Lipschitz continuity of $g$, for almost every $r \in (0, \sqrt{m/(2\pi)})$ we then get

$$\begin{aligned}
&P(\Omega_\lambda; B_r(x)) + P(\Omega_\lambda; B_r^c(x)) + \lambda \mathcal{I}_1(\Omega_\lambda) + \int_{\Omega_\lambda} g(y)\,\mathrm{d}y = E_\lambda(\Omega_\lambda) \\
&\quad \leq E_\lambda(\Omega_{\lambda,r}) = \sqrt{\frac{m}{m - v(r)}} \Big( P(\Omega_\lambda; B_r^c(x)) + \mathcal{H}^1(\partial B_r(x) \cap \Omega_\lambda) \Big) \\
&\qquad + \lambda \sqrt{\frac{m - v(r)}{m}} \mathcal{I}_1(\Omega_\lambda \setminus B_r(x)) + \frac{m}{m - v(r)} \int_{\Omega_\lambda \setminus B_r(x)} g\left( \sqrt{\frac{m}{m - v(r)}}\, y \right) \mathrm{d}y \\
&\quad \leq P(\Omega_\lambda; B_r^c(x)) + \frac{dv}{dr}(r) + \lambda \mathcal{I}_1(\Omega_\lambda \setminus B_r(x)) + \int_{\Omega_\lambda} g(y)\,\mathrm{d}y + C v(r),
\end{aligned} \tag{6.3}$$

where the constant $C > 0$ depends only on $m$, $\lambda$ and $g$.

After some simplifications, (6.3) reads

$$P(\Omega_\lambda; B_r(x)) \leq \frac{dv}{dr}(r) + C v(r) + \lambda \mathcal{I}_1(\Omega_\lambda \setminus B_r(x)) - \lambda \mathcal{I}_1(\Omega_\lambda). \tag{6.4}$$

Applying Lemma 13 with $U = \Omega_\lambda \setminus B_r(x)$ and $V = \Omega_\lambda \cap B_r(x)$, we then obtain

$$\begin{aligned}
P(\Omega_\lambda; B_r(x)) &\leq \frac{dv}{dr}(r) + C v(r) + \frac{\lambda\pi}{4|\Omega_\lambda \setminus B_r(x)|} P(\Omega_\lambda \cap B_r(x)) \\
&\leq \frac{dv}{dr}(r) + C' v(r) + \frac{\lambda\pi}{4m} P(\Omega_\lambda \cap B_r(x)),
\end{aligned} \tag{6.5}$$

where $C' > 0$ depends only on $m$, $\lambda$ and $g$.

Since for almost every $r > 0$ there holds

$$P(\Omega_\lambda; B_r(x)) + \frac{dv}{dr}(r) = P(\Omega_\lambda \cap B_r(x)), \tag{6.6}$$

by adding the quantity $\frac{dv}{dr}(r)$ to both sides of (6.5) we obtain

$$P(\Omega_\lambda \cap B_r(x)) \left( 1 - \frac{\lambda\pi}{4m} \right) \leq 2 \frac{dv}{dr}(r) + C' v(r). \tag{6.7}$$

Thanks to the isoperimetric inequality, for almost every $r \in (0, \sqrt{m/(2\pi)})$ we then get

$$2\sqrt{\pi} \left( 1 - \frac{\lambda\pi}{4m} \right) \sqrt{v(r)} \leq 2 \frac{dv}{dr}(r) + C' v(r). \tag{6.8}$$

Recalling that $\lambda < 4m/\pi$, there exists $r_0 \in (0, \sqrt{m/(2\pi)})$, depending only on $m$, $\lambda$ and $g$, such that

$$2\sqrt{\pi} \left( 1 - \frac{\lambda\pi}{4m} \right) \sqrt{v(r)} - C' v(r) \geq \sqrt{\pi} \left( 1 - \frac{\lambda\pi}{4m} \right) \sqrt{v(r)} \qquad \text{for all } 0 < r \leq r_0, \tag{6.9}$$

which gives

$$\frac{dv}{dr}(r) \geq \frac{\sqrt{\pi}}{2}\left(1-\frac{\lambda\pi}{4m}\right)\sqrt{v(r)} \qquad \text{for a.e. } 0 < r \leq r_0. \tag{6.10}$$

After a direct integration, this inequality implies that

$$v(r) \geq \frac{\pi}{16}\left(1-\frac{\lambda\pi}{4m}\right)^2 r^2 \qquad \text{for a.e. } 0 \leq r \leq r_0, \tag{6.11}$$

which gives the first inequality in (2.16). This concludes the proof. □

## 7. Asymptotic Shape of Minimizers: Proof of Theorem 6

*Proof of Theorem 6.* The first assertion is a direct consequence of Theorem 3, since $\lambda_k < \lambda_c(m_k)$ for $k$ large enough.

We now prove the second assertion. Let $\Omega_k$ be a minimizer of $E_{\lambda_k}$ over $\mathcal{K}_{m_k}$. Recalling Remark 5, without loss of generality we can assume that

$$P(\Omega_k) = \mathcal{H}^1(\partial\Omega_k). \tag{7.1}$$

By a change of variables $x = r_k\tilde{x}$, with $r_k := \sqrt{m_k/\pi}$ we obtain that

$$E_{\lambda_k}(\Omega_k) = r_k F_k(\widetilde{\Omega}_k), \tag{7.2}$$

where $\widetilde{\Omega}_k := r_k^{-1}\Omega_k$, so that, in particular, $|\widetilde{\Omega}_k| = \pi$, and

$$F_k(\Omega) := P(\Omega) + \frac{\lambda_k\pi}{m_k}\mathcal{I}_1(\Omega) + r_k\int_\Omega g\,(r_k\tilde{x})\,d\tilde{x}. \tag{7.3}$$

Observe that since $g \in \mathcal{G}$, there exists $x_0 \in \mathbb{R}^2$ such that $g(x_0) = \min g$, and without loss of generality we may assume that $x_0 = 0$. By the minimality of $\Omega_k$ we have that

$$P(\widetilde{\Omega}_k) + \frac{\lambda_k\pi}{m_k}\mathcal{I}_1(\widetilde{\Omega}_k) + r_k\int_{\widetilde{\Omega}_k} g\,(r_k\tilde{x})\,d\tilde{x} \leq P(B_1(0)) + \frac{\lambda_k\pi}{m_k}\mathcal{I}_1(B_1(0)) + r_k\int_{B_1(0)} g\,(r_k\tilde{x})\,d\tilde{x}. \tag{7.4}$$

Notice also that, since the gradient of $g$ is locally bounded, we have

$$0 \leq \int_{B_1(0)} (g\,(r_k\tilde{x}) - g(0))\,d\tilde{x} \leq Cr_k, \tag{7.5}$$

where $C > 0$ depends only on $g$, for all $k$ large enough. From (7.4) and (7.5) we then get

$$P(\widetilde{\Omega}_k) + \frac{\lambda_k\pi}{m_k}\mathcal{I}_1(\widetilde{\Omega}_k) + r_k\int_{\widetilde{\Omega}_k} (g\,(r_k\tilde{x}) - g(0))\,d\tilde{x} \leq P(B_1(0)) + \frac{\lambda_k\pi}{m_k}\mathcal{I}_1(B_1(0)) + Cr_k^2, \tag{7.6}$$

and we note that the integral in the left-hand side is non-negative.

