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Existence of Traveling Waves of Invasion for Ginzburg–Landau-type Problems in Infinite Cylinders

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Abstract

We study a class of systems of reaction–diffusion equations in infinite cylinders which arise within the context of Ginzburg–Landau theories and describe the kinetics of phase transformation in second-order or weakly first-order phase transitions with non-conserved order parameters. We use a variational characterization to study the existence of a special class of traveling wave solutions which are characterized by a fast exponential decay in the direction of propagation. Our main result is a simple verifiable criterion for existence of these traveling waves under the very general assumptions of non-linearities. We also prove boundedness, regularity, and some other properties of the obtained solutions, as well as several sufficient conditions for existence or non-existence of such traveling waves, and give rigorous upper and lower bounds for their speed. In addition, we prove that the speed of the obtained solutions gives a sharp upper bound for the propagation speed of a class of disturbances which are initially sufficiently localized. We give a sample application of our results using a computer-assisted approach.

1. Introduction

This paper is concerned with the study of traveling wave solutions of reaction–diffusion systems of the gradient type

$$u_t = \Delta u + f(u), \quad f(u) = -\nabla_u V(u). \tag{1.1}$$

Here, $u = u(x,t) \in \mathbb{R}^m$, $V : \mathbb{R}^m \to \mathbb{R}$, $x = (y,z) \in \Sigma = \Omega \times \mathbb{R}$, $\Omega \subset \mathbb{R}^{n-1}$ is a bounded domain, so Σ is an infinite cylinder. Either Neumann or Dirichlet boundary conditions can be chosen:

$$(n \cdot \nabla u)|_{\partial\Sigma} = 0, \quad \text{or} \quad u|_{\partial\Sigma} = 0, \tag{1.2}$$

where n is the outward normal to $\partial\Sigma$ (in fact, one could treat more complicated boundary conditions in a similar way). We assume that $f(0) = 0$, and so

$$u = 0 \tag{1.3}$$

is the trivial solution of Equations (1.1) and (1.2).

Equation (1.1) is a prototypical equation in the theory of phase transition kinetics. Systems undergoing second-order or weakly first-order phase transitions are characterized by the presence of a "soft mode" near the transition temperature. This allows one to introduce the concept of the "order parameter" to describe the thermodynamic state of the system near the transition point [29]. The order parameter is generally a vector field and can physically describe, for example, the magnitude of the spontaneous polarization, magnetization, or a structural change in a crystal. If the order parameter is a non-conserved quantity, as is the case in ferroelectrics and ferromagnets, for example, the relaxation of the soft mode toward equilibrium may be modeled as a gradient flow in $L^2(\Omega; \mathbb{R}^m)$ down the Ginzburg–Landau free energy (see, for example, [7,26,31])

$$u_t = -\frac{\delta F}{\delta u}, \quad F[u] = \int_\Omega \left(\frac{1}{2} \sum_{i=1}^m |\nabla u_i|^2 + V(u) \right) \mathrm{d}x. \tag{1.4}$$

Here $F[u]$ is a free energy functional, in which $V(u)$ is a local thermodynamic potential, typically obtained via a Taylor expansion and symmetry arguments (see, for example, [30,46]), and the gradient term penalizes spatial variations of the order parameter [29,30] (for the effect of anisotropy, see the end of Section 6).

We note that equations of the Ginzburg–Landau type can sometimes be systematically derived from more microscopic theories, such as kinetic Monte Carlo models, etc. [8,14,27]). For example, the scalar ($m = 1$) Ginzburg–Landau equation can be derived by performing a gradient expansion of the non-local evolution equation obtained for the long-range Ising model subject to Glauber dynamics near the phase transition point [8]. Let us also point out that the choice of the boundary conditions is also dictated by the physics at the surface and is, therefore, problem-dependent. For example, in the context of coarse-grained spin systems with long-range interactions mentioned above, the Dirichlet boundary conditions will be more appropriate, as opposed to the more conventional choice of Neumann boundary conditions in Ginzburg–Landau-type problems.

As an example, if u_i are the three components of the magnetization vector in a ferromagnetic crystal with cubic symmetry near the Curie temperature, and h_i are the components of the applied field, the kinetics of u may be described by the following Ginzburg–Landau equation:

$$\tau \frac{\partial u_i}{\partial t} = g\Delta u_i + h_i + a u_i - b_1 u_i^3 - b_2 u_i \sum_{i \neq j} u_j^2, \tag{1.5}$$

where a, b_1, g, τ are all positive constants, and $b_2 > -\frac{b_1}{2}$, in three space dimensions [30]. Note that Ginzburg–Landau-type equations often arise as a result of the normal form expansion near a bifurcation point for partial differential equations (see, for example, [11]). Let us also point out that scalar reaction-diffusion equations, which automatically fall into the category of gradient systems, arise in a wide variety of applications, most notably in biology [38].

Traveling wave solutions are special solutions of Equation (1.1) of the form $u(y, z, t) = \bar{u}(y, z - ct)$ with $c \in \mathbb{R}$ which describe uniformly translating "phase change regions", moving with speed c. In the following, we will only consider the solutions invading the $u = 0$ equilibrium from the left; hence we will assume $c > 0$ and $\bar{u} \to 0$ as $z \to +\infty$ everywhere below. This is an important class of solutions of Equation (1.1) which is believed to describe the long-time asymptotics of the solutions of the initial value problem for Equation (1.1) with sufficiently localized initial data (for recent developments, see [40–42]). In fact, it was recently shown that under certain assumptions only a special class of traveling wave solutions can be selected as the long-time asymptotic solution for the initial value problem [37]. These so-called *variational traveling waves* are characterized by a fast exponential decay ahead of the traveling wave solution and admit an interesting variational characterization which allows one to establish a number of their properties. This paper will be concerned with the problem of the existence of such traveling wave solutions.

Substituting the traveling wave ansatz into Equation (1.1), we obtain the following elliptic problem for $\bar{u}$:

$$\bar{u}_{zz} + \Delta_y \bar{u} + c\bar{u}_z + f(\bar{u}) = 0, \tag{1.6}$$

with the boundary conditions from Equation (1.2). This equation attracted a great deal of attention, starting with the early works of Fisher [19] and Kolmogorov, Petrovskii and Piskunov [28]. The case of scalar equations (that is, $m = 1$) has been extensively analyzed (see [5,17,50] for reviews, and more recent work in [6,24,32,34]). In particular, the fact that (1.6) can be recast into a variational form was first noted by Heinze [23,24].

Much less is known about the solutions of Equation (1.6) for systems, that is, when $m > 1$. Let us point out that it is possible to use dynamical systems techniques to obtain very general existence results for solutions of Equation (1.6) in cylinders [16,35]. The price to pay, however, is that very little information about the solutions, in particular, about their limiting behavior at the ends of the cylinder, is available [16]. So far general existence results for solutions connecting prescribed equilibria were limited to the case of monotone systems, for which the maximum principle holds [49], and gradient systems with bistable non-linearities in one space dimension [10,36,39,45].

Here we are going to establish existence of variational traveling wave solutions for Equation (1.1) with the gradient-type non-linearity under very general assumptions. By variational traveling wave, we mean a non-trivial solution of Equation (1.6) for some c that also lies in the exponentially weighted Sobolev space $H^1_c(\Sigma; \mathbb{R}^m)$ [37]. Our main existence result is contained in the following theorem (for definitions and statements of hypotheses, see Section 2).

Theorem 1.1. *Under hypotheses (H1)–(H3), there exist* $c^\dagger \geqq c > 0$*(where c is the "trial velocity" given by assumption (H3)) and* $\bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$*,* $\bar{u} \not\equiv 0$*, satisfying*

Equation (1.6) *with* $c = c^{\dagger}$. *Furthermore,* $\bar{u}$ *is a classical solution of* (1.6)*,* $\bar{u}(x) \in \mathcal{K}$ *for all* $x \in \Sigma$ *(where* $\mathcal{K}$ *is given in assumption (H2)) and* $|\bar{u}(y,z)| \leqq C e^{-\lambda z}$*, for some* $C > 0$ *and* $\lambda > 0$.

Let us give a summary of our paper here. In Section 2 we introduce the functional spaces, the exponentially weighted Sobolev spaces of vector-valued functions $H^1_c(\Sigma; \mathbb{R}^m)$, and the main variational problem, problem (P), to be analyzed. Here we present the three main hypotheses on the non-linearity in Equation (1.6) and discuss their significance. Then, in Section 3, under the assumption of existence, we establish a number of properties of the minimizers of problem (P). In particular, we establish boundedness, regularity, and global gradient estimates for the minimizers, as well as uniqueness of their speed. Going further, in Section 4 we introduce a constrained variational problem, problem (P′), which will be used to establish existence of minimizers for problem (P). Here we show that existence of solutions for problem (P′) implies that for problem (P).

Then, in Section 5 we prove existence of minimizers for problem (P′). This result is established via a sequence of lemmas associated with the properties of the exponentially weighted Sobolev spaces $H^1_c(\Sigma; \mathbb{R}^m)$. We first obtain a uniform estimate that allows one to obtain information on the exponential decay of functions obeying the constraint and uniform estimates on the $||\cdot||_{1,c}$-norm. The crucial piece of the proof is establishing lower semicontinuity of the considered functional. This is done by estimating the measure of "bad" sets, the sets $\Omega_+(z)$, for functions in balls in $H^1_c(\Sigma, \mathbb{R}^m)$, as $z \to +\infty$, via an application of relative isoperimetric inequality and the co-area formula.

In Section 6 we establish several criteria of existence and non-existence of the considered type of the traveling waves. We also prove a number of properties of the minimizers, such as their one-dimensionality in the case of Neumann boundary conditions, or the fact that for the potentials V that depend only on the magnitude of the vector u, the minimizers are essentially scalar (up to a constant vector). We conclude this section by proving that in a certain class of solutions of the original parabolic problem the speed of the minimizers is in fact a sharp upper bound on the speed of propagation of disturbances. Finally, in Section 7 we consider a two-variable Ginzburg–Landau model as a sample application, for which we explicitly verify various assumptions of the analysis using a computer-assisted approach.

Remark 1. Under suitable regularity assumptions, our results can be straightforwardly extended to a general class of equations in which Δ_y is replaced by a strictly elliptic second-order operator in divergence form, and both this operator and the non-linearity are allowed to depend on the transverse coordinate y.

Notations

Throughout the paper, u_i denote the components of $u \in \mathbb{R}^m$; C^k, C^{∞}_0, $C^{k,\alpha}$ denote the usual spaces of continuous functions with k continuous derivatives, smooth functions with compact support, continuously differentiable functions with Hölder-continuous derivatives of order k for $\alpha \in (0, 1)$ (or Lipschitz-continuous when $\alpha = 1$), respectively. Unless it is otherwise clear from the context, "·" denotes a scalar product and $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$ (occasionally, when there can be no confusion, we use this notation to denote the same quantities in $\mathbb{R}^m$). The symbol ∇ is reserved for the gradient in $\mathbb{R}^n$, while ∇_y stands for the gradient in $\Omega \subset \mathbb{R}^{n-1}$ (we use ∇_u to denote the gradient in $\mathbb{R}^m$). Similarly, the symbol Δ stands for the Laplacian in $\mathbb{R}^n$, and Δ_y for the Laplacian in Ω. By a classical solution of Equation (1.6), we mean a function $u \in (C^2(\Sigma) \cap C^1(\bar{\Sigma}))^m$ that satisfies this equation with a given value of $c \in \mathbb{R}$ and the boundary conditions in Equation (1.2). For any domain $\omega \subseteq \Omega$, the quantity $|\omega|$ denotes the Lebesgue measure of $\omega \subseteq \mathbb{R}^{n-1}$ (with the convention that $|\Omega| = 1$ for $n = 1$), and $|\partial\omega|$ that of the boundary of ω. The numbers C, K, M, λ, etc., will denote generic positive constants.

2. Preliminaries and variational formulation

In this section, we introduce a few basic definitions and state our main assumptions. Throughout this paper it is assumed that Ω is a bounded domain with boundary of class C^2 whenever $n \geqq 3$. We now list some assumptions on the regularity and growth of $V(u)$.

(H1) The function $V : \mathbb{R}^m \to \mathbb{R}$ satisfies

$$V \in C^0(\mathbb{R}^m), \quad V(0) = \nabla_u V(0) = 0, \quad V(u) \geqq -C|u|^2 \tag{2.1}$$

for some $C \geqq 0$.

(H2) There exists a convex compact set $\mathcal{K} \subset \mathbb{R}^m$ which contains the origin, such that $V \in C^{1,1}(\mathcal{K})$ and for all $u \notin \mathcal{K}$

$$V(u) \geqq V(\Pi_{\mathcal{K}}(u)), \tag{2.2}$$

where $\Pi_{\mathcal{K}} : \mathbb{R}^m \to \mathbb{R}^m$ is the projection on the set $\mathcal{K}$, that is, $\Pi_{\mathcal{K}}(u)$ is the closest point to u which lies in $\mathcal{K}$.

Let us point out that our results remain valid if V is defined only in $\mathcal{K}$, together with the condition

$$(\nu \cdot \nabla_u V)|_{\partial\mathcal{K}} \geqq 0, \tag{2.3}$$

where ν is any outward normal to $\partial\mathcal{K}$, holds in place of Equation (2.2) in hypothesis (H2). Indeed, we can always consider the following continuous extension of $V(u)$ to the whole of $\mathbb{R}^m$:

$$\widetilde{V}(u) = V(\Pi_{\mathcal{K}}(u)) + \nabla_u V(\Pi_{\mathcal{K}}(u)) \cdot (u - \Pi_{\mathcal{K}}(u)). \tag{2.4}$$

By construction, $\widetilde{V}(u)$ is Lipschitz continuous on the whole $\mathbb{R}^m$ and, furthermore, is continuously differentiable up to the boundary of $\mathcal{K}$ [13]. Clearly, by Equation (2.3) hypothesis (H2) holds for $\widetilde{V}$. Also, since $\Pi_{\mathcal{K}}$ is 1-Lipschitz, $\widetilde{V}$ satisfies the condition $\widetilde{V}(u) \geqq -C|u|^2$, and so hypothesis (H1) is also met by $\widetilde{V}$.