We recall from Lemma 9 the inequality

$$P(B_1(0)) + \lambda\mathcal{I}_1(B_1(0)) \leq \mathcal{H}^1(\widetilde{\Omega}_k) + \lambda\mathcal{I}_1(\widetilde{\Omega}_k) = P(\widetilde{\Omega}_k) + \lambda\mathcal{I}_1(\widetilde{\Omega}_k), \tag{7.7}$$

where the last equality follows from Remark 5, for all $\lambda \leq 4$. Hence we get

$$\mathcal{I}_1(B_1(0)) - \mathcal{I}_1(\widetilde{\Omega}_k) \leq \frac{1}{\lambda}\left(P(\widetilde{\Omega}_k) - P(B_1(0))\right). \tag{7.8}$$

From (7.6) and (7.8) with $\lambda = 4$ we then obtain

$$\left(1 - \frac{\lambda_k\pi}{4m_k}\right)\left(P(\widetilde{\Omega}_k) - P(B_1(0))\right) + r_k\int_{\widetilde{\Omega}_k}\left(g\left(r_k\tilde{x}\right) - g(0)\right)\,d\tilde{x} \leq Cr_k^2. \tag{7.9}$$

By the isoperimetric inequality in quantitative form [16], there exist $\tilde{x}_k \in \mathbb{R}^2$ such that

$$|\widetilde{\Omega}_k \Delta B_1(\tilde{x}_k)|^2 \leq Cm_k\left(1 - \frac{\lambda_k\pi}{4m_k}\right)^{-1}, \tag{7.10}$$

for all $k$ small enough, for some constant $C > 0$ depending only on $g$. Hence, recalling the assumption on $\lambda_k$, $m_k$, we obtain

$$\lim_{k\to+\infty}|\widetilde{\Omega}_k \Delta B_1(\tilde{x}_k)| = 0, \tag{7.11}$$

implying that $\widetilde{\Omega}_k$ converge to $B_1(0)$ in the $L^1$-sense. Hausdorff convergence of $\widetilde{\Omega}_k$ and of their complements then follows from the fact that the density estimates in Theorem 4 can be easily seen to hold for $\widetilde{\Omega}_k$ uniformly in $k$. We conclude, since Hausdorff convergence of a sequence of sets and of their complements entails Hausdorff convergence of the boundaries of the sequence (see, for instance, [23, Theorem 2.7]).

Similarly, from (7.9) written in the original unscaled variables, and with the help of the isoperimetric inequality we infer that

$$\frac{1}{m_k}\int_{\mathbb{R}^2}\chi_{\Omega_k}(x)\bar{g}(x)\,\mathrm{d}x - g(0) \leq Cm_k^{1/2}, \tag{7.12}$$

where $\bar{g}(x) = \min\{g(x), g(0) + 1\}$ and $\chi_{\Omega_k}$ are the characteristic functions of $\Omega_k$. At the same time, defining $x_k := r_k\tilde{x}_k$ and using (7.10) we have

$$\left|\int_{\mathbb{R}^2}(\chi_{\Omega_k} - \chi_{B_{r_k}(x_k)})\bar{g}\,\mathrm{d}x\right| \leq (g(0) + 1)|\Omega_k \Delta B_{r_k}(x_k)| \leq Cm_k^{3/2}, \tag{7.13}$$

for $C > 0$ depending only on $g$ and all $k$ large enough. Thus, we have that (7.12) also holds with $\chi_{\Omega_k}$ replaced with $\chi_{B_{r_k}(x_k)}$, and by Lipschitz continuity of $\bar{g}$ we obtain

$$\bar{g}(x_k) = \frac{1}{m_k}\int_{\mathbb{R}^2}\chi_{B_{r_k}(x_k)}(x)\bar{g}(x_k)\,\mathrm{d}x \leq \frac{1}{m_k}\int_{\mathbb{R}^2}\chi_{B_{r_k}(x_k)}(x)\bar{g}(x)\,\mathrm{d}x + Cm_k^{1/2}$$

$$\leq \frac{1}{m_k} \int_{\mathbb{R}^2} \chi_{\Omega_k}(x)\bar{g}(x)\,\mathrm{d}x + C' m_k^{1/2} \leq g(0) + C'' m_k^{1/2}, \tag{7.14}$$

for some $C, C', C'' > 0$ depending only on $g$ and all $k$ large enough. In particular, $\bar{g}(x_k) = g(x_k)$ for all $k$ sufficiently large, and by coercivity of $g$ the sequence of $x_k$ is bounded. Thus, it is the desired sequence.

Finally, to prove the third assertion of the theorem, we pass to the limit $k \to \infty$ in (7.14), after extracting a convergent subsequence, and use continuity of $g$. □

## 8. The Euler–Lagrange Equation: Proof of Theorem 7

The aim of this section is to obtain the Euler–Lagrange equation satisfied by regular critical points of the functional $E_\lambda$. In order to do this, we first compute the first variation of an auxiliary functional which will be shown to be related to the capacitary energy.

Given an open set $\Omega \subset \mathbb{R}^2$, not necessarily bounded, and a function $f \in L^{\frac{4}{3}}(\mathbb{R}^2)$, we define

$$I_{\Omega,f}(v) := \begin{cases} \frac{1}{2}\|v\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} - \int_{\mathbb{R}^2} f v \,\mathrm{d}x & \text{if } v \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2) \text{ and } v|_{\Omega^c} = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{8.1}$$

Notice that since the space $\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ continuously embeds into $L^4(\mathbb{R}^2)$, the functional $I_{\Omega,f}$ admits a unique minimizer $u_{\Omega,f} \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$, which satisfies

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u_{\Omega,f} = f & \text{on } \Omega, \\ u_{\Omega,f} = 0 & \text{on } \Omega^c, \end{cases} \tag{8.2}$$

in the distributional sense, namely (see [26, Eq. (4.14)]):

$$\int_\Omega u_{\Omega,f} (-\Delta)^{\frac{1}{2}} \varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x \qquad \forall \varphi \in C_c^\infty(\Omega), \tag{8.3}$$

where

$$(-\Delta)^{\frac{1}{2}} \varphi(x) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{2\varphi(x) - \varphi(x-y) - \varphi(x+y)}{|y|^3} \,\mathrm{d}y \qquad x \in \mathbb{R}^2, \tag{8.4}$$

with the usual convention of extending $\varphi$ by zero outside $\Omega$. Furthermore, when $u_{\Omega,f}|_\Omega \in C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ for some $\alpha \in (0,1)$, we also have that (8.2) holds pointwise in $\Omega$, with the definition of $(-\Delta)^{\frac{1}{2}}$ in (8.4) extended to such functions [34, Section 3]. In addition, in this case we have

$$J_f(\Omega) := \min I_{\Omega,f} = -\frac{1}{2} \int_\Omega u_{\Omega,f} f \,\mathrm{d}x = -\frac{1}{2} \int_\Omega u_{\Omega,f} \,(-\Delta)^{\frac{1}{2}} u_{\Omega,f} \,\mathrm{d}x. \tag{8.5}$$

The following lemma gives a basic regularity result for the Dirichlet problem in (8.2).

**Lemma 23.** *Let* $f \in L^\infty(\mathbb{R}^2) \cap L^{\frac{4}{3}}(\mathbb{R}^2)$*, let* $\Omega \subset \mathbb{R}^2$ *be an open set and let* $u_{\Omega,f}$ *be the minimizer of* $I_{\Omega,f}$*. Then there exists a constant* $C > 0$ *depending only on* $f$ *such that* $\|u_{\Omega,f}\|_{L^\infty(\mathbb{R}^2)} \le C$.
*If in addition* $f|_\Omega \in C^\alpha_{\mathrm{loc}}(\Omega)$ *for some* $\alpha \in (0,1)$ *then* $u_{\Omega,f}|_\Omega \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

*Proof.* Let $\varphi \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ be the unique solution to $(-\Delta)^{\frac{1}{2}}\varphi = -f$ in $\mathbb{R}^2$ (for a detailed discussion of the notion and the representations of the solution, see [26, Section 4]). In particular, since by assumption $f \in L^{\frac{4}{3}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, from [26, Lemma 4.1] it follows that $\varphi \in L^\infty(\mathbb{R}^2)$. Furthermore, since we have

$$\frac{1}{2}\|v\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} - \int_{\mathbb{R}^2} vf\,\mathrm{d}x = \frac{1}{2}\|v+\varphi\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} - \frac{1}{2}\|\varphi\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \qquad \text{for any } v \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2), \tag{8.6}$$

we get that the function $w_{\Omega,f} := u_{\Omega,f} + \varphi$ solves the minimum problem

$$w_{\Omega,f} = \operatorname{argmin}\left\{\|w\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \;:\; w \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2),\ w|_{\Omega^c} = \varphi\right\}. \tag{8.7}$$

By an explicit computation, for any $w \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ we have $\|\bar{w}\|_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \le \|w\|_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}$, where

$$\bar{w} := \min(\max(w, -\|\varphi\|_{L^\infty(\mathbb{R}^2)}), \|\varphi\|_{L^\infty(\mathbb{R}^2)}). \tag{8.8}$$

It then follows that $\|w_{\Omega,f}\|_{L^\infty(\mathbb{R}^2)} \le \|\varphi\|_{L^\infty(\mathbb{R}^2)}$, yielding

$$\|u_{\Omega,f}\|_{L^\infty(\mathbb{R}^2)} \le \|w_{\Omega,f}\|_{L^\infty(\mathbb{R}^2)} + \|\varphi\|_{L^\infty(\mathbb{R}^2)} \le 2\,\|\varphi\|_{L^\infty(\mathbb{R}^2)}. \tag{8.9}$$

Finally, Hölder regularity of the derivative of $u$ is an immediate consequence of [34, Eq. (6.2)] (see also the references therein). □

We now recall the definition of the normal $\frac{1}{2}$-derivative of the function $u_{\Omega,f}$ vanishing at points of $\partial\Omega$:

$$\partial_\nu^{1/2} u_{\Omega,f}(x) := \lim_{s\to 0^+} \frac{u_{\Omega,f}(x - s\nu(x))}{s^{1/2}} \qquad x \in \partial\Omega, \tag{8.10}$$

where $\nu(x)$ is the outward unit normal vector. We have the following result that will be crucial for the computation of the shape derivative of $J_f(\Omega)$.

**Lemma 24.** *Let* $\Omega_n, \Omega_\infty \subset \mathbb{R}^2$*,* $n \in \mathbb{N}$*, be open sets whose boundaries are uniformly bounded and uniformly of class* $C^{1,\alpha}$ *for some* $\alpha \in (0,1/2)$*. Let* $f \in L^\infty(\mathbb{R}^2) \cap L^{\frac{4}{3}}(\mathbb{R}^2)$ *and assume that* $\Omega_n \to \Omega_\infty$*, as* $n \to \infty$*, in the Hausdorff distance. Then, for all* $n \in \mathbb{N} \cup \{\infty\}$ *the function* $\partial_\nu^{1/2} u_{\Omega_n,f}$ *can be continuously extended to a function* $\overline{D}_n \in C^\alpha(\mathbb{R}^2)$ *such that* $\overline{D}_n \to \overline{D}_\infty$ *as* $n \to \infty$*, locally uniformly in* $\mathbb{R}^2$.

*Proof.* Denote $u_n := u_{\Omega_n,f}$ for simplicity. Let $R_1 > 2R_0 > 0$ be such that $\partial\Omega_n \subset B_{R_0/2}(0)$ and $B_{R_0}(0) \subset B_{R_1/2}(x_0)$ for all $n \in \mathbb{N} \cup \{\infty\}$ and $x_0 \in \partial\Omega_n$. Let also $\widetilde{\Omega}_n := \Omega_n \cap B_{R_0+R_1}(0)$.