We note that in the context of Equation (1.1) the set $\mathcal{K}$, together with an assumption like the one in Equation (2.3), plays the role of an invariant region, and its existence ensures global existence of solutions for the initial value problem associated with Equation (1.1) (see, for example, [43,44]).

We now introduce the definition of the exponentially weighted Sobolev spaces in which we will be working:

Definition 1. For $c > 0$, denote by $H^1_c(\Sigma; \mathbb{R}^m)$ the completion of the restrictions of $(C_0^\infty(\mathbb{R}^n))^m$ to Σ with respect to the norm

$$||u||^2_{1,c} = ||u||^2_{L^2_c(\Sigma;\mathbb{R}^m)} + ||\nabla u||^2_{L^2_c(\Sigma;\mathbb{R}^m)}, \quad ||u||^2_{L^2_c(\Sigma;\mathbb{R}^m)} = \sum_{i=1}^m \int_\Sigma \mathrm{e}^{cz} |u_i|^2 \, \mathrm{d}x.$$

For Dirichlet boundary conditions, replace $C_0^\infty(\mathbb{R}^n)$ with $C_0^\infty(\Sigma)$ above.

The weight appearing in the definition of the spaces $H^1_c(\Sigma; \mathbb{R}^m)$ arises quite naturally in the context of propagation for Equation (1.1) [18,37,40]. Indeed, Equation (1.1) written in the reference frame moving with speed c loses a variational structure of Equation (1.4) because of the appearance of the term containing a first derivative. However, by multiplying this equation by an appropriate weight (e^{cz}) we obtain an equation which again has a variational structure [23,24,37].

Let us mention an important general property of the spaces $H^1_c(\Sigma; \mathbb{R}^m)$ which is an analogue of the Poincaré inequality and will be needed to establish the existence result.

Lemma 2.1. *For all $u \in H^1_c(\Sigma; \mathbb{R}^m)$, we have*

$$\frac{c^2}{4} \int_\Sigma \mathrm{e}^{cz} \sum_{i=1}^m u_i^2 \, \mathrm{d}x \leqq \int_\Sigma \mathrm{e}^{cz} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial z} \right)^2 \mathrm{d}x. \tag{2.5}$$

Proof. The proof follows from the estimate in Equation (5.1) of Lemma 5.1 below, in the limit $R \to -\infty$. □

For $u \in H^1_c(\Sigma; \mathbb{R}^m)$ define two functionals

$$\Phi_c[u] = \int_\Sigma \mathrm{e}^{cz} \left(\frac{1}{2} \sum_{i=1}^m |\nabla u_i|^2 + V(u) \right) \mathrm{d}x, \tag{2.6}$$

$$\Gamma_c[u] = \frac{1}{2} \int_\Sigma \mathrm{e}^{cz} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial z} \right)^2 \mathrm{d}x. \tag{2.7}$$

Clearly, by hypothesis (H1) the functional $\Phi_c : H^1_c(\Sigma; \mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$ is well defined for all $u \in H^1_c(\Sigma; \mathbb{R}^m)$.

At least formally, Equation (1.6) describing the traveling wave solutions of Equation (1.1) is the Euler–Lagrange equation associated with the functional Φ_c [37]. A major difficulty, however, is the fact that the speed c of the traveling wave is also part of the solution and must, therefore, be determined simultaneously. Our approach to this question is via the following variational problem:

(P) *Find a non-trivial minimizer $\bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ of $\Phi_{c^\dagger}$ for some $c^\dagger > 0$.*

Now the speed $c = c^\dagger$ is part of the solution of problem (P), and we have

Proposition 2.2. *Let $\bar{u}$ be a solution of problem (P), with $\bar{u}(x) \in \mathcal{K}$ for all $x \in \Sigma$. Then $\bar{u}$ satisfies Equation (1.6) weakly in $H^1_c(\Sigma; \mathbb{R}^m)$ with $c = c^\dagger$.*

Proof. Observe that by hypotheses (H1) and (H2) for $u(x) \in \mathcal{K}$ the functional $\Phi_{c^\dagger}[u]$ is of class C^1 on $H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$. Therefore, if $\bar{u}$ is a minimizer of $\Phi_{c^\dagger}$, then for any $\varphi \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$

$$\int_\Sigma e^{c^\dagger z} \sum_{i=1}^m \left(\frac{\partial \bar{u}_i}{\partial z} \frac{\partial \varphi_i}{\partial z} + \nabla_y \bar{u}_i \cdot \nabla_y \varphi_i + \frac{\partial V(\bar{u})}{\partial u_i} \varphi_i \right) dx = 0, \tag{2.8}$$

which is a weak version of Equation (1.6) with $c = c^\dagger$. □

We point out that under hypotheses (H1) and (H2) we will further prove regularity of the solutions of problem (P) (see Section 3 below). So these solutions are classical solutions of Equation (1.6). Let us also mention that several other variational approaches to traveling waves exist [3,21,25,50].

Before turning to the analysis of problem (P), let us introduce the following two constants:

$$\nu_0 = \mu_0 + \liminf_{|u| \to 0} \frac{2V(u)}{|u|^2}, \quad \mu_- = \min_{u \in \mathcal{K}} \frac{2V(u)}{|u|^2}, \tag{2.9}$$

where μ_0 is the smallest eigenvalue of $-\Delta_y$ with the boundary conditions as in Equation (1.2), and the lim inf is taken over $u \in \mathcal{K}$. Clearly, in view of hypothesis (H1), both are well defined. These quantities play a crucial role in the existence of solutions of problem (P), as we will show below. To motivate their introduction, let us consider in more detail the decay of the solutions of Equation (1.6) at plus infinity (see also [37]). To this end, let us linearize Equation (1.6) around $u = 0$ at large z. Then the solutions of Equation (1.6) that decay as $z \to +\infty$ are expected to be approximately a superposition of functions $u_k(y, z) = e^{-\lambda_k z} v_k(y)$, where λ_k satisfy

$$\lambda_k^2 - c\lambda_k - \nu_k = 0, \tag{2.10}$$

and $v_k(y) \in \mathbb{R}^m$ and $\nu_k \in \mathbb{R}$ are the eigenfunctions and the eigenvalues defined by the equality

$$-\Delta_y v_k + H(0) v_k = \nu_k v_k, \quad H(u) = (\nabla_u \otimes \nabla_u) V(u), \tag{2.11}$$

where $H(u)$ is the Hessian of the potential $V(u)$ (here we assume that V is twice differentiable at the origin), provided $\operatorname{Re} \lambda_k > 0$. We note that ν_k can, in turn, be broken up into a sum of the eigenvalue μ_k of $-\Delta_y$ in Ω with the boundary conditions from Equation (1.2), and the eigenvalues of a symmetric matrix $H(0)$, implying that ν_k are all real, bounded from below, and increasing as $k \to \infty$.

Equation (2.10) can be trivially solved to give

$$\lambda_k^\pm(c) = \frac{c \pm \sqrt{c^2 + 4\nu_k}}{2}, \tag{2.12}$$

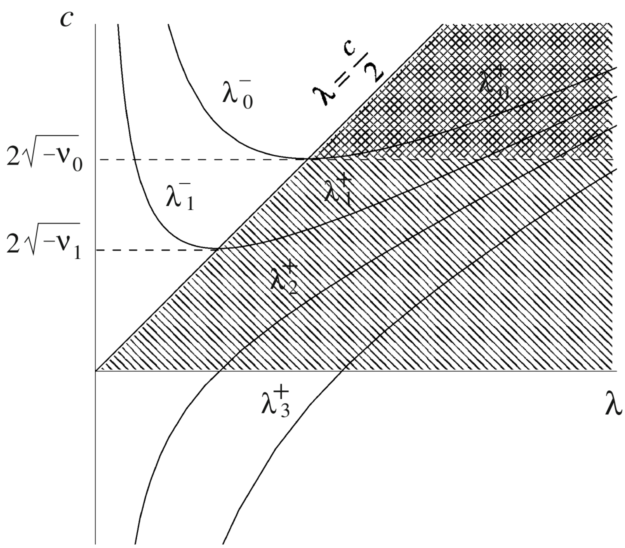

Fig. 2.1. A qualitative form of the dependencies $\lambda_k^\pm(c)$ for $\nu_0 < \nu_1 < 0 < \nu_2 < \cdots$

so for each $\nu_k \neq 0$ there is at least one solution with $\operatorname{Re}\lambda_k > 0$. Thus, in the case of twice-differentiable $V(u)$ the value of ν_0 determines the slowest possible rate of decay of the solutions of Equation (1.6) at plus infinity, corresponding to the plus sign in Equation (2.12), while μ_- gives a lower bound for ν_0.

We now state the third assumption needed to establish existence of solutions of problem (P).

(H3) There exist $c > 0$ such that $c^2 + 4\nu_0 > 0$, and $u \in H_c^1(\Sigma; \mathbb{R}^m)$, $u \not\equiv 0$, such that $\Phi_c[u] \leqq 0$.

Let us explain the meaning of this assumption. The condition $c^2 + 4\nu_0 > 0$ ensures the weak lower semicontinuity of the functional Φ_c on $H_c^1(\Sigma; \mathbb{R}^m)$ (see Proposition 5.5); hence it is crucial in proving existence of minimizers for Φ_c. The condition $\Phi_c[u] \leqq 0$ for some $u \not\equiv 0$, guarantees that the minimizer is not identically equal to zero. Due to Proposition 3.5, this assumption is necessary in order to have traveling wave solutions of Equation (1.6) lying in $H_c^1(\Sigma; \mathbb{R}^m)$.

Observe that, if $\nu_0 \geqq 0$, the first condition in (H3) is automatically satisfied, and the second condition can be expressed only in terms of z-independent functions (see Proposition 6.2). On the other hand, if $\nu_0 < 0$, there exists a finite set of k's, for which $\nu_k < 0$. In turn, for those k's and $c^2 > -4\nu_0 \geqq -4\nu_k$ there are two values of $\lambda_k = \lambda_k^\pm > 0$ that solve Equation (2.10), with $\lambda_k^- < \frac{c}{2} < \lambda_k^+$, see Equation (2.12). As an illustration, consider the case $\nu_0 < \nu_1 < 0 < \nu_2 < \cdots$, in Fig. 2.1. Here we show schematically the locations of the curves $\lambda_k^\pm$ as functions of c for the first four values of k. Since the solution of problem (P) belongs to $H_c^1(\Sigma; \mathbb{R}^m)$, it must decay faster than $e^{-cz/2}$ (cross-hatched area in Fig. 2.1), and so all the solutions of Equation (1.6) decaying asymptotically as $e^{-\lambda_k^- z}$ at plus infinity are automatically excluded. Hence, the hypothesis (H3), together with the existence of solutions of problem (P), implies the existence of traveling waves with fast exponential decay. These are special solutions of Equation (1.6), since generically one would expect the decay with the slower rate $e^{-\lambda_k^- z}$ at some k for which $\nu_k < 0$ (see also [11,32,37,41,47]). Under hypothesis (H3) we will be looking for the traveling waves moving with speed $c^\dagger \geqq c > 2\sqrt{-\nu_0}$ when $\nu_0 < 0$ (see below), this region corresponds to the cross-hatched area in Fig. 2.1.

3. Properties of minimizers

Before proceeding to the construction of solutions to problem (P), we investigate a number of their properties. First, observe that both Φ_c and Γ_c transform similarly under translations.

Lemma 3.1. *Let $u \in H_c^1(\Sigma;\mathbb{R}^m)$ and $u_a(y,z) := u(y,z-a)$. Then, $u_a \in H_c^1(\Sigma;\mathbb{R}^m)$ also, and*

$$\Phi_c[u_a] = e^{ca}\Phi_c[u] \quad \text{and} \quad \Gamma_c[u_a] = e^{ca}\Gamma_c[u]. \tag{3.1}$$

From this lemma, which is verified by direct inspection of the respective functionals, we obtain the following important result:

Proposition 3.2. *If $\bar{u}$ is a solution of problem (P), then $\Phi_{c^\dagger}[u] \geqq 0$ for all $u \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$ and $\Phi_{c^\dagger}[\bar{u}] = 0$.*

Proof. The first statement is an obvious consequence of the fact that $\bar{u}$ is a minimizer, if the second statement holds. To prove the latter, we first note that $\inf_{u\in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)} \Phi_{c^\dagger}[u] \leqq 0$, since zero is in $H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$. On the other hand, if $\Phi_{c^\dagger}[u] < 0$ for some $u \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$, then $\Phi_{c^\dagger}[u_a] < \Phi_{c^\dagger}[u]$, where $u_a \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$ is as in Lemma 3.1, with $a > 0$; hence, there are no minimizers of $\Phi_{c^\dagger}$. □

In other words, the assumption about the existence of a non-trivial $u \in H_c^1(\Sigma;\mathbb{R}^m)$ such that $\Phi_c[u] \leqq 0$ in hypothesis (H3) is in fact necessary, since the solution of problem (P) has this property for $c = c^\dagger$ by Proposition 3.2.

Next we establish a priori bounds on $\bar{u}$ and $\nabla\bar{u}$ for the solutions of problem (P).

Proposition 3.3. *If $\bar{u}$ is a solution of problem (P), then*

(i) *$\bar{u}(x) \in \mathcal{K}$ for all $x \in \Sigma$.*
(ii) *$\bar{u} \in (C^2(\Sigma) \cap C^1(\overline{\Sigma}))^m$, and $\nabla\bar{u} \in (L^\infty(\Sigma))^{mn} \cap H^1_{c^\dagger}(\Sigma;\mathbb{R}^{mn})$.*
(iii) *For all $x = (y,z) \in \Sigma$, we have $|\bar{u}(y,z)| \leqq Ce^{-\lambda z}$ for some $C > 0$ and $\lambda > 0$.*

Proof. (i) Let $\Pi_{\mathcal{K}} : \mathbb{R}^m \to \mathbb{R}^m$ be the projection on the convex set $\mathcal{K}$, as in hypothesis (H2). Recall that [13]

$$\Pi_{\mathcal{K}}(u) = u - d_{\mathcal{K}}(u)\nabla_u d_{\mathcal{K}}(u), \tag{3.2}$$

where $d_{\mathcal{K}}(u)$ is the distance to $u \in \mathbb{R}^m$ from the set $\mathcal{K}$. Then, if we replace $\bar{u}$ by $\tilde{u} := \Pi_{\mathcal{K}}(\bar{u}) \in H^1_c(\Sigma; \mathbb{R}^m)$, we have $V(\tilde{u}) \leqq V(\bar{u})$ by (2.2) and $\sum_{i=1}^m |\nabla \tilde{u}_i|^2 \leqq \sum_{i=1}^m |\nabla \bar{u}_i|^2$ since $\Pi_{\mathcal{K}}$ is a 1-Lipschitz function. Let $W \subset \Sigma$ be defined as $W := \{x \in \Sigma : \bar{u}(x) \notin \mathcal{K}\}$ and assume, by contradiction, that W has positive measure. Then, since the function $d_{\mathcal{K}}(\bar{u})$ is not constant on W, there exist a set $W' \subset W$ of positive measure and a constant $\delta > 0$ such that $|\nabla d_{\mathcal{K}}(\bar{u}(x))| \geqq \delta$ almost everywhere on W'. Approximating $\bar{u}$ with smooth functions in $H^1_c(\Sigma; \mathbb{R}^m)$ and differentiating (3.2), we obtain

$$\int_{W'} \mathrm{e}^{cz} \sum_{i=1}^m |\nabla \tilde{u}_i|^2 \, \mathrm{d}x < \int_{W'} \mathrm{e}^{cz} \sum_{i=1}^m |\nabla \bar{u}_i|^2 \, \mathrm{d}x,$$

which implies $\Phi_c[\tilde{u}] < \Phi_c[\bar{u}]$ and contradicts the minimality of $\bar{u}$. So, $\bar{u}(x) \in \mathcal{K}$ for almost every $x \in \Sigma$. Then, the statement of the proposition follows from the regularity result below.

(ii) Since by the above result $\bar{u} \in (L^\infty(\Sigma))^m$ and f is continuous on the essential range of $\bar{u}$, we have $f_i \in L^p_{\mathrm{loc}}(\Sigma)$, for any $p \geqq 1$ and for all $1 \leqq i \leqq m$. So, choosing p sufficiently large and applying the De Giorgi–Nash theory to each component $\bar{u}_i$ of $\bar{u}$, we obtain $\bar{u}_i \in C^{0,\alpha}(\Sigma)$, $1 \leqq i \leqq m$ with some $\alpha \in (0, 1)$ (see, for example, [20, Theorem 8.22]). Then, since $f \in C^{0,1}(\mathcal{K})$ by hypothesis (H2), it follows from the Schauder theory [20] that $\bar{u} \in (C^{2,\alpha}(\Sigma))^m$.

To obtain $C^{1,\alpha}$ regularity of $\bar{u}$ up to the boundary of Σ and a uniform estimate for $\nabla \bar{u}$ in $(L^\infty(\Sigma))^{mn}$, we apply to each component $\bar{u}_i$ the classical $W^{2,p}$ regularity theory (see, for example, [1,33]), which can be easily adapted to the case of a fixed slice of the cylinder Σ. We shall give the proof in detail in the case of Dirichlet boundary condition; for Neumann boundary conditions, using the the estimates of [1] (see also [33]) instead of the estimates of [20] and recalling that $\partial\Sigma$ is uniformly of class C^2, one can easily adapt the same proof.

By setting $v_i := \bar{u}_i \mathrm{e}^{cz/2}$, one can see that after a change of variables Equation (1.6) with $\bar{u} \in H^1_c(\Sigma; \mathbb{R}^m)$ is equivalent to

$$\Delta v_i - \frac{c^2}{4} v_i = f_i(\bar{u}) \mathrm{e}^{cz/2}, \quad v_i \in H^1(\Sigma), \quad v|_{\partial\Sigma} = 0, \tag{3.3}$$

where $H^1(\Sigma)$ is the usual Sobolev space.

For fixed $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in \mathbb{R}$, with $[z_1, z_2] \subset (\tilde{z}_1, \tilde{z}_2)$, consider the slices $\Sigma_0 := \Omega \times (z_1, z_2)$ and $\tilde{\Sigma}_0 := \Omega \times (\tilde{z}_1, \tilde{z}_2)$ of the cylinder Σ. Since $\bar{u}_i \in L^\infty(\Sigma)$ (by part (i)), the right-hand side of Equation (3.3) is in $L^p(\tilde{\Sigma}_0)$ for all $p > 1$. By standard regularity theory (see, for example, [20, Theorem 8.12 and Theorem 9.16]), we deduce $v_i \in W^{2,p}(\tilde{\Sigma}_0)$. Moreover, the a priori estimate given by [20, Theorem 9.13] to Equation (3.3) on the domains $\tilde{\Sigma}_0$ and Σ_0 yields

$$\begin{aligned} \|v_i\|_{W^{2,p}(\Sigma_0)} &\leqq C \left(\|v_i\|_{L^p(\tilde{\Sigma}_0)} + \left\| f_i(\bar{u}) \mathrm{e}^{cz/2} \right\|_{L^p(\tilde{\Sigma}_0)} \right) \\ &= C \left(\left\| u_i \mathrm{e}^{cz/2} \right\|_{L^p(\tilde{\Sigma}_0)} + \left\| f_i(\bar{u}) \mathrm{e}^{cz/2} \right\|_{L^p(\tilde{\Sigma}_0)} \right), \end{aligned} \tag{3.4}$$

where C depends on the parameters $n,\ p,\ c$ and the geometry of $\Sigma_0,\ \tilde{\Sigma}_0$. Since both $\bar{u}_i,\ f_i(\bar{u}) \in L^\infty(\Sigma)$, we can set $M = \max\{||\bar{u}_i||_{L^\infty(\Sigma)}, ||f_i(\bar{u})||_{L^\infty(\Sigma)}\}$ to obtain

$$\|v_i\|_{W^{2,p}(\Sigma_0)} \leqq 2MC \left\| e^{cz/2} \right\|_{L^p(\tilde{\Sigma}_0)} \leqq 2MC|\tilde{\Sigma}_0| e^{c\tilde{z}_2/2}.$$

By choosing $p > n/2$ and applying [20, Theorem 7.26], we deduce that

$$||v_i||_{C^1(\overline{\Sigma}_0)} \leqq 2MCS|\tilde{\Sigma}_0|^{\frac{1}{n}-\frac{1}{p}+1} e^{c\tilde{z}_2/2}, \tag{3.5}$$

where $S = S(n, p)$ is the Sobolev imbedding constant. Hence, coming back to the function u and using the inequality in Equation (3.5), we obtain $\bar{u}_i \in C^1(\overline{\Sigma}_0)$, and for any $(y, z) \in \overline{\Sigma}_0$

$$\begin{aligned} |\nabla \bar{u}_i(y, z)| = |\nabla(v_i e^{-cz/2})| &\leqq |e^{-cz/2} \nabla v_i| + \frac{c}{2} |e^{-cz/2} v_i| \\ &\leqq (2 + c)MCS|\tilde{\Sigma}_0|^{\frac{1}{n}-\frac{1}{p}+1} e^{c(\tilde{z}_2 - \tilde{z}_1)/2} = C', \end{aligned}$$

where the constant C' is invariant with respect to translations of the slice along z. So, translating the slices $\Sigma_0,\ \tilde{\Sigma}_0$ simultaneously along z, we obtain the estimate for all $x \in \Sigma$. Finally, the fact that $\nabla \bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^{mn})$ follows directly from (3.4) with $p = 2$ and the inequality $|f_i(\bar{u})| \leqq C|\bar{u}|$.

(iii) Now we prove the uniform exponential decay of u as $z \to +\infty$. Suppose, to the contrary, there exists a sequence $x_k = (y_k, z_k) \in \Sigma$, such that $z_k \to +\infty$ and $|\bar{u}(x_k)| e^{\lambda z_k} \to \infty$ for all $\lambda > 0$. Since $\partial\Omega$ is Lipschitz continuous, Σ satisfies the uniform interior cone property. So there exists a cone $\mathcal{C}_\Sigma$ (with finite height) such that each point $x_k \in \Sigma$ is the vertex of a cone $\mathcal{C}_k$ congruent to $\mathcal{C}_\Sigma$ that lies in $\bar{\Sigma}$. Up to a subsequence, we can further assume that $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for all $i \neq j$. By the previous result, we have $\nabla \bar{u} \in (L^\infty(\Sigma))^{mn}$, so $|\bar{u}(x)| \geqq \frac{1}{2}|\bar{u}(x_k)|$ for all $x \in \tilde{\mathcal{C}}_k$, where $\tilde{\mathcal{C}}_k$ is a smaller cone similar to $\mathcal{C}_k$, with the same vertex and $|\tilde{\mathcal{C}}_k| = \min\{|\mathcal{C}_k|, \varepsilon|\bar{u}(x_k)|^n\}$ for some $\varepsilon > 0$ (recall that $n = \dim \Sigma$). By assumption we have $|u(x_k)| \geqq e^{-\lambda z_k}$ for all $k \geqq N$ for some integer N, and also we can choose N large enough that $|\tilde{\mathcal{C}}_k| \geqq \varepsilon e^{-n\lambda z_k}$. But this implies

$$\int_\Sigma e^{c^\dagger z} |\bar{u}|^2 \, dx \geqq \sum_{k=1}^\infty \int_{\tilde{\mathcal{C}}_k} e^{c^\dagger z} |\bar{u}|^2 \, dx \geqq \frac{\varepsilon}{4} \sum_{k=N}^\infty e^{(c^\dagger - \lambda(2+n))z_k} = \infty$$

for $\lambda = c^\dagger/(2+n)$, which contradicts the fact that $\bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$. □

Let us point out that the obtained value of $\lambda = c^\dagger/(2+n)$ in the proof above is, of course, not sharp. It should be possible to obtain precise estimates for λ by studying the asymptotic behavior of solutions of Equation (1.6) at plus infinity. However, for $n \leqq 3$ the rate of decay of the solution can be estimated by noting that from Proposition 3.3(ii) and the Sobolev embedding theorem we obtain $\|v_i\|_{C^{0,\alpha}(\Sigma)} \leqq C$. This, in turn, implies that $|u(y, z)| \leqq C e^{-cz/2}$ for any $c < c^\dagger$, which is sharp.

A crucial property of the considered variational problem is uniqueness of the speed $c^\dagger$ (this point was already briefly discussed in [37]).

Proposition 3.4. *The value of $c^\dagger$ in the solution of problem (P) is unique.*

Proof. Assume by contradiction that there exist $c_2^\dagger > c_1^\dagger$, together with $\bar{u}^{(1)} \in H^1_{c_1^\dagger}(\Sigma;\mathbb{R}^m)$ and $\bar{u}^{(2)} \in H^1_{c_2^\dagger}(\Sigma;\mathbb{R}^m)$ such that

$$\Delta \bar{u}^{(1,2)} + c_{1,2}^\dagger \frac{\partial \bar{u}^{(1,2)}}{\partial z} + f(\bar{u}^{(1,2)}) = 0.$$

Let us first show that $\bar{u}^{(2)} \in H^1_{c_1^\dagger}(\Sigma;\mathbb{R}^m)$. Since $c_2^\dagger > c_1^\dagger$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_\Omega e^{c_1^\dagger z}|\bar{u}^{(2)}|^2\,dy\,dz &= \int_{-\infty}^{0}\int_\Omega e^{c_1^\dagger z}|\bar{u}^{(2)}|^2\,dy\,dz + \int_{0}^{+\infty}\int_\Omega e^{c_1^\dagger z}|\bar{u}^{(2)}|^2\,dy\,dz\\ &\leqq \frac{M|\Omega|}{c_1^\dagger} + \int_0^\infty\int_\Omega e^{c_2^\dagger z}|\bar{u}^{(2)}|^2\,dy\,dz\\ &\leqq \frac{M|\Omega|}{c_1^\dagger} + \int_\Sigma e^{c_2^\dagger z}|\bar{u}^{(2)}|^2\,dx < \infty,\end{aligned}$$

for some $M > 0$, where we took into account that the solutions of problem (P) are uniformly bounded by Lemma 3.3. Since, in turn, by Proposition 3.3, we have $\nabla\bar{u}^{(1,2)} \in (L^\infty(\Sigma))^{mn}$ as well, this argument can be repeated for the gradient. So $\bar{u}^{(2)} \in H^1_{c_1^\dagger}(\Sigma;\mathbb{R}^m)$.