Notice that from Lemma 23 it follows that $\|u_n\|_{L^\infty(\mathbb{R}^2)} \le C$ for some constant $C > 0$ independent of $n$. Then by [35, Proposition 1.1], applied with $\Omega$ replaced by $\widetilde{\Omega}_n$ and $B_1(0)$ replaced by $B_{R_1}(x_0)$ for some $x_0 \in \partial\Omega_n$, the sequence $(u_n)$ is uniformly bounded in $C^{1/2}(B_{R_0}(0))$. We observe that the $C^{1/2}$-estimate in [35] is uniform in $n$ since the involved constants depend only on the $C^{1,\alpha}$-norm of the boundary of $\partial\widetilde{\Omega}_n$. As a consequence, by Arzelà-Ascoli Theorem, up to passing to a subsequence, the functions $u_n$ converge as $n \to \infty$ to $u^*$ uniformly in $\overline{B}_{R_0}(0)$.

To identify the limit function $u^*$, we establish the $\Gamma$-convergence of the functional $I_{\Omega_n,f}$ to $I_{\Omega_\infty,f}$ with respect to the weak convergence in $\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$. The latter is the natural topology, since the minimizers of $I_{\Omega_n,f}$ are uniformly bounded in $\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ independently of $n$. Indeed, by Hölder inequality we have

$$0 = I_{\Omega_n,f}(0) \ge \frac{1}{2}\|u_n\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} - \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|u_n\|_{L^4(\mathbb{R}^2)}, \tag{8.11}$$

and the last term is dominated by the first term in the right-hand side by fractional Sobolev inequality [11, Theorem 6.5]. The $\Gamma - \liminf$ follows from lower-semicontinuity of the $\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$-norm and the continuity of the linear term, together with the fact that the limit function vanishes a. e. in $\Omega^c_\infty$ by compact embedding of $\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ into $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p < 4$ [11, Corollary 7.2]. Finally, the $\Gamma - \limsup$ follows by approximating the limit function by a function from $C^\infty_c(\Omega_\infty)$, for which we have pointwise convergence of $I_{\Omega_n,f}$, and a diagonal argument. As a corollary to this result, we have that $u_n \rightharpoonup u_\infty$ in $\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ and, hence, by uniqueness of the minimizer of $I_{\Omega_\infty,f}$, we also have $u_n \to u_\infty$ a. e. in $\mathbb{R}^2$. In particular, $u^* = u_\infty$ a. e. in $B_{R_0}(0)$.

We now consider the functions $D_n : \widetilde{\Omega}_n \to \mathbb{R}$, $D_n(x) := u_n(x)/d_n^{1/2}(x)$, where $d_n(x) := \mathrm{dist}(x, \widetilde{\Omega}^c_n)$ and $n \in \mathbb{N} \cup \{\infty\}$. Then by [35, Theorem 1.2] (see also [10]), applied as before with $\Omega$ replaced by $\widetilde{\Omega}_n$ and $B_1(0)$ replaced by $B_{R_1}(x_0)$ for some $x_0 \in \partial\Omega_n$, the sequence $(D_n)$ is uniformly bounded in $C^\alpha(\overline{B}_{R_0}(0))$. By classical extension theorems (see for instance [19, Theorem 6.38]) for all $n \in \mathbb{N}$ we can extend $D_n$ to a function $\overline{D}_n : \overline{B}_{R_0}(0) \to \mathbb{R}$ such that

$$\|\overline{D}_n\|_{C^\alpha(\overline{B}_{R_0}(0))} \le C_0\|D_n\|_{C^\alpha(\overline{B}_{R_0}(0))} \le C, \tag{8.12}$$

where the constants $C_0,\ C$ are independent of $n$. Again by Arzelà-Ascoli Theorem, up to passing to a subsequence, the functions $\overline{D}_n$ converge as $n \to \infty$ to a function $\overline{D}^* \in C^\alpha(\overline{B}_{R_0}(0))$ uniformly. Moreover, from the convergence of $u_n$ to $u_\infty$ we get that $\overline{D}^*|_{\widetilde{\Omega}_\infty \cap \overline{B}_{R_0}(0)} = D_\infty$.

Finally, we observe that $\overline{D}_n$ is a continuous extension of $\partial^{1/2}_\nu u_n$ for all $n \in \mathbb{N} \cup \{\infty\}$, since we have, for any $x \in \partial\Omega_n$,

$$\overline{D}_n(x) = \lim_{s\to 0^+} D_n(x - s\nu(x)) = \lim_{s\to 0^+} \frac{u_n(x - s\nu(x))}{d_n(x - s\nu_{\Omega_n}(x))^{1/2}} = \partial^{1/2}_\nu u_n(x), \tag{8.13}$$

concluding the proof. □

**Corollary 25.** *Under the assumptions of Lemma 24, let* $x_n \in \partial\Omega_n$ *and* $x \in \partial\Omega_\infty$ *be such that* $x_n \to x \in \partial\Omega_\infty$. *Then* $\partial_\nu^{1/2} u_n(x_n) \to \partial_\nu^{1/2} u_\infty(x)$ *as* $n \to +\infty$.

*Proof.* Consider the extensions $\overline{D}_n$, $n \in \mathbb{N} \cup \{\infty\}$, constructed in the proof of the previous lemma. Then we have

$$\begin{aligned} |\partial_\nu^{1/2} u_n(x_n) - \partial_\nu^{1/2} u_\infty(x)| &= |\overline{D}_n(x_n) - \overline{D}_\infty(x)| \\ &\leq |\overline{D}_n(x_n) - \overline{D}_n(x)| + |\overline{D}_n(x) - \overline{D}_\infty(x)| \end{aligned} \tag{8.14}$$

and the right-hand side of the latter inequality converges to 0 as $n \to +\infty$. □

We now compute the first variation of the functional $J_f$. We note that for bounded domains and under stronger regularity assumptions such a computation was carried out in [8], with a relatively long and technical proof. Here we provide an alternative, shorter proof, that also covers the case of unbounded domains and weaker assumptions on the regularity of $f$ and $\partial\Omega$.

**Theorem 26.** *Let* $f \in L^\infty(\mathbb{R}^2) \cap L^{\frac{4}{3}}(\mathbb{R}^2)$ *be such that* $f|_\Omega \in C^\alpha_{\mathrm{loc}}(\Omega)$ *for some* $\alpha \in (0, 1)$. *Let* $\Omega$ *be an open set with compact boundary of class* $C^2$, *and let* $u_{\Omega,f}$ *be the unique minimizer of* $I_{\Omega,f}$. *Let* $\zeta \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, *and let* $(\Phi_t)_{t\in\mathbb{R}}$ *be a smooth family of diffeomorphisms of the plane satisfying* $\Phi_0 = \mathrm{Id}$ *and* $\frac{d}{dt}\Phi_t\big|_{t=0} = \zeta$. *Then, if* $\nu$ *is the outward pointing normal vector to* $\partial\Omega$, *the normal* $\frac{1}{2}$*-derivative* $\partial_\nu^{1/2} u_{\Omega,f}$ *is well-defined and belongs to* $C^\beta(\partial\Omega)$ *for any* $\beta \in (0, 1/2)$. *Moreover, we have*

$$\frac{d}{dt} J_f(\Phi_t(\Omega))\bigg|_{t=0} = -\frac{\pi}{8} \int_{\partial\Omega} (\partial_\nu^{1/2} u_{\Omega,f}(x))^2 \zeta(x) \cdot \nu(x)\, d\mathcal{H}^1(x). \tag{8.15}$$

*Proof.* Let $\Omega_t := \Phi_t(\Omega)$. Since $\partial\Omega$ is of class $C^2$, for all $x \in \partial\Omega_t$ and $|t|$ small enough we can write

$$\Phi_t^{-1}(x) = x + t\rho_t(x)\nu_t(x), \tag{8.16}$$

where $\rho_t \in C^2(\partial\Omega_t)$ is a scalar function and $\nu_t$ is the unit outward normal to $\partial\Omega_t$. Furthermore, the right-hand side of (8.16) establishes a bijection between $\partial\Omega_t$ and $\partial\Omega$, and we have

$$\rho_0(x) := \lim_{t\to 0} \rho_t(x) = -\zeta(x) \cdot \nu(x) \qquad \forall x \in \partial\Omega. \tag{8.17}$$

For $t > 0$ sufficiently small, let $\Omega_t \subset \Omega$ be a regular inward deformation of $\Omega$, namely, $\Omega_t$ is such that (8.16) holds true with some $\rho_t \geq 0$. Note that it is enough to consider inward perturbations, since for outward perturbations one would simply interchange the roles of $\Omega_t$ and $\Omega$ in the argument below.

We denote $u := u_{\Omega,f}$ and $u_t := u_{\Omega_t,f}$ for simplicity. Recall that $u$ and $u_t$ solve pointwise

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{8.18}$$

and

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u_t = f & \text{in} \quad \Omega_t, \\ u_t = 0 & \text{in} \quad \Omega_t^c. \end{cases} \tag{8.19}$$

In particular, by [35, Theorem 1.2] we have $|u(x)| \le C\sqrt{\operatorname{dist}(x, \partial\Omega)}$ for some constant $C > 0$, which in turn implies that $|(-\Delta)^{\frac{1}{2}} u(x)| \le C/\sqrt{\operatorname{dist}(x, \partial\Omega)}$, and the same holds for the function $u_t$, with $\Omega$ replaced by $\Omega_t$ and the constant $C$ independent of $t$ for all small enough $t$. These estimates justify all the computations of integrals involving $u$ and $u_t$ below.

From (8.18), (8.19) and (8.5) we get

$$\begin{aligned} J_f(\Omega_t) - J_f(\Omega) &= \frac{1}{2}\int_{\mathbb{R}^2} u(-\Delta)^{\frac{1}{2}} u \, dx - \frac{1}{2}\int_{\mathbb{R}^2} u_t(-\Delta)^{\frac{1}{2}} u_t \, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^2} (u_t + u)(-\Delta)^{\frac{1}{2}}(u_t - u) \, dx \\ &= -\frac{1}{2}\int_{\Omega\setminus\Omega_t} u(-\Delta)^{\frac{1}{2}} u_t \, dx + \frac{1}{2}\int_{\Omega\setminus\Omega_t} uf \, dx. \end{aligned} \tag{8.20}$$

The last term in (8.20) satisfies

$$\left| \frac{1}{2}\int_{\Omega\setminus\Omega_t} uf \, dx \right| \le C \, \|f\|_{L^\infty(\Omega\setminus\Omega_t)} \, \|\rho_t\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}} \, t^{\frac{1}{2}} \, |\Omega \setminus \Omega_t| \le C' t^{\frac{3}{2}} = o(t). \tag{8.21}$$

We thus focus on the first term of the right-hand side of (8.20). We have

$$\begin{aligned} -\frac{1}{2}\int_{\Omega\setminus\Omega_t} u(-\Delta)^{\frac{1}{2}} u_t \, dx &= -\frac{1}{8\pi}\int_{\Omega\setminus\Omega_t}\int_{\mathbb{R}^2} u(x) \frac{2u_t(x) - u_t(x-y) - u_t(x+y)}{|y|^3} \, dy \, dx \\ &= \frac{1}{4\pi}\int_{\Omega\setminus\Omega_t}\int_{\Omega_t} \frac{u(x)u_t(y)}{|x-y|^3} \, dy \, dx. \end{aligned} \tag{8.22}$$

Next we split the integral over $\Omega_t$ in (8.22) into integrals over $\Omega^R$ and $\Omega_t \setminus \Omega^R$, where

$$\Omega^R := \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > R\} \tag{8.23}$$

and $R > 0$ is such that $\partial\Omega^R$ is of class $C^2$ and $\Omega_t \setminus \Omega^R$ consists of a union of disjoint strip-like domains. We have

$$\begin{aligned} &-\frac{1}{2}\int_{\Omega\setminus\Omega_t} u(-\Delta)^{\frac{1}{2}} u_t \, dx \\ &= \frac{1}{4\pi}\int_{\Omega\setminus\Omega_t}\int_{\Omega^R} \frac{u(x)u_t(y)}{|x-y|^3} \, dy \, dx + \frac{1}{4\pi}\int_{\Omega\setminus\Omega_t}\int_{\Omega_t\setminus\Omega^R} \frac{u(x)u_t(y)}{|x-y|^3} \, dy \, dx \\ &= \frac{1}{4\pi}\int_{\Omega\setminus\Omega_t}\int_{\Omega^R} \frac{u(x)u_t(y)}{|x-y|^3} \, dy \, dx + \frac{1}{4\pi} \\ &\quad \times \int_{\partial\Omega_t}\int_{\partial\Omega_t}\int_0^{t\rho_t(x)}\int_0^R \frac{u(x + s\nu_t(x))u_t(y - s'\nu_t(y))}{|x - y + s\nu_t(x) + s'\nu_t(y)|^3} \\ &\quad \times (1 + s\kappa(x))(1 - s'\kappa(y)) \, ds' \, ds \, d\mathcal{H}^1(y) \, d\mathcal{H}^1(x), \end{aligned} \tag{8.24}$$

where $\kappa$ is the curvature of $\partial\Omega_t$, positive if $\Omega_t$ is convex. As above, one can check that the first term on the right-hand side of (8.24) is $O(t^{3/2}R^{-2})$ for all $t$ small enough, hence we can focus again just on the second term.