In order to obtain a contradiction, let us scalar multiply Equation (1.6), with $\bar{u} = \bar{u}^{(2)}$ and $c = c_2^\dagger$, by $e^{c_1^\dagger z}\frac{\partial \bar{u}^{(2)}}{\partial z}$ and integrate over Σ. This is justified since $\bar{u}^{(2)}$ is a classical solution of Equation (1.6) by Proposition 3.3; hence we can integrate the expression over the domain $\Sigma_R := \Omega \times (-R, R)$ and then let $R \to +\infty$ on a suitable sequence. After a number of integrations by parts, we obtain

$$\begin{aligned}0 &= \int_\Sigma e^{c_1^\dagger z}\sum_{i=1}^m \frac{\partial \bar{u}_i^{(2)}}{\partial z}\left(\frac{\partial^2 \bar{u}_i^{(2)}}{\partial z^2} + c_2^\dagger \frac{\partial \bar{u}_i^{(2)}}{\partial z} + \Delta_y \bar{u}_i^{(2)} + f_i(\bar{u}^{(2)})\right)dx\\ &= c_1^\dagger \Phi_{c_1^\dagger}[\bar{u}^{(2)}] + 2(c_2^\dagger - c_1^\dagger)\Gamma_{c_1^\dagger}[\bar{u}^{(2)}],\end{aligned} \tag{3.6}$$

which implies that $\Phi_{c_1^\dagger}[\bar{u}^{(2)}] < 0$, contradicting Proposition 3.2. □

We now extend the result in Proposition 3.2 to any classical solution $\bar{u}$ of Equation (1.6) which lies in $H^1_c(\Sigma;\mathbb{R}^m)$.

Proposition 3.5. *Let $\bar{u} \in H^1_c(\Sigma;\mathbb{R}^m)$ be a classical solution of Equation (1.6). Then*

$$\Phi_c[\bar{u}] = 0. \tag{3.7}$$

Proof. We scalar multiply as above Equation (1.6) by $e^{cz}\frac{\partial \bar{u}}{\partial z}$ and integrate over Σ. The result then follows exactly as in Equation (3.6). □

Observe that $\bar{u}$ in Proposition 3.5 may or may not be a solution of problem (P). In the first case we have $c = c^\dagger$ and all the critical points of Φ_c in $H_c^1(\Sigma; \mathbb{R}^m)$ are solutions of problem (P). In the second case we have $c < c^\dagger$, and $\bar{u}$ is only a critical point of Φ_c, and not a minimizer. This means that the solutions of Equation (1.6) obtained by solving problem (P) are the fastest moving traveling waves within a class of sufficiently rapidly decaying solutions. This also means that, under hypotheses (H1) and (H2), if there exists a traveling wave solution $\bar{u} \in H_c^1(\Sigma; \mathbb{R}^m)$ with speed c satisfying $c^2 + 4\nu_0 > 0$, then problem (P) has a solution. This follows from the fact that in this case $\bar{u}$ is the function whose existence is required by (H3) [37].

4. Constrained minimization problem

To proceed with establishing the existence of solutions for problem (P), let us make a simple, but crucial observation about the translational invariance of Equation (1.6) in the z-direction, which leads to a natural loss of compactness. From the variational viewpoint, under the assumption of the existence of a non-trivial $u \in H_c^1(\Sigma; \mathbb{R}^m)$ such that $\Phi_c[u] \leqq 0$ in hypothesis (H3), which is necessary for existence of solution of problem (P), one cannot expect any kind of coercivity for the functional $\Phi_c[u]$, since the sequence of $u_n(z, y) := \bar{u}(z - n, y)$ has the property that $\Phi_c[u_n] \leqq 0$, while $||u_n||_{1,c} \to \infty$ by Proposition 3.2.

To deal with this issue, we follow the idea of Heinze [24] and introduce an auxiliary constrained variational problem. Define

$$\mathcal{B}_c := \{u \in H_c^1(\Sigma; \mathbb{R}^m) : \Gamma_c[u] = 1\}. \tag{4.1}$$

Then consider the following variational problem:

(P′) *Find* $u_c \in \mathcal{B}_c$ *satisfying:* $\Phi_c[u_c] = \inf_{\mathcal{B}_c} \Phi_c[u] \leqq 0.$

It is easy to see that the constraint $\mathcal{B}_c$ gives a natural way to fix translations along the axis of the cylinder. In particular, the functional Φ_c becomes coercive on $\mathcal{B}_c$ (see Lemma 5.2). Let us note that up to a transformation, problem (P′) is equivalent to the constrained variational problem considered by Heinze [24].

In the following, we will show that existence of solutions of problem (P′) implies the same for problem (P). Let us begin by proving that the solutions of the problem (P′) also lie within $\mathcal{K}$.

Lemma 4.1. *Let u_c be a solution of problem (P′). Then $u(x) \in \mathcal{K}$ for almost everywhere $x \in \Sigma$.*

Proof. We use the same projection argument as in Proposition 3.3. Namely, suppose that $u_c(x)$ is not in $\mathcal{K}$ in a set of non-zero measure. Then, repeating the arguments of Proposition 3.3, we obtain $\Phi_c[\tilde{u}] < \Phi_c[u_c] \leqq 0$, where $\tilde{u} := \Pi_{\mathcal{K}}(u_c)$. Similarly, $\Gamma_c[u_c] \geqq \Gamma_c[\tilde{u}] > 0$, where the last inequality follows from the fact that $\Phi_c[\tilde{u}] < 0$. So, by Lemma 3.1 there exists a constant $a \geqq 0$ such that $\tilde{u}_a(y, z) := \tilde{u}(y, z - a)$ is in $\mathcal{B}_c$, and $\Phi_c[\tilde{u}_a] \leqq \Phi_c[\tilde{u}]$. Therefore, u_c is not a minimizer of problem (P′), leading to contradiction. □

The following proposition establishes the connection between the solutions of problems (P) and (P′).

Proposition 4.2. *If u_c is a solution of problem (P'), then*

$$\bar{u}(y,z) = u_c(y, z\sqrt{1-\Phi_c[u_c]}) \quad \text{and} \quad c^\dagger = c\sqrt{1-\Phi_c[u_c]}, \tag{4.2}$$

are those for problem (P).

Proof. First of all, as in Proposition 2.2, we have Φ_c and Γ_c of class C^1 on $H^1_c(\Sigma;\mathbb{R}^m)$. Let $D\Gamma_c[u]v$ be the Fréchet derivative of Γ_c at u acting on v. Since

$$D\Gamma_c[u]u = \int_\Sigma e^{cz}\sum_{i=1}^m \left(\frac{\partial u_i}{\partial z}\right)^2 dx = 2, \quad \forall u \in \mathcal{B}_c,$$

we obtain $D\Gamma_c[u] \not\equiv 0$ on the constraint. Thus, applying the Lagrange multiplier theorem (see, for example, [9, Section 3.5]), we obtain

$$\int_\Sigma e^{c^\dagger z}\sum_{i=1}^m \left((1-\mu)\frac{\partial u_{c,i}}{\partial z}\frac{\partial \varphi_i}{\partial z} + \nabla_y u_{c,i}\cdot\nabla_y\varphi_i + \frac{\partial V(u_c)}{\partial u_i}\varphi_i\right) dx = 0. \tag{4.3}$$

where μ is the Lagrange multiplier.

Let us now show that $\mu < 1$. Indeed, suppose the opposite is true. Fix $a > 0$, and consider $u_a(x) := e^{-ca/2}u_c(x) \in \mathcal{K}$ almost everywhere (recall that $\mathcal{K}$ is convex, and $0 \in \mathcal{K}$). Then, for $v = u_a - u_c \in H^1_c(\Sigma;\mathbb{R}^m)$ the Fréchet derivatives of Φ_c and Γ_c on u_c satisfy

$$D\Phi_c[u_c]v = \mu D\Gamma_c[u_c]v \leqq 2(e^{-ca/2}-1) < 0, \tag{4.4}$$

where we recalled that $u_c \in \mathcal{B}_c$. Therefore, since Φ_c is of class C^1, there exists a sufficiently small $a > 0$ such that $\Phi_c[u_a] < \Phi_c[u_c] \leqq 0$. Now, consider $\tilde{u}_a(y,z) := u_a(y, z-a)$. A straightforward calculation then shows that $\tilde{u}_a \in \mathcal{B}_c$. However, by Lemma 3.1 and the fact that $\Phi_c[u_a] < 0$, we obtain $\Phi_c[\tilde{u}_a] < \Phi_c[u_c]$, contradicting the fact that u_c is a minimizer.

So, $\mu < 1$, and from Lemma 4.1 and the argument of Proposition 3.3 we deduce that $u_c \in (C^2(\Sigma))^m$ and satisfies

$$(1-\mu)\left(\frac{\partial^2 u_c}{\partial z^2} + c\frac{\partial u_c}{\partial z}\right) + \Delta_y u_c + f(u_c) = 0. \tag{4.5}$$

Now, we scalar multiply Equation (4.5) by $e^{cz}\frac{\partial \bar{u}}{\partial z}$ as in Proposition 3.4, and integrate over Σ to obtain

$$0 = \int_\Sigma e^{cz}\sum_{i=1}^m \left[(1-\mu)\frac{\partial u_{c,i}}{\partial z}\frac{\partial^2 u_{c,i}}{\partial z^2} + \nabla_y u_{c,i}\cdot\frac{\partial}{\partial z}\nabla_y u_{c,i} + \frac{\partial V(u_c)}{\partial u_i}\frac{\partial u_{c,i}}{\partial z}\right] dx$$
$$= c(\mu - \Phi_c[u_c]),$$

where we recalled that $\Gamma_c[u_c] = 1$. This means that

$$\mu = \Phi_c[u_c]. \tag{4.6}$$

To show that $\bar{u}$ and $c^\dagger$ are solutions of problem (P), first fix $u \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ and introduce $\tilde{u}(y, \zeta) = u\left(y, \frac{\zeta}{\sqrt{1-\mu}}\right)$, which is possible since $\mu < 1$. Then

$$\int_{-\infty}^{+\infty}\int_{\Omega} e^{c\zeta}\tilde{u}_i^2 \,dy\,d\zeta = \sqrt{1-\mu}\int_{-\infty}^{+\infty}\int_{\Omega} e^{c^\dagger z}u_i^2 \,dy\,dz, \tag{4.7}$$

$$\int_{-\infty}^{+\infty}\int_{\Omega} e^{c\zeta}|\nabla_y\tilde{u}_i|^2 \,dy\,d\zeta = \sqrt{1-\mu}\int_{-\infty}^{+\infty}\int_{\Omega} e^{c^\dagger z}|\nabla_y u_i|^2 \,dy\,dz, \tag{4.8}$$

$$\int_{-\infty}^{+\infty}\int_{\Omega} e^{c\zeta}\left(\frac{\partial\tilde{u}_i}{\partial z}\right)^2 dy\,d\zeta = \frac{1}{\sqrt{1-\mu}}\int_{-\infty}^{+\infty}\int_{\Omega} e^{c^\dagger z}\left(\frac{\partial u_i}{\partial z}\right)^2 dy\,dz. \tag{4.9}$$

Therefore, $\tilde{u} \in H^1_c(\Sigma; \mathbb{R}^m)$, and

$$\begin{aligned}\Phi_{c^\dagger}[u] &= \int_{-\infty}^{+\infty}\int_{\Omega} e^{c^\dagger z}\left(\frac{1}{2}\sum_{i=1}^{m}\left[\left(\frac{\partial u_i}{\partial z}\right)^2 + |\nabla_y u_i|^2\right] + V(u)\right) dy\,dz \\ &= \frac{1}{\sqrt{1-\mu}}\int_{-\infty}^{+\infty}\int_{\Omega} e^{c\zeta}\left(\frac{1}{2}\sum_{i=1}^{m}\left[(1-\mu)\left(\frac{\partial\tilde{u}_i}{\partial\zeta}\right)^2 + |\nabla_y\tilde{u}_i|^2\right] + V(\tilde{u})\right) dy\,d\zeta \\ &= \frac{1}{\sqrt{1-\mu}}\left(\Phi_c[\tilde{u}] - \mu\Gamma_c[\tilde{u}]\right). \end{aligned} \tag{4.10}$$

Now we claim that if the solution of problem (P′) exists, then

$$\Phi_c[\tilde{u}] \geqq \mu\Gamma_c[\tilde{u}]. \tag{4.11}$$

Indeed, if $\Gamma_c[\tilde{u}] = 0$, then by Lemma 2.1 and hypothesis (H1) we have $\Phi_c[\tilde{u}] = 0$ also, so Equation (4.11) holds trivially. On the other hand, if $\Gamma_c[\tilde{u}] > 0$, then there exists a constant $a \in \mathbb{R}$ such that the translated function $\tilde{u}_a(y, z) := \tilde{u}(y, z - a)$ of $\tilde{u}$ is in $\mathcal{B}_c$. Hence, $\Phi_c[\tilde{u}_a] \geqq \Phi_c[u_c] = \mu\Gamma_c[\tilde{u}_a]$, and by Lemma 3.1 the inequality in Equation (4.11) holds for $\tilde{u}$, with equality achieved when $\tilde{u} = u_c$. Hence, by Equation (4.10) we have $\Phi_{c^\dagger}[u] \geqq 0$ for all $u \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$, and $\bar{u}$ gives a solution of problem (P). □

5. Existence of constrained minimizers

To proceed with the proof of existence of constrained minimizers, we need to establish some compactness properties of the sublevel sets of Φ_c. Since we will work in the weak topology of $H^1_c(\Sigma; \mathbb{R}^m)$, which is a Hilbert space, it is enough to show that Φ_c has bounded sublevel sets (that is, it is coercive). This, however, may be false for general Φ_c, even after eliminating translations in some way. As a simple example, consider $\Phi_c[u] = \frac{1}{2}\int_{\mathbb{R}} e^{cz}(u_z^2 - u^2 + u^4)\,dz$, with $u : \mathbb{R} \to \mathbb{R}$. It is easy to see that this functional is not coercive on $H^1_c(\mathbb{R})$ when $c = 2$. Indeed, consider a sequence of functions $u_n \in H^1_c(\mathbb{R})$ defined as $u_n(z) = e^{-(1+1/n)z}$ for $z \geqq 0$ and $u_n(z) = 1$ for $z < 0$. Clearly, the sequence (u_n) is not bounded in $H^1_c(\mathbb{R})$. However, a straightforward calculation shows that $\Phi_c[u_n] = (3n^2+5n+2)/(2n(n+2)) < \infty$.