To estimate the last integral in the right-hand side of (8.24), we first observe that the curvature contributions inside the brackets can be bounded by $O(R)$ and, therefore, a posteriori give rise to errors of order $O(Rt)$ for all $t$ small enough, as the integral itself will be shown to be $O(t)$. Thus, we have

$$
\begin{aligned}
-\frac{1}{2}\int_{\Omega\setminus\Omega_t} u(-\Delta)^{\frac{1}{2}}u_t\,\mathrm{d}x = {} & \frac{1}{4\pi} \\
& \times \int_{\partial\Omega_t}\int_{\partial\Omega_t}\int_0^{t\rho_t(x)} \\
& \int_0^R \frac{u(x+s\nu_t(x))u_t(y-s'\nu_t(y))}{|x-y+s\nu_t(x)+s'\nu_t(y)|^3}\,ds'\,ds\,d\mathcal{H}^1(y)\,d\mathcal{H}^1(x) \\
& +O(t^{3/2}R^{-2})+O(Rt).
\end{aligned} \tag{8.25}
$$

For $x\in\partial\Omega_t$, we let

$$
F(x) := \int_{\partial\Omega_t}\int_0^{t\rho_t(x)}\int_0^R \frac{u(x+s\nu_t(x))u_t(y-s'\nu_t(y))}{|x-y+s\nu_t(x)+s'\nu_t(y)|^3}\,ds'\,ds\,d\mathcal{H}^1(y), \tag{8.26}
$$

and split the integral over $y$ into a near-field part

$$
F_R(x) := \int_{\partial\Omega_t\cap B_R(x)}\int_0^{t\rho_t(x)}\int_0^R \frac{u(x+s\nu_t(x))u_t(y-s'\nu_t(y))}{|x-y+s\nu_t(x)+s'\nu_t(y)|^3}\,ds'\,ds\,d\mathcal{H}^1(y), \tag{8.27}
$$

and the far field part $F(x)-F_R(x)$. As with (8.25), the latter may be estimated to be $O(t^{3/2}R^{-2})$, so we focus on the computation of $F_R(x)$. To that end, we let $y=y(\sigma)$ be the arc-length parametrization of $\partial\Omega_t\cap B_R(x)$ relative to $x$ and observe that

(i) $(y(\sigma)-x)\cdot\nu_t(x)=O(\sigma^2)$,
(ii) $\nu_t(x)\cdot\nu_t(y(\sigma))=1+O(\sigma^2)$,
(iii) $|y(\sigma)-x|=\sigma+O(\sigma^3)$,

uniformly in $x$ and $t$. Therefore,

$$
\begin{aligned}
|y(\sigma)-x-s\nu_t(x)-s'\nu_t(y(\sigma))|^2 &= |y(\sigma)-x-(s+s')\nu_t(x)-s'(\nu_t(y(\sigma)-\nu_t(x)))|^2 \\
&= |y(\sigma)-x|^2+(s+s')^2-2(y(\sigma)-x)\cdot\nu_t(x)(s+s') \\
&\quad +|s'|^2|\nu_t(y(\sigma))-\nu_t(x)|^2 \\
&\quad -2s'(y(\sigma)-x)\cdot(\nu_t(y(\sigma))-\nu_t(x))+2(s+s')s'\nu_t(x)\cdot(\nu_t(y(\sigma))-\nu_t(x)) \\
&= \sigma^2+(s+s')^2+O(\sigma^4)+O(\sigma^2R)+O(\sigma^2R^2) \\
&= (\sigma^2+(s+s')^2)(1+O(R)),
\end{aligned} \tag{8.28}
$$

again, uniformly in $x$ and $t$, for all $R$ small enough. Thus we have

$$
|y(\sigma)-x-s\nu_t(x)-s'\nu_t(y(\sigma))|^{-3} = (\sigma^2+(s+s')^2)^{-\frac{3}{2}}(1+O(R)). \tag{8.29}
$$

By the uniform convergence of $\partial_\nu^{1/2} u_t$ to $\partial_\nu^{1/2} u$ as $t \to 0$, and the fact that $\partial_\nu^{1/2} u_t$ is of class $C^\beta(\partial\Omega_t)$ for all $\beta \in (0, 1/2)$ (by Lemma 24), we have that

$$\begin{aligned} u(x + s\nu_t(x)) &= (1 + o_t(1))\, \partial_\nu^{1/2} u(x + t\rho_t(x)\nu_t(x))\sqrt{t\rho_t(x) - s} \\ &= (1 + o_t(1))\, \partial_\nu^{1/2} u_t(x)\sqrt{t\rho_t(x) - s} \end{aligned} \tag{8.30}$$

and

$$\begin{aligned} u_t(y(\sigma) - s'\nu_t(y(\sigma))) &= (1 + o_t(1))\, \partial_\nu^{1/2} u_t(y(\sigma))\sqrt{s'} \\ &= (1 + o_t(1) + o_R(1))\, \partial_\nu^{1/2} u_t(x)\sqrt{s'}. \end{aligned} \tag{8.31}$$

Plugging (8.29), (8.30) and (8.31) into (8.27), we get

$$F_R(x) = (1 + o_t(1) + o_R(1)) \int_{\sigma_R^-(x)}^{\sigma_R^+(x)} \int_0^{t\rho_t(x)} \int_0^R \frac{|\partial_\nu^{1/2} u_t(x)|^2 \sqrt{(t\rho_t(x) - s)s'}}{(\sigma^2 + (s + s')^2)^{3/2}}\, ds'\, ds\, d\sigma, \tag{8.32}$$

where $\sigma_R^\pm(x) = \pm R + O(R^3)$.

Observe that $F_R(x) = 0$ if $\rho_t(x) = 0$. If $\rho_t(x) > 0$, we can perform the change of variables

$$z = \frac{s}{t\rho_t(x)}, \qquad z' = \frac{s'}{t\rho_t(x)}, \qquad \zeta = \frac{\sigma}{t\rho_t(x)}, \tag{8.33}$$

to obtain

$$\begin{aligned} F_R(x) &= (1 + o_t(1) + o_R(1))\, t\rho_t(x) |\partial_\nu^{1/2} u_t(x)|^2 \\ &\quad \times \int_{\sigma_R^-(x)/(t\rho_t(x))}^{\sigma_R^+(x)/(t\rho_t(x))} \int_0^1 \int_0^{R/(t\rho_t(x))} \frac{\sqrt{(1 - z)z'}}{(\zeta^2 + (z + z')^2)^{3/2}}\, \mathrm{d}z'\, \mathrm{d}z\, \mathrm{d}\zeta, \end{aligned} \tag{8.34}$$

which is also valid if $\rho_t(x) = 0$. By the Dominated Convergence Theorem, as $t \to 0$ the integral in the right-hand side converges to

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_0^1 \int_0^\infty \frac{\sqrt{(1 - z)z'}}{(\zeta^2 + (z + z')^2)^{3/2}}\, \mathrm{d}z'\, \mathrm{d}z\, \mathrm{d}\zeta \\ &\quad = 2 \int_0^1 \int_0^\infty \frac{\sqrt{(1 - z)z'}}{(z + z')^2}\, \mathrm{d}z'\, \mathrm{d}z = \frac{\pi^2}{2}. \end{aligned} \tag{8.35}$$

We thus have

$$-\frac{1}{2} \int_{\Omega \setminus \Omega_t} u(-\Delta)^{\frac{1}{2}} u_t\, \mathrm{d}x = (1 + o_t(1) + o_R(1)) \frac{\pi t}{8} \int_{\partial\Omega_t} |\partial_\nu u_t(x)|^2 \rho_t(x)\, d\mathcal{H}^1(x),$$

so that by the above estimates and the uniform continuity of $\rho_t$ and $\partial_\nu^{1/2} u_t$ in $t$, we get

$$\lim_{t \to 0} \frac{J_f(\Omega_t) - J_f(\Omega)}{t} = (1 + o_R(1))\, \frac{\pi}{8} \int_{\partial\Omega} |\partial_\nu u(x)|^2 \rho_0(x) d\mathcal{H}^1(x). \tag{8.36}$$

Finally, by (8.17) the thesis follows by sending $R \to 0$ in (8.36) and Lemma 24. □

*Remark 27.* As was mentioned earlier, for bounded domains and under stronger regularity assumptions the result in Theorem 26 was obtained in [8] with a different proof. In [8, Theorem 1], the first variation is stated with a non-explicit constant, but an analysis of the proof shows that their constant agrees with ours, as it should. Our proof exploits the boundary regularity for non-local elliptic problems developed in [35] (see also the survey [34] and [10]), which simplifies the proof even for bounded domains.

We are now in a position to compute the first variation of the functional $\mathcal{I}_1$ on $C^2$–regular bounded sets, and consequently the Euler–Lagrange equation for $E_\lambda$ for such sets. Recalling (3.6), it is enough to compute the first variation of the $\frac{1}{2}$-capacity $\mathrm{cap}_1(\Omega)$, which follows directly from Theorem 26, as we show below.

**Theorem 28.** *Let $\Omega$ be a compact set with boundary of class $C^2$, let $\nu$ be the outward pointing normal vector to $\partial\Omega$ and let $u_\Omega$ be the $\frac{1}{2}$-capacitary potential of $\Omega$ defined in (3.11). Then, the $1/2$-derivative $\partial_\nu^{1/2} u_\Omega$ is well-defined and belongs to $C^\beta(\partial\Omega)$ for any $\beta \in (0, 1/2)$. Moreover, letting $\zeta$ and $\Phi_t$ be as in Theorem 26, there holds*

$$\frac{d}{dt}\mathrm{cap}_1(\Phi_t(\Omega))\bigg|_{t=0} = \frac{\pi}{4}\int_{\partial\Omega}(\partial_\nu^{1/2}u_\Omega(x))^2\zeta(x)\cdot\nu(x)\,d\mathcal{H}^1(x). \tag{8.37}$$

*Proof.* Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ be such that $\varphi = 1$ in an open neighborhood of $\Omega$. Observe that our choice of $\varphi$ implies that, if

$$f(x) := -(-\Delta)^{\frac{1}{2}}\varphi(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{2\varphi(x)-\varphi(x-y)-\varphi(x+y)}{|y|^3}\,\mathrm{d}y, \tag{8.38}$$

then $\in L^\infty(\mathbb{R}^2)\cap L^{\frac{4}{3}}(\mathbb{R}^2)\cap C^{0,1}(\mathbb{R}^2)$. Notice also that any test function $u$ in the definition of $\mathrm{cap}_1(\Omega)$ such that $u = 1$ on $\Omega$ can be put in correspondence with a test function $v = u - \varphi$ in the definition of the auxiliary functional $I_{\Omega^c,f}$ in (8.1). Moreover, since

$$\frac{1}{2}\|u\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} = \frac{1}{2}\|v\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} + \frac{1}{2}\|\varphi\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} - \int_{\mathbb{R}^2} vf\,\mathrm{d}x, \tag{8.39}$$

we get that

$$\frac{1}{2}\mathrm{cap}_1(\Omega) = \frac{1}{2}\|\varphi\|^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} + J_f(\Omega^c), \tag{8.40}$$

and the minimizer $u_\Omega$ satisfies $u_\Omega = v_{\Omega^c,f} + \varphi$, where $v_{\Omega^c,f}$ is the minimizer of $I_{\Omega^c,f}$. Observing also that $\partial_\nu^{1/2}u_\Omega = \partial_\nu^{1/2}v_{\Omega^c,f}$, the conclusion follows by Theorem 26. □

Finally, Theorem 7 is a direct consequence of Theorem 28, together with (3.6) and (3.11).

*Acknowledgements.* The work of CBM was partially supported by NSF via grants DMS-1614948 and DMS-1908709. MN has been supported by GNAMPA-INdAM and by the University of Pisa via grant PRA 2017-18. BR was partially supported by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR) and the GNAMPA-INdAM Project 2019 "Ottimizzazione spettrale non lineare".