This difficulty in fact is not merely technical, and puts certain limitations on the applicability of our variational approach. In particular, as can be seen from the example just mentioned, it cannot be used directly to characterize the minimal speed of traveling waves in systems with Fisher-type non-linearities (see also Proposition 6.1 and [37]).

Even if coercivity of Φ_c may not hold in general, in Lemma 5.2 we show that it does hold if we consider the intersection of the sublevel sets of Φ_c with the set $\mathcal{B}_c$ defined in (4.1). So, establishing existence for problem (P′) amounts to proving weak sequential lower semicontinuity of Φ_c. Here, again, there is a difficulty, since Σ is an unbounded domain and $V(u)$ is allowed to be negative, so the standard theory [12] does not apply. In the following we will establish sequential lower semicontinuity of the functional Φ_c under the assumption $c^2+4\nu_0>0$ from hypothesis (H3). This assumption is also essential, as it is possible to construct sequences in $H^1_c(\Sigma;\mathbb{R}^m)$ on which Φ_c "jumps up", if this condition is not satisfied (see the discussion following Proposition 5.5).

We begin by proving the following lemma about a Poincaré-type inequality in the weighted Sobolev space $H^1_c(\Sigma;\mathbb{R}^m)$.

Lemma 5.1. *Let* $u\in H^1_c(\Sigma;\mathbb{R}^m)$. *Then*

$$\frac{c^2}{4}\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}\sum_{i=1}^m u_i^2\,\mathrm{d}y\,\mathrm{d}z \leqq \int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}\sum_{i=1}^m\left(\frac{\partial u_i}{\partial z}\right)^2\mathrm{d}y\,\mathrm{d}z, \tag{5.1}$$

$$\int_\Omega \sum_{i=1}^m u_i^2(y,R)\,\mathrm{d}y \leqq \frac{\mathrm{e}^{-cR}}{c}\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}\sum_{i=1}^m\left(\frac{\partial u_i}{\partial z}\right)^2\mathrm{d}y\,\mathrm{d}z, \tag{5.2}$$

for any $R\in\mathbb{R}$.

Proof. Let us first prove Equation (5.1):

$$\frac{c}{2}\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}u_i^2\,\mathrm{d}y\,\mathrm{d}z = -\frac{1}{2}\mathrm{e}^{cR}\int_\Omega u_i^2(y,R)\,\mathrm{d}y - \int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}u_i\frac{\partial u_i}{\partial z}\,\mathrm{d}y\,\mathrm{d}z$$
$$\leqq \left(\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}u_i^2\,\mathrm{d}y\,\mathrm{d}z\right)^{1/2}\left(\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}\left(\frac{\partial u_i}{\partial z}\right)^2\mathrm{d}y\,\mathrm{d}z\right)^{1/2},$$

which implies (5.1).

Turn to Equation (5.2) now. Since

$$\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}\left(\sqrt{c}u_i+\frac{1}{\sqrt{c}}\frac{\partial u_i}{\partial z}\right)^2\mathrm{d}y\,\mathrm{d}z\geqq 0,$$

we obtain

$$\frac{1}{c}\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}\left(\frac{\partial u_i}{\partial z}\right)^2\mathrm{d}y\,\mathrm{d}z \geqq -2\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}u_i\frac{\partial u_i}{\partial z}\,\mathrm{d}y\,\mathrm{d}z - c\int_R^{+\infty}\int_\Omega \mathrm{e}^{cz}u_i^2\,\mathrm{d}y\,\mathrm{d}z$$
$$= \mathrm{e}^{cR}\int_\Omega u_i^2(y,R)\,\mathrm{d}y,$$

which gives Equation (5.2). □

The following lemma estimates the norm of $\nabla_y u$ on $\mathcal{B}_c$ and, via Lemma 2.1, establishes coercivity of the functional Φ_c on $\mathcal{B}_c$.

Lemma 5.2. *Let V satisfy hypotheses (H1) and (H2), and let $u \in \mathcal{B}_c$. Then*

$$\int_\Sigma e^{cz} \sum_{i=1}^m |\nabla_y u_i|^2 \, dx \leqq 2\Phi_c[u] + \frac{8|\mu_-|}{c^2}, \tag{5.3}$$

where μ_- is defined in Equation (2.9).

Proof. By Equations (2.9) and Lemma 2.1, we have

$$\int_\Sigma e^{cz} V(u) \, dx \geqq -\frac{|\mu_-|}{2} \int_\Sigma e^{cz} \sum_{i=1}^m u_i^2 \, dx \geqq -\frac{2|\mu_-|}{c^2} \int_\Sigma e^{cz} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial z}\right)^2 dx.$$

Hence, for $\Gamma_c[u] = 1$ we have

$$\frac{1}{2} \int_\Sigma e^{cz} |\nabla_y u|^2 \, dx \leqq \Phi_c[u] - \int_\Sigma e^{cz} V(u) \, dx \leqq \Phi_c[u] + \frac{4|\mu_-|}{c^2},$$

which is equivalent to Equation (5.3). □

We now turn to the question of lower semicontinuity. Let us introduce the following notation:

$$\Phi_c[u, (a, b)] = \int_a^b \int_\Omega e^{cz} \left(\frac{1}{2} \sum_{i=1}^m |\nabla u_i|^2 + V(u) \right) dy dz. \tag{5.4}$$

We will analyze the behavior of $\Phi_c[u, (-\infty, R)]$ and $\Phi_c[u, (R, +\infty)]$ on a weakly converging sequence and take the limit $R \to +\infty$. To this end, we first establish the sequential lower semicontinuity of $\Phi_c[u, (-\infty, R)]$ for all $R \in \mathbb{R}$, with respect to the weak topology of $H_c^1(\Sigma; \mathbb{R}^m)$.

Lemma 5.3. *Let V satisfy hypotheses (H1) and (H2), and let $u_n \rightharpoonup u$ in $H_c^1(\Sigma; \mathbb{R}^m)$. Then,*

$$\liminf_{n\to\infty} \Phi_c[u_n, (-\infty, R)] \geqq \Phi_c[u, (-\infty, R)]$$

for any $R \in \mathbb{R}$.

Proof. This follows by standard semicontinuity results (see, for example, [12, Propositions 2.1, 2.2]) by considering $v := e^{cz/2} u \in H^1(\Sigma; \mathbb{R}^m)$ and using the fact that by hypothesis (H2) $V(u)$ is bounded from below, and $\int_{-\infty}^R \int_\Omega e^{cz} \, dy \, dz < \infty$. □

To proceed, we need to establish the following key estimate.

Lemma 5.4. *Let V satisfy hypotheses (H1) and (H2), and let $c^2 + 4\nu_0 > 0$. Then, for any $\varepsilon > 0$ and $C > 0$, there exists $R = R(\varepsilon, C)$ such that*

$$\Phi_c[u, (R, +\infty)] \geqq -\varepsilon,$$

for any $u \in H_c^1(\Sigma; \mathbb{R}^m)$ such that $||u||_{1,c} \leqq C$.

Proof. Since $(C_0^\infty(\mathbb{R}^n))^m$ is dense in $H_c^1(\Sigma;\mathbb{R}^m)$, in the following arguments we can assume that $u \in (C_0^\infty(\mathbb{R}^n))^m$. We prove this lemma via a sequence of steps.

Step 1. In view of Equation (5.1), we have

$$\Phi_c[u,(R,+\infty)] \geqq \int_R^{+\infty}\int_\Omega e^{cz}\left(\frac{1}{2}\left(\frac{c^2}{4}+\mu_0\right)\sum_{i=1}^m u_i^2 + V(u)\right) dy\,dz, \quad (5.5)$$

where, as in Equation (2.9), $\mu_0 \geqq 0$ is the smallest eigenvalue of $-\Delta_y$ in Ω with the corresponding boundary conditions. From the definition of ν_0 in Equation (2.9), for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|u| < \delta$

$$V(u) \geqq \frac{1}{2}\sum_{i=1}^m(\nu_0 - \mu_0 - \varepsilon)u_i^2.$$

Therefore, if $c^2 + 4\nu_0 > 0$, the integrand in Equation (5.5) is non-negative for all $|u| < \delta$, with some positive δ.

Note that if $n = 1$, then from Equation (5.2) it follows that $u \to 0$ uniformly as $R \to +\infty$; so from the argument above it immediately follows that $\Phi_c[u,(R,+\infty)] \geqq 0$ for sufficiently large R, and the statement of the lemma is proved (see also [32]). So, in the following we will assume that $n \geqq 2$.

Step 2. Define

$$v(y,z) = -\frac{1}{2}\left(\frac{c^2}{4}+\mu_0\right)\sum_{i=1}^m u_i^2(y,z) - V(u(y,z)) \quad (5.6)$$

and introduce

$$\Omega_+(z) = \{y \in \Omega : v(y,z) > 0\}. \quad (5.7)$$

By the result of Step 1, we have $|u(y,z)| \geqq \delta$ whenever $y \in \Omega_+(z)$. Therefore

$$|\Omega_+(z)|\delta^2 \leqq \sum_{i=1}^m \int_{\Omega_+(z)} u_i^2(y,z)\,dy \leqq \sum_{i=1}^m \int_\Omega u_i^2(y,z)\,dy. \quad (5.8)$$

Combining this with Equation (5.2) and taking into account that $\Gamma_c[u] = 1$, we obtain

$$|\Omega_+(z)| \leqq \frac{2e^{-cz}}{c\delta^2} \to 0 \quad \text{as} \quad z \to +\infty. \quad (5.9)$$

Step 3. Now we want to estimate the integral in Equation (5.5). First, observe that $v(y,z) = 0$ whenever $y \in \partial\Omega_+(z) \cap \Omega$. From Equations (5.5) and (5.6), we have

$$\Phi_c[u,(R,+\infty)] \geqq -\int_R^{+\infty}\int_{\Omega_+(z)} e^{cz} v\,dy\,dz. \quad (5.10)$$

Let us introduce the level sets (for simplicity of notation, we suppress the z-dependence in the definition)

$$\omega(t) = \{y \in \Omega_+(z) : v(y,z) > t\}. \tag{5.11}$$

In view of Equation (5.9), we have $|\omega(t)| \leqq |\Omega_+(z)| \leqq \frac{1}{2}|\Omega|$ for sufficiently large R. Then, by the relative isoperimetric inequality [15] there exists a constant C_Ω which depends only on Ω and not on ω, such that (recall that $\dim \Omega = n-1$)

$$|\omega|^{\frac{n-2}{n-1}} \leqq C_\Omega |\partial\omega_0|, \quad \partial\omega_0 = \partial\omega \cap \Omega. \tag{5.12}$$

Then, using the Cavalieri principle and then the co-area formula [15], we obtain

$$\begin{aligned}\int_{\Omega_+(z)} v\,\mathrm{d}y &= \int_0^\infty |\omega(t)|\,\mathrm{d}t \leqq |\Omega_+(z)|^{\frac{1}{n-1}} \int_0^\infty |\omega(t)|^{\frac{n-2}{n-1}}\,\mathrm{d}t \\ &\leqq C_\Omega |\Omega_+(z)|^{\frac{1}{n-1}} \int_0^\infty |\partial\omega_0(t)|\,\mathrm{d}t = C_\Omega |\Omega_+(z)|^{\frac{1}{n-1}} \int_{\Omega_+(z)} |\nabla_y v|\,\mathrm{d}y. \end{aligned} \tag{5.13}$$

Let us now multiply the last integral in Equation (5.13) by e^{cz} and integrate over $(R, +\infty)$. Then, using the definition of v in Equation (5.6), chain rule, hypothesis (H1) and the Schwarz inequality, we obtain

$$\begin{aligned}&\left(\int_R^{+\infty}\int_{\Omega_+(z)} \mathrm{e}^{cz}|\nabla_y v|\,\mathrm{d}y\mathrm{d}z\right)^2 \\ &= \left(\int_R^{+\infty}\int_{\Omega_+(z)} \mathrm{e}^{cz}\left|\sum_{i=1}^m \left[\left(\frac{c^2}{4}+\mu_0\right)u_i + \frac{\partial V}{\partial u_i}\right]\nabla_y u_i\right|\mathrm{d}y\,\mathrm{d}z\right)^2 \\ &\leqq M\left[\int_R^{+\infty}\int_{\Omega_+(z)} \mathrm{e}^{cz}\left(\sum_{i=1}^m u_i^2\right)^{1/2}\left(\sum_{j=1}^m |\nabla_y u_j|^2\right)^{1/2}\mathrm{d}y\,\mathrm{d}z\right]^2 \\ &\leqq M\int_R^{+\infty}\int_{\Omega_+(z)} \mathrm{e}^{cz}\sum_{i=1}^m u_i^2\,\mathrm{d}y\,\mathrm{d}z \int_R^{+\infty}\int_{\Omega_+(z)} \mathrm{e}^{cz}\sum_{i=1}^m |\nabla_y u_i|^2\,\mathrm{d}y\,\mathrm{d}z \\ &\leqq \frac{8M}{c^2}\int_{-\infty}^{+\infty}\int_\Omega \mathrm{e}^{cz}\sum_{i=1}^m |\nabla_y u_i|^2\,\mathrm{d}y\,\mathrm{d}z, \end{aligned} \tag{5.14}$$

where M is a constant independent of R and u, and in the last step we used Equation (5.1) and the fact that $u \in \mathcal{B}_c$. Combining this with Equations (5.9) and (5.13) yields

$$\int_R^{+\infty}\int_{\Omega_+(z)} \mathrm{e}^{cz} v\,\mathrm{d}y\,\mathrm{d}z \leqq K\mathrm{e}^{-\frac{cR}{n-1}}\left(\int_{-\infty}^{+\infty}\int_\Omega \mathrm{e}^{cz}\sum_{i=1}^m |\nabla_y u_i|^2\,\mathrm{d}y\,\mathrm{d}z\right)^{1/2}, \tag{5.15}$$

where K is a constant independent of R and u.