## References

1. Ambrosio, L., Caselles, V., Masnou, S., Morel, J.M.: Connected components of sets of finite perimeter and applications to image processing. *J. Eur. Math. Soc.* **3**(1), 39–92, 2001
2. Ambrosio, L., Tilli, P.: Topics on Analysis in Metric Spaces. Oxford Lecture Series in Mathematics and its Applications, 25. Oxford University Press, Oxford (2004)
3. Barrero, A., Loscertales, I.G.: Micro- and nanoparticles via capillary flows. *Annu. Rev. Fluid Mech.* **39**, 89–106, 2007
4. Basaran, O.A., Scriven, L.E.: Axisymmetric shapes and stability of isolated charged drops. *Phys. Fluids A* **1**, 795–798, 1989
5. Burton, J.C., Taborek, P.: Simulations of Coulombic fission of charged inviscid drops. *Phys. Rev. Lett.* **106**, 144501, 2011
6. Castro-Hernandez, E., García-Sánchez, P., Tan, S.H., Gañán-Calvo, A.M., Baret, J.-C., Ramos, A.: Breakup length of AC electrified jets in a microfluidic flow focusing junction. *Microfluid. Nanofluid.* **19**, 787–794, 2015
7. Choksi, R., Muratov, C.B., Topaloglu, I.: An old problem resurfaces nonlocally: Gamow's liquid drops inspire today's research and applications. *Not. Am. Math. Soc.* **64**, 1275–1283, 2017
8. Dalibard, A.-L., Gérard-Varet, D.: On shape optimization problems involving the fractional Laplacian. *ESAIM Control Optim. Calc. Var.* **19**, 976–1013, 2013
9. De Philippis, G., Hirsch, J., Vescovo, G.: Regularity of minimizers for a model of charged droplets. Preprint to appear on Commun. Math. Phys. (2019)
10. De Silva, D., Savin, O.: Boundary Harnack estimates in slit domains and applications to thin free boundary problems. *Rev. Mat. Iberoam.* **32**(3), 891–912, 2016
11. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136**(5), 521–573, 2012
12. Esposito, L., Fusco, N.: A remark on a free interface problem with volume constraint. *Convex Anal.* **18**(2), 417–426, 2011
13. Fernández de la Mora, J.: The fluid dynamics of Taylor cones. *J. Ann. Rev. Fluid Mech.* **39**, 217–243, 2007
14. Figalli, A., Maggi, F.: On the shape of liquid drops and crystals in the small mass regime. *Arch. Ration. Mech. Anal.* **201**(1), 143–207, 2011
15. Fontelos, M.A., Friedman, A.: Symmetry-breaking bifurcations of charged drops. *Arch. Ration. Mech. Anal.* **172**, 267–294, 2004
16. Fusco, N., Maggi, F., Pratelli, A.: The sharp quantitative isoperimetric inequality. *Ann. Math.* **2**(168), 941–980, 2008
17. Garzon, M., Gray, L.J., Sethian, J.A.: Numerical simulations of electrostatically driven jets from nonviscous droplets. *Phys. Rev. E* **89**, 033011, 2014
18. Gaskell, S.J.: Electrospray: principles and practice. *J. Mass Spectrom.* **32**, 677–688, 1997
19. Gilbarg, D., Trudinger, N.: *Elliptic partial differential equations of second order. Second edition.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224. Springer, Berlin, 1983

20. Goldman, M., Novaga, M., Ruffini, B.: Existence and stability for a non-local isoperimetric model of charged liquid drops. *Arch. Ration. Mech. Anal.* **217**, 1–36, 2015
21. Goldman, M., Novaga, M., Ruffini, B.: On minimizers of an isoperimetric problem with long-range interactions and convexity constraint. *Anal. PDE* **11**(5), 1113–1142, 2018
22. Hofmann, S., Mitrea, M., Taylor, M.: Geometric and transformational properties of Lipschitz domains, Semmes–Kenig–Toro domains, and other classes of finite perimeter domains. *J. Geom. Anal.* **17**(4), 593–647, 2007
23. Iglesias, J.A., Mercier, G.: Convergence of level sets in total variation denoising through variational curvatures in unbounded domains. *SIAM J. Math. Anal.* **53**(2), 1509–1545, 2021
24. Landkof, N.S.: Foundations of Modern Potential Theory. Springer, New York (1972)
25. Lieb, E.H., Loss, M.: Analysis. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI (2001)
26. Lu, J., Moroz, V., Muratov, C.: Orbital-free density functional theory of out-of-plane charge screening in graphene. *J. Nonlinear Sci.* **25**(6), 1391–1430, 2015
27. Maggi, F.: Sets of Finite Perimeter and Geometric Variational Problems. An Introduction to Geometric Measure Theory. Cambridge Studies in Advanced Mathematics, 135. Cambridge University Press, Cambridge (2012)
28. Mennucci, A.: On perimeters and volumes of fattened sets. *Int. J. Math. Math. Sci.* **2019**, 8283496, 2019
29. Muratov, C., Novaga, M.: On well-posedness of variational models of charged drops. *Proc. R. Soc. Lond. A* **472**, 20150808, 2016
30. Muratov, C., Novaga, M., Ruffini, B.: On equilibrium shapes of charged flat drops. *Commun. Pure Appl. Math.* **71**(6), 1049–1073, 2018
31. Novaga, M., Ruffini, B.: Brunn–Minkowski inequality for the 1-Riesz capacity and level set convexity for the $1/2$-Laplacian. *J. Convex Anal.* **22**, 1125–1134, 2015
32. Pólya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. Princeton University Press, Princeton (1951)
33. Rayleigh, Lord: On the equilibrium of liquid conducting masses charged with electricity. *Philos. Mag.* **14**, 184–186, 1882
34. Ros-Oton, X.: Nonlocal elliptic equations in bounded domains: a survey. *Publ. Mat.* **60**(1), 3–26, 2016
35. Ros-Oton, X., Serra, J.: Boundary regularity estimates for nonlocal elliptic equations in $C^1$ and $C^{1,\alpha}$ domains. *Ann. Mat. Pura Appl.* **196**(5), 1637–1668, 2017
36. Schmidt, T.: Strict interior approximation of sets of finite perimeter and functions of bounded variation. *Proc. Am. Math. Soc.* **143**(5), 2069–2084, 2015
37. Taylor, G.: Disintegration of water drops in an electric field. *Proc. R. Soc. Lond. A* **280**, 383–397, 1964

C. B. Muratov
Department of Mathematical Sciences,
New Jersey Institute of Technology,
Newark
NJ
07102 USA.
e-mail: muratov@njit.edu

and

M. Novaga
Department of Mathematics,
University of Pisa,
Largo B. Pontecorvo 5,
56127 Pisa
Italy.
e-mail: matteo.novaga@unipi.it

and

B. Ruffini
Institut Montpelliérain Alexander Grothendieck,
Université de Montpellier,
place Eugene Bataillon,
34095 Montpellier Cedex 5
France.

and

Dipartimento di Matematica,
Alma Mater Studiorum – Università di Bologna,
Piazza di Porta San Donato 5,
40126 Bologna
Italy.
e-mail: berardo.ruffini@unibo.it

(*Received June 16, 2020 / Accepted November 25, 2021*)
*Published online February 2, 2022*