Finally, by assumption the integral in the right-hand side of Equation (5.15) is bounded by C, so its left-hand side can be made arbitrarily small by choosing large enough R. In view of Equation (5.10), this proves the statement of the lemma. □

Combining the two lemmas above, we obtain the following:

Proposition 5.5. *Let V satisfy hypotheses (H1) and (H2), and let $c^2 + 4\nu_0 > 0$. Then, the functional Φ_c is sequentially weakly lower semicontinuous on $H^1_c(\Sigma; \mathbb{R}^m)$.*

Proof. Let $u_n \rightharpoonup u$ in $H^1_c(\Sigma; \mathbb{R}^m)$. Hence, (u_n) is bounded in $H^1_c(\Sigma; \mathbb{R}^m)$, and by Lemmas 5.3 and 5.4

$$\begin{aligned}\liminf_{n\to\infty} \Phi_c[u_n] &\geqq \liminf_{n\to\infty}\{\Phi_c[u_n, (-\infty, R)]\} + \liminf_{n\to\infty}\{\Phi_c[u_n, (R, +\infty)]\}\\ &\geqq \Phi_c[u, (-\infty, R)] - \varepsilon\\ &= \Phi_c[u] - \Phi_c[u, (R, +\infty)] - \varepsilon, \end{aligned} \tag{5.16}$$

for large enough R. Now, by noting that $\Phi_c[u, (R, +\infty)] \leqq \varepsilon$ for sufficiently large R, Equation (5.16) leads to

$$\liminf_{n\to\infty} \Phi_c[u_n] \geqq \Phi_c[u] - 2\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, we conclude that $\Phi_c[u] \leqq \liminf_{n\to\infty} \Phi_c[u_n]$. □

We notice that the assumption $c^2 + 4\nu_0 \geqq 0$ is also necessary to ensure the lower semicontinuity of Φ_c. Indeed, assume by contradiction that Φ_c is sequentially weakly lower semicontinuous with $c^2 + 4\nu_0 < 0$, and consider the sequence $u_n \in H^1_c(\Sigma; \mathbb{R}^m)$, defined as $u_n(y, z) := \frac{v_0(y)}{\sqrt{n}} \mathrm{e}^{-cz/2 - z^2/n^2}$, where $v_0 \neq 0$ is an eigenvector of the operator $-\Delta_y + \nabla^2_u V(0)$, corresponding to the eigenvalue ν_0 (here we assume for simplicity that V is twice differentiable in 0). It is easy to see that the sequence (u_n) is bounded in $H^1_c(\Sigma; \mathbb{R}^m)$ and converges weakly to 0. However, a simple calculation shows that

$$\lim_{n\to\infty} \Phi_c[u_n] = \frac{\sqrt{\pi}}{4\sqrt{2}}(c^2 + 4\nu_0) \int_\Omega v_0^2 \, \mathrm{d}y < 0 = \Phi_c[0],$$

which gives a contradiction.

We are ready to prove our main existence result.

Proposition 5.6. *Let V satisfy hypotheses (H1) and (H2), and suppose that there exists $u \in \mathcal{B}_c$ such that $\Phi_c[u] \leqq 0$, for some c satisfying $c^2 + 4\nu_0 > 0$. Then problem (P′) has a solution.*

Proof. Let (u_n) be a minimizing sequence for problem (P′), i.e $u_n \in \mathcal{B}_c$ with $\Phi_c[u_n] \to \inf_{\mathcal{B}_c} \Phi_c$. By assumption, $\inf_{\mathcal{B}_c} \Phi_c \leqq 0$, and without the loss of generality we may assume that $\Phi_c[u_n] \leqq 0$. Since $\Gamma_c[u_n] = 1$, from inequality (2.5) we obtain $\int_\Sigma \mathrm{e}^{cz}|u_n|^2 \, \mathrm{d}x \leqq \frac{8}{c^2}$. Also, from Lemma 5.2 we obtain a similar bound for the norm of $\nabla_y u$. Thus, the sequence (u_n) is bounded in $H^1_c(\Sigma)$; therefore, up to a subsequence, it converges weakly to some $u \in H^1_c(\Sigma)$.

If $\inf_{\mathcal{B}_c} \Phi_c = 0$, we deduce that u in the assumption of this proposition is a minimizer. Therefore, let us assume that $\inf_{\mathcal{B}_c} \Phi_c < 0$. Then, by lower semicontinuity of Φ_c established in Proposition 5.5 we have $\Phi_c[u] \leqq \inf_{\mathcal{B}_c} \Phi_c < 0$, so $u \not\equiv 0$. Also, since by standard semicontinuity results [12]

$$1 = \liminf_{n\to\infty} \Gamma_c[u_n] \geqq \Gamma_c[u] > 0,$$

we can, by using Lemma 3.1, choose $a \geqq 0$ such that

$$\Gamma_c[u_a] = 1 \quad \text{with} \quad u_a(y, z) := u(y, z - a).$$

Since $\inf_{\mathcal{B}_c} \Phi_c < 0$ and $a \geqq 0$, we derive

$$\Phi_c[u_a] = \mathrm{e}^{ca}\Phi_c[u] \leqq \Phi_c[u] \leqq \inf_{\mathcal{B}_c} \Phi_c$$

with the first inequality being strict when $a > 0$. Therefore, $a = 0$, meaning that $\Gamma_c[u] = 1$ and $\Phi_c[u] = \inf_{\mathcal{B}_c} \Phi_c$, so u solves problem (P′). □

Let us point out that, for one-dimensional problems ($n = 1$), in which the functional Γ_c generates an equivalent norm in $H^1_c(\mathbb{R})$, the minimizing sequence (u_n) converges to u strongly in $H^1_c(\mathbb{R})$.

6. Further properties of minimizers

In this section we analyze problem (P) and its solutions in more detail. Our first result, based on the application of Theorem 1.1, is a general non-existence result for the solutions of problem (P) with sufficiently large c (see also [37,32]).

Proposition 6.1. *Let V satisfy hypotheses (H1) and (H2), and let $c^2+4(\mu_0+\mu_-) > 0$, where μ_0 is the smallest eigenvalue of $-\Delta_y$ in Ω with boundary conditions from Equation (1.2), and μ_- is given by Equation (2.9). Then problem (P) has no solutions.*

Proof. Let $\bar{u} \in H^1_c(\Sigma; \mathbb{R}^m)$ be a solution of problem (P). By Propositions 3.3 and 2.2 we know that $\bar{u}(x) \in \mathcal{K}$ and $\bar{u}(\cdot, z) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$. Since

$$\int_\Omega |\nabla_y \bar{u}_i(y, z)|^2 \,\mathrm{d}y \geqq \mu_0 \int_\Omega \bar{u}_i^2(y, z)\,\mathrm{d}y, \quad z \in \mathbb{R},$$

we obtain, using Lemma 2.1,

$$\begin{aligned}\Phi_c[\bar{u}] &\geqq \frac{1}{2}\sum_{i=1}^m \int_{-\infty}^{+\infty}\int_\Omega \mathrm{e}^{cz}\left[\left(\frac{\partial \bar{u}_i}{\partial z}\right)^2 + (\mu_0+\mu_-)\bar{u}_i^2\right]\mathrm{d}y\,\mathrm{d}z \\ &\geqq \frac{1}{2}\left(\frac{c^2}{4}+\mu_0+\mu_-\right)\int_{-\infty}^{+\infty}\int_\Omega \mathrm{e}^{cz}\sum_{i=1}^m \bar{u}_i^2\,\mathrm{d}y\,\mathrm{d}z > 0,\end{aligned}$$

unless $\bar{u} = 0$. But this contradicts Proposition 3.2. □

Naturally, in view of the discussion at the end of Section 3, this implies that under the assumptions of Proposition 6.1 there are *no* traveling wave solutions lying in $H^1_c(\Sigma; \mathbb{R}^m)$. A simple example of such a situation is the Fisher's equation in one space dimension, for which it is known that all the traveling wave solutions decay at infinity with the rate $e^{-\lambda_- z}$ (see Equation (2.10) with $\nu_k = 0$) and, therefore, cannot lie in $H^1_c(\mathbb{R})$ [2,32].

Let us point out that we will have $\mu_- \geqq 0$ if $V(u) \geqq 0$ throughout $\mathcal{K}$, so a necessary condition for existence of solutions of problem (P), which is familiar from the analysis of the one-dimensional scalar problem [17], is that $V(u) < 0$ somewhere in $\mathcal{K}$. In that case, if also $\mu_0 + \mu_- < 0$, problem (P) may have solutions only with $c \leqq c_{\max}$, where $c_{\max} = 2\sqrt{-\mu_0 - \mu_-}$ [37].

Next we establish the following necessary and sufficient condition for existence of traveling wave solutions for potentials with linearly stable equilibrium at $u = 0$. Let us introduce the functional

$$E[v] := \int_\Omega \left(\frac{1}{2} \sum_{i=1}^m |\nabla_y v_i|^2 + V(v) \right) \mathrm{d}y, \quad v \in H^1(\Omega; \mathbb{R}^m), \tag{6.1}$$

where $H^1(\Omega; \mathbb{R}^m)$ is the Sobolev space of functions with values in $\mathbb{R}^m$ (for Dirichlet boundary conditions, take $H^1_0(\Omega; \mathbb{R}^m)$ instead). Under the hypotheses (H1) and (H2), this functional is known to have a minimizer $\bar{v} \in H^1(\Omega; \mathbb{R}^m)$ (see [12]) which satisfies the corresponding boundary conditions and such that

$$\Delta_y \bar{v} + f(\bar{v}) = 0. \tag{6.2}$$

Observe that for Neumann boundary conditions $\bar{v}$ is constant and is simply a minimum of the potential V. It turns out that this functional can be used to characterize the existence of solutions of Equation (1.6).

Proposition 6.2. *Let V satisfy (H1) and (H2), and assume $\nu_0 \geqq 0$. Then Equation (1.6) has a solution $\bar{u} \in H^1_c(\Sigma; \mathbb{R}^m)$ if and only if*

$$\inf E[v] < 0, \tag{6.3}$$

where the inf *is taken over the functions $v \in H^1(\Omega; \mathbb{R}^m)$ that satisfy the boundary conditions in Equation (1.2).*

Proof. Let us first prove that this assumption is sufficient. If Equation (6.3) is satisfied, then choose a trial function

$$u_\lambda(y, z) = \begin{cases} \bar{v}(y), & z < 0, \\ \bar{v}(y)e^{-\lambda z}, & z \geqq 0, \end{cases} \tag{6.4}$$

where $\bar{v}$ is a minimizer of E. Clearly, $u_\lambda \in H^1_c(\Sigma; \mathbb{R}^m)$ if $\lambda > \frac{c}{2}$. Substituting this into the definition of Φ_c, we find that

$$\Phi_c[u_\lambda] \leqq \frac{1}{c} E[\bar{v}] + \frac{1}{2(2\lambda - c)} \int_\Omega \sum_{i=1}^m \left((\lambda^2 + C)\bar{v}_i^2(y) + |\nabla_y \bar{v}_i|^2 \right) \mathrm{d}y, \tag{6.5}$$

where we used hypothesis (H1). Noting that $E[\bar{v}] < 0$, for fixed λ it is then possible to choose c so small that the right-hand side of this expression is negative. Then, u_λ will satisfy hypothesis (H3), which ensures the existence of a solution $\bar{u}$ of problem (P) by Theorem 1.1.

Let us prove that the assumption (6.3) is also necessary. Suppose on the contrary that $E[v] \geqq 0$ for all $v \in H^1(\Omega; \mathbb{R}^m)$. Then also

$$\int_\Sigma e^{cz} \left(\frac{1}{2} \sum_{i=1}^m |\nabla_y u_i|^2 + V(u) \right) dx \geqq 0$$

for all $u \in H_c^1(\Sigma; \mathbb{R}^m)$. Using this and Lemma 2.1, we then obtain

$$\Phi_c[u] \geqq \int_\Sigma e^{cz} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial z} \right)^2 dx \geqq \frac{c^2}{8} \int_\Sigma e^{cz} \sum_{i=1}^m u_i^2 \, dx. \tag{6.6}$$

Therefore, from Proposition 3.5 we conclude that there are no non-trivial solutions of Equation (1.6) which lie in $H_c^1(\Sigma; \mathbb{R}^m)$. □

Notice that inequality (6.6) shows that, if Equation (1.6) has a non-trivial solution in $H_c^1(\Sigma; \mathbb{R}^m)$, then $\inf E[v] < 0$, without any assumption on the sign of ν_0.

In the case of Neumann boundary conditions, we have the following result.

Proposition 6.3. *Let $\bar{u}$ be a solution of problem (P) with Neumann boundary conditions. Then $\bar{u}$ depends only on the variable z.*

Proof. Let us consider the function $g : \Omega \to \mathbb{R}$ defined as

$$g(y) := \int_\mathbb{R} e^{cz} \left(\frac{1}{2} \sum_{i=1}^m |\nabla \bar{u}_i|^2 + V(\bar{u}) \right) dz,$$

so that by Proposition 3.2 we have $\Phi_c[\bar{u}] = \int_\Omega g(y)\, dy = 0$. Assume first that the function g is not constant almost everywhere in Ω. Hence, we can choose $\bar{y} \in \Omega$ such that $g(\bar{y}) < 0$. By Fubini's theorem, we can also assume that the function $\tilde{u}(y, z) := \bar{u}(\bar{y}, z)$ belongs to $H_c^1(\Sigma; \mathbb{R}^m)$. However, clearly $\Phi_c[\tilde{u}] \leqq g(\bar{y})|\Omega| < 0$, contradicting Proposition 3.2.

If the function g is constant almost everywhere on Ω but $\bar{u}$ depends on y, then we can choose $\bar{y} \in \Omega$ such that

$$\int_\mathbb{R} e^{cz} |\nabla_y \bar{u}(\bar{y}, z)|^2 \, dz > 0.$$

Defining $\tilde{u}$ as above, we obtain $\Phi_c[\tilde{u}] < \Phi_c[\bar{u}] = 0$, which gives again a contradiction. □

Next we establish the fact that the solutions of problem (P) are essentially scalar functions, if the potential V depends only on the modulus of u.

Proposition 6.4. *Assume $V(u) = V(|u|)$, that is, the function V depends only on the modulus of u, and let $\bar{u} \in H^1_c(\Sigma; \mathbb{R}^m)$ be a solution of problem (P). Then, there exists a vector $v \in \mathbb{R}^m$ and a function $\eta \in C^2(\Sigma) \cap C^1(\overline{\Sigma})$, $\eta(x) > 0$, such that $\bar{u}(x) = \eta(x)v$ for any $x \in \Sigma$.*

Proof. Consider the non-empty open set $\Sigma' \subseteq \Sigma$ on which $|\bar{u}| > 0$. Introduce $\eta(x) = |\bar{u}(x)|$ on Σ and $n(x) = \bar{u}(x)/|\bar{u}(x)|$ on Σ'. The latter has the meaning of the director of the vector field u, and so we have $|n| = 1$. From these definitions $\bar{u} = \eta n$ in Σ' and $\nabla \bar{u} = 0$ almost everywhere in $\Sigma \backslash \Sigma'$. So a straightforward calculation shows that

$$\sum_{i=1}^{m} |\nabla \bar{u}_i|^2 = |\nabla \eta|^2 + \eta^2 \sum_{i=1}^{m} |\nabla n_i|^2 \geqq |\nabla \eta|^2. \tag{6.7}$$

Now consider $\tilde{u}(x) = (\eta(x), 0, \dots, 0) \in H^1_c(\Sigma; \mathbb{R}^m)$. If the last inequality in Equation (6.7) is strict, then

$$\Phi_c[\tilde{u}] < \Phi_c[\bar{u}],$$

since by assumption $V(\tilde{u}) = V(\eta) = V(|\bar{u}|) = V(\bar{u})$, and this contradicts the minimality of $\bar{u}$. So, $\nabla n = 0$ in Σ' and $\tilde{u}$ is also a minimizer, and, therefore, is regular by Proposition 3.3. Therefore, η is a classical solution of the scalar equation

$$\Delta \eta + c\eta_z - V'(\eta) = 0, \tag{6.8}$$

and, furthermore, $\eta(x) \geqq 0$. Then, we have in fact $\eta(x) > 0$ everywhere in Σ, and so $\Sigma' = \Sigma$. Indeed, define the function $c^{\pm}(x) = \left[\frac{V'(\eta(x))}{\eta(x)}\right]^{\pm}$, where $[v]^- = -\min\{v, 0\}$ and $[v]^+ = \max\{v, 0\}$, for all $x \in \Sigma'$, and set $c^{\pm}(x) = 0$ otherwise. Note that by hypothesis (H2) we have $c^{\pm} \in L^{\infty}(\Sigma)$. Then Equation (6.8) can be rewritten as

$$\Delta \eta + c\eta_z - c^+(x)\eta = -c^-(x)\eta \leqq 0.$$

So, by the strong maximum principle [20, Theorem 3.5], we conclude that $\eta(x) > 0$ for all $x \in \Sigma$. It then follows that n is a constant vector throughout Σ, which concludes the proof. □

In other words, to find the solution of problem (P) under the above assumption, one only needs to consider the scalar equation whose solutions lie in the considered exponentially weighted Sobolev spaces. Notice that for constant sign solutions of Equation (6.8) precise estimates of the decay of the solution as $z \to +\infty$ can be obtained [48]. Since, in addition, our solutions lie in the spaces $H^1_c(\Sigma; \mathbb{R}^m)$, it follows that $u = O(e^{-\lambda_0^+ z})$, where λ_0^+ is defined in Equation (2.12), for large positive z. Thus, generally these solutions are special in the sense that they have a non-generic fast exponential decay at $+\infty$ (see also [32]).

Our next group of results concerns the behavior of solutions of problem (P) as $z \to -\infty$. Our main tool here is the familiar energy estimate for gradient systems.

Lemma 6.5. *Let $\bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ be a solution of problem (P), then $\bar{u}_z \in L^2(\Sigma; \mathbb{R}^m)$.*

Proof. Scalar multiplying Equation (1.6) by $\bar{u}_z$ and integrating over $\Sigma_R := \Omega \times (-R, R)$, $R > 0$, we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^m \int_{\Sigma_R} \frac{\partial \bar{u}_i}{\partial z} \left(\frac{\partial^2 \bar{u}_i}{\partial z^2} + \Delta_y \bar{u}_i + c^\dagger \frac{\partial \bar{u}_i}{\partial z} + f_i(\bar{u}) \right) \mathrm{d}x \\ &= c^\dagger \int_{\Sigma_R} \sum_{i=1}^m \left(\frac{\partial \bar{u}_i}{\partial z} \right)^2 \mathrm{d}x \\ &\quad + \left[\int_\Omega \left(\frac{1}{2} \sum_{i=1}^m \left(\frac{\partial \bar{u}_i}{\partial z} \right)^2 - \frac{1}{2} \sum_{i=1}^m |\nabla_y \bar{u}_i|^2 - V(\bar{u}) \right) \mathrm{d}y \right]_{-R}^{R}, \end{aligned} \tag{6.9}$$

where we used the boundary conditions in Equation (1.2) to erase the boundary term

$$\int_{\partial\Omega \times (-R \times R)} (\nabla_y \bar{u}_i \cdot n_{\partial\Sigma}) \frac{\partial \bar{u}_i}{\partial z} \, \mathrm{d}\sigma \, \mathrm{d}z.$$

Recalling that by Proposition 3.3, we have $\bar{u}_i \in W^{1,\infty}(\Sigma)$, passing to the limit in the equality (6.9) for $R \to +\infty$, we obtain the thesis. □

For any $R \in \mathbb{R}$, let $\widetilde{\Sigma}_R := \Omega \times (R, R+1)$. By the results of part (ii) of Proposition 3.3 we have that the functions $\bar{u}_i$ are uniformly bounded in $W^{2,p}(\widetilde{\Sigma}_R)$, with $p > n$, independently of R. It then follows that $\bar{u}_z$ is bounded and uniformly continuous on Σ, hence from Lemma 6.5 we obtain

$$\lim_{z \to \pm\infty} \bar{u}_z(y, z) = 0 \quad \text{uniformly in } y \in \Omega. \tag{6.10}$$

On the other hand, by Proposition 3.3 we know that $\bar{u}(z, y) \to 0$ uniformly in $y \in \Omega$ as $z \to +\infty$. Then, by the same $W^{2,p}(\widetilde{\Sigma}_R)$ estimate and Sobolev imbedding theorem we obtain

$$\lim_{z \to +\infty} |\nabla_y \bar{u}(y, z)| = 0 \quad \text{uniformly in } y \in \Omega. \tag{6.11}$$

In the following proposition we characterize the possible limits (that is, the α-limit set) of $\bar{u}(\cdot, z)$ for $z \to -\infty$ (we refer the reader also to [22] for related results using dynamical systems techniques).

Proposition 6.6. *Let $\bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ be a solution of problem (P), then there exists a sequence $z_n \to -\infty$ and a function $v \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$, satisfying the same boundary conditions as $\bar{u}$, such that*

$$\begin{aligned} \lim_{n \to +\infty} \bar{u}(\cdot, z_n) &= v \quad \text{in } (C^1(\overline{\Omega}))^m. \\ \Delta_y v + f(v) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{6.12}$$

Conversely, let v be any function such that $\lim_{n\to\infty} \bar{u}(\cdot, z_n) = v$ in $(C^1(\overline{\Omega}))^m$, for some sequence $z_n \to -\infty$. Then $v \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$, v satisfies the same boundary conditions as $\bar{u}$, and Equation (6.12) holds.

Proof. Let $\phi \in H^1(\Omega;\mathbb{R}^m)$ be a test function (we further assume $\phi \in H_0^1(\Omega;\mathbb{R}^m)$ if we have Dirichlet boundary conditions). Scalar multiplying Equation (1.6) by $\phi(y)$ and integrating over $\widetilde{\Sigma}_R$, we obtain

$$0 = \left[\sum_{i=1}^m \int_\Omega \phi_i \frac{\partial \bar{u}_i}{\partial z}\,\mathrm{d}y\right]_R^{R+1} + c\sum_{i=1}^m \int_{\widetilde{\Sigma}_R} \phi_i \frac{\partial \bar{u}_i}{\partial z}\,\mathrm{d}x - \sum_{i=1}^m \int_{\widetilde{\Sigma}_R} \left(\nabla_y \bar{u}_i \cdot \nabla_y \phi_i - f_i(\bar{u})\phi_i\right)\,\mathrm{d}x. \tag{6.13}$$

Since $\bar{u}_z \to 0$ in $C^0(\widetilde{\Sigma}_R)$ for $R \to -\infty$, we have

$$\lim_{R\to-\infty} \sum_{i=1}^m \left(\left[\int_\Omega \phi_i \frac{\partial \bar{u}_i}{\partial z}\,\mathrm{d}y\right]_R^{R+1} + c\int_{\widetilde{\Sigma}_R} \phi_i \frac{\partial \bar{u}_i}{\partial z}\,\mathrm{d}x\right) = 0. \tag{6.14}$$

Note that the family of functions $\bar{u}(y, z+R)$ is equibounded in $(C^1(\overline{\Sigma}_0))^m$, where $\Sigma_0 := \Omega \times (0,1)$. Indeed, from the estimates of Proposition 3.3, we obtain a uniform bound on $\bar{u}_i(y, z+R)$ in $W^{2,p}(\Sigma_0)$, with $p > n$. So, by the Ascoli–Arzelà theorem there exists a sequence $R_n \to -\infty$ and a function $\tilde{v}$ such that $\bar{u}(y, z+R_n) \to \tilde{v}$ in $(C^1(\overline{\Sigma}_0))^m$. Moreover, since $\lim_{R\to-\infty} \bar{u}_z(y, z+R) = 0$ uniformly on $\overline{\Sigma}_0$, we obtain $\tilde{v}_z = 0$, that is, the function $\tilde{v}$ depends only on y. Setting $v(y) := \tilde{v}(y,z)$, we then obtain $\lim_{n\to\infty} \bar{u}(\cdot, z_n) = v$ in $(C^1(\Omega))^m$, for example, for $z_n = R_n$.

From Equations (6.13) and (6.14), it then follows

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \sum_{i=1}^m \int_{\widetilde{\Sigma}_{R_n}} \left(\nabla_y \bar{u}_i \cdot \nabla_y \phi_i - f_i(\bar{u})\phi_i\right)\,\mathrm{d}x \\ &= \sum_{i=1}^m \int_\Omega \left(\nabla_y v_i \cdot \nabla_y \phi_i - f_i(v)\phi_i\right)\,\mathrm{d}y, \end{aligned} \tag{6.15}$$

for any $\phi \in H^1(\Omega;\mathbb{R}^m)$ (responsible for any $\phi \in H_0^1(\Omega;\mathbb{R}^m)$), which implies $v \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$, v satisfies the same boundary conditions as $\bar{u}$ on $\partial\Omega$, and $\Delta_y v + f(v) = 0$ in Ω.

Conversely, let us assume that there exists a function v such that $\lim_{n\to\infty} \bar{u}(\cdot, z_n) = v$ in $(C^1(\overline{\Omega}))^m$, for some sequence $z_n \to -\infty$. Then, reasoning exactly as above with $R_n = z_n$ we obtain $v \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$, v satisfies the same boundary conditions as $\bar{u}$ on $\partial\Omega$, and $\Delta_y v + f(v) = 0$ in Ω. □

We note that, by regularity of $\bar{u}$, a weak form of convergence (such as weak in $L^2(\Omega;\mathbb{R}^m)$, for example) implies a stronger $(C^1(\Omega))^m$–convergence in the second part of Proposition 6.6.

Let $E[v]$ be the functional defined in (6.1) and introduce

$$W := \left\{v \in H^1(\Omega;\mathbb{R}^m):\ v(y) \in \mathcal{K} \text{ for all } y \in \Omega,\ \text{and } E[v] < 0\right\}.$$

Again, in the case of the Dirichlet boundary conditions replace $H^1(\Omega;\mathbb{R}^m)$ with $H_0^1(\Omega;\mathbb{R}^m)$. Taking $R=-z_n$ in (6.9) and letting $n\to+\infty$, from Proposition 6.6 and Equations (6.10) and (6.11), we obtain the following:

Corollary 6.7. *Let v be as in Proposition 6.6. Then $v\in W$, and in particular, $v\neq 0$.*

Under some extra assumptions on the critical points of $E[v]$, it is possible to give more precise information on the asymptotic behavior of the solutions of problem (P) at $z=-\infty$.

Corollary 6.8. *Assume that any critical point of E in W is isolated in the strong topology of $H^1(\Omega;\mathbb{R}^m)$. Then the limit in Proposition 6.6 is a full limit, that is,*

$$\lim_{z\to-\infty}\bar{u}(\cdot,z)=v\quad\text{in } (C^1(\overline{\Omega}))^m,$$

with $v\in W$.

Proof. Note that the mapping $z\mapsto\bar{u}(\cdot,z)$ is a continuous mapping from $\mathbb{R}$ to $H^1(\Omega;\mathbb{R}^m)$. Suppose that the full limit of $\bar{u}(\cdot,z)$ does not exist. By continuity of this mapping, Proposition 6.6 and Corollary 6.7, there exists $\varepsilon>0$ and a sequence $z_n'\to-\infty$ such that $\varepsilon\leqq||\bar{u}(\cdot,z_n')-v||_{H^1(\Omega;\mathbb{R}^m)}\leqq 2\varepsilon$, where $v\in W$ is some limit from Proposition 6.6, and the 2ε-neighborhood of v does not contain any other elements of W. By regularity of $\bar{u}$ we can pass to a subsequence, still labeled (z_n') that converges strongly in $H^1(\Omega;\mathbb{R}^m)$. Therefore, if $v'=\lim_{n\to\infty}\bar{u}(\cdot,z_n')$, then $\varepsilon\leqq||v'-v||_{H^1(\Omega;\mathbb{R}^m)}\leqq 2\varepsilon$, too. But, by Proposition 6.6 and Corollary 6.7 every convergent sequence in $(C^1(\Omega))^m$ has a limit that is in W, which contradicts the assumption that there are no elements of W in the 2ε-neighborhood of v that are distinct from v. □

Note that a sufficient condition for a critical point of E to be isolated is that it is non-degenerate (that is, that the second variation of E does not have zero eigenvalues). Also note that in the case of Neumann boundary conditions we know from Proposition 6.3 that the function $\bar{u}$ is independent of $y\in\Omega$, which implies that the function v is a constant. Therefore, we obtain the full limit in Proposition 6.6 simply if we assume that any critical point of V in the open set $\{u\in\mathbb{R}^m : V(u)<0\}\subset\mathbb{R}^m$ is isolated.

We conclude this section by showing that, under suitable assumptions, the solutions of Equation (1.1) propagate along Σ with asymptotic speed bounded by $c^\dagger$. Let us note that to address this question in full generality we need a suitable existence theory for the initial value problem given by Equation (1.1). This, however, would go beyond the scope of our paper. On the other hand, it is possible to show that a large class of initial data for Equation (1.1) will generate solutions in the class $\mathcal{Q}_c(\Sigma,\mathbb{R}^+)$ introduced in [37], a natural target space for the solutions of Equation (1.1):

Definition 2. We will say that $u\in\mathcal{Q}_c(\Sigma,\mathbb{R}^+)$, if $u\in C^\infty(\Sigma\times\mathbb{R}^+)$, $u(x,t)\in\mathcal{K}$, and there exists $\lambda>\frac{c}{2}$ such that for any $T>t_0>0$ and multi-index α there exists a constant $C_\alpha=C_\alpha(t_0,T)$ such that $|D^\alpha u(\cdot,t)|<C_\alpha(1+\mathrm{e}^{-\lambda z})$ for all $t\in[t_0,T]$.

Notice first that in the context of Equation (1.1) the set $\mathcal{K}$ has a meaning of an invariant region, whose existence assures global in time existence of solutions for Equation (1.1), and by standard parabolic theory we obtain uniform bounds on the derivatives (see, for example, [33]). So, what the classes $\mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ control is mainly the rate of exponential decay of the solution, quantified by the value of c. Notice that the assumption that the solution of Equation (1.1) lies in $\mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ (even with arbitrary $c > 0$) can be easily satisfied, for example, whenever $u(\cdot, 0)$ takes values in $\mathcal{K}$ and has compact support.

We now state our result.

Proposition 6.9. *Suppose that problem* (*P*) *has a solution, and let* $u(x,t) \in \mathcal{Q}_{c^\dagger}(\Sigma, \mathbb{R}^+)$ *be a solution of Equation* (1.1)*. Then, for any* $c' > c^\dagger$*, it holds that* $u(y, z + c't, t) \to 0$ *as* $t \to \infty$*, uniformly on compact subsets of* Σ.

Proof. Fix a constant c'' such that $c^\dagger < c'' < 2\lambda$, with λ from the definition of $\mathcal{Q}_{c^\dagger}(\Sigma, \mathbb{R}^+)$, then $u(\cdot, t) \in H^1_{c''}(\Sigma; \mathbb{R}^m)$. Differentiating $\Phi_{c''}[u(y, z + c''t, t)]$ in t and integrating by parts, which is justified by the uniform estimates for $u \in \mathcal{Q}_{c^\dagger}(\Sigma, \mathbb{R}^+)$, we obtain for all $t > 0$

$$\frac{\mathrm{d}\Phi_{c''}[u(y, z + c''t, t)]}{\mathrm{d}t} = -\int_\Sigma \mathrm{e}^{c''z} \sum_{i=1}^m \left(\Delta u_i + c'' \frac{\partial u_i}{\partial z} + f_i(u) \right)^2 \mathrm{d}x \leqq 0. \tag{6.16}$$

Since also $c'' > c^\dagger$, we have $0 \leqq \Phi_{c''}[u(y, z + c''t, t)] \leqq \Phi_{c''}[u(y, z, t_0)]$, $t_0 > 0$, and by Lemma 3.1 we obtain

$$\Phi_{c''}[u(y, z + c't, t)] = \mathrm{e}^{-c''(c'-c'')t} \Phi_{c''}[u(y, z + c''t, t)] \to 0 \tag{6.17}$$

as $t \to \infty$. On the other hand, letting $\tilde{u}(y, \zeta, t) := u\left(y, \frac{c^\dagger}{c''}\zeta, t\right)$ and retracing the arguments of Equations (4.7)–(4.9), we obtain $\tilde{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ and

$$\begin{aligned}
\Phi_{c''}[u] &= \int_\Sigma \mathrm{e}^{c''z} \left(\frac{1}{2} \sum_{i=1}^m \left[\left(\frac{\partial u_i}{\partial z} \right)^2 + |\nabla_y u_i|^2 \right] + V(u) \right) \mathrm{d}y\, \mathrm{d}z \\
&= \frac{c^\dagger}{c''} \int_\Sigma \mathrm{e}^{c^\dagger \zeta} \left(\frac{1}{2} \sum_{i=1}^m \left[\left(\frac{c''}{c^\dagger} \right)^2 \left(\frac{\partial \tilde{u}_i}{\partial \zeta} \right)^2 + |\nabla_y \tilde{u}_i|^2 \right] + V(\tilde{u}) \right) \mathrm{d}y\, \mathrm{d}\zeta \\
&= \frac{c''^2 - c^{\dagger 2}}{c'' c^\dagger} \Gamma_{c^\dagger}[\tilde{u}] + \frac{c^\dagger}{c''} \Phi_{c^\dagger}[\tilde{u}] \geqq \frac{c''^2 - c^{\dagger 2}}{c'' c^\dagger} \Gamma_{c^\dagger}[\tilde{u}] = \frac{c''^2 - c^{\dagger 2}}{c''^2} \Gamma_{c''}[u],
\end{aligned}$$

since $\Phi_{c^\dagger}[u] \geqq 0$ for all $u \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ by Proposition 3.2. But then, using Lemma 2.1, we have

$$\Phi_{c''}[u(y, z + c't, t)] \geqq \frac{c''^2 - c^{\dagger 2}}{8} \int_\Sigma \mathrm{e}^{c''z} \sum_{i=1}^m u_i^2(y, z + c't, t)^2 \mathrm{d}y\, \mathrm{d}z. \tag{6.18}$$

Therefore, $u(y, z + c't, t) \to 0$ in $L^2_{c^\dagger}(\Sigma; \mathbb{R}^m)$ as $t \to \infty$. Since $u(\cdot, t) \in (L^\infty(\Sigma))^m$ we have $\nabla u(\cdot, t) \in (L^\infty(\Sigma))^{mn}$ uniformly for any $t \geqq t_0$, with $t_0 > 0$ (see [33]); hence $u(y, z + c't, t) \to 0$ uniformly on compact subsets of Σ. □

Let us emphasize that the result in Proposition 6.9 implies that the speed $c^\dagger$ obtained in problem (P) has a special significance for the solutions of the original parabolic problem. Indeed, $c^\dagger$ is the maximum speed with which solutions may propagate (for example, in the sense of the speed of the leading edge [37,40]). On the other hand, observe that this is also a sharp upper bound, since existence of solutions of problem (P) obviously implies existence of solutions of Equation (1.1) which propagate with speed $c^\dagger$.

Finally, let us note that in general the free energy functional in Equation (1.4) may include the effect of anisotropy [29,30], that is, the gradient square term in $F[u]$ can be replaced by a quadratic form generated by a symmetric positive-definite constant $n \times n$ matrix G. Then the analogue of Equation (1.1) becomes

$$u_t = \nabla \cdot (G \nabla u) - \nabla_u V(u). \tag{6.19}$$

Similarly, the boundary conditions for this equation should be modified from Equation (1.2) and become

$$(\nu \cdot G\, \nabla u)|_{\partial \Sigma} = 0 \quad \text{or} \quad u|_{\partial \Sigma} = 0. \tag{6.20}$$

One can naturally ask whether the above problem admits traveling wave solutions, too. Indeed, it is not difficult to see that Equation (6.19) with the boundary conditions from Equation (6.20) can be reduced to Equation (1.1), with the boundary conditions from Equation (1.2), by the linear change of variables

$$x' = G^{-1/2} x.$$

In this way we obtain a problem of the type considered above on a modified cylinder Σ', which can then be treated in the same fashion.

7. An application

In this section, we consider a sample application problem, for which various assumptions of the theorems above can be explicitly verified, and demonstrate the practical utility of our methods. For a particular example we will use a computer-assisted approach to obtain the necessary estimates for existence. Note that with a bit of extra work these types of results can be made completely rigorous. This, however, is beyond the purpose of this section, which is to illustrate our theorems.

As a sample problem, we will consider Equation (1.5) with $\tau = 1$, $g = 1$, $a = 3$, $b_1 = 1$, $b_2 = \frac{3}{2}$, $h_1 = \frac{11}{20}$, and $h_2 = 0$. For simplicity, we will consider the case $m = 2$ and $n = 1$ (implying that $\Sigma = \mathbb{R}$), so that the vector character of the problem is preserved. Let us mention that in one space dimension existence of traveling wave solutions in gradient systems can be also studied by topological techniques [36,39,45].

Thus, with $u = (u_1, u_2)$, this problem has the following expression for the potential V:

$$V(u_1, u_2) = -\tfrac{11}{20} u_1 - \tfrac{3}{2}\left(u_1^2 + u_2^2\right) + \tfrac{1}{4}\left(u_1^4 + u_2^4\right) + \tfrac{3}{4} u_1^2 u_2^2. \tag{7.1}$$

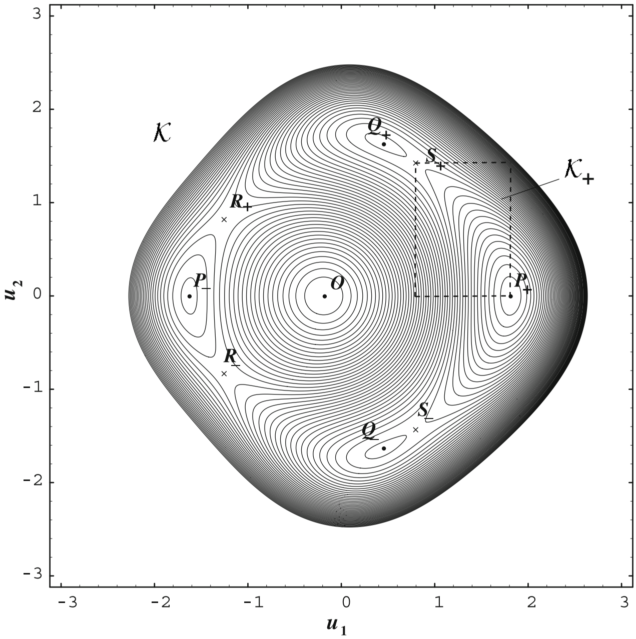

Fig. 7.2. The level curves of the potential V in Equation (7.1). The outermost contour corresponds to $V = \frac{1}{4}$ and shows the boundary of the set $\mathcal{K}$. The set $\mathcal{K}_+$ is enclosed by the dashed lines

The plot of the level curves of V is presented in Fig. 7.2. An inspection of this figure shows that V has one local maximum $O(p_0, 0)$, four local minima $P_\pm(p_\pm, 0)$ and $Q_\pm(p_1, \pm q_1)$, and four saddle points $R_\pm(p_2, \pm q_2)$ and $S_\pm(p_3, \pm q_3)$, respectively (see Fig. 7.2). It is easy to see that the set $\mathcal{K} := \{(u_1, u_2) \in \mathbb{R}^2 : V(u_1, u_2) \leqq \frac{1}{4}\}$ has the required properties, being convex and satisfying Equation (2.3). There is also a rectangle $\mathcal{K}_+ = \{(u_1, u_2) \in \mathbb{R}^2 : p_3 \leqq u_1 \leqq p_+,\ 0 \leqq u_2 \leqq q_3\}$, which is also convex and satisfies Equation (2.3).

We are going to study existence of several types of traveling waves which connect to different equilibria, namely to O, P_-, and S_+. Each such case leads to a different variational problem, since in order to satisfy hypothesis (H1), one needs to subtract from V its value at the equilibrium point reached at $z = +\infty$. So, we will consider each such problem separately and establish existence and non-existence of variational traveling waves, as well as the upper and lower bounds for the speed. To simplify the notation, we will still say that u lies in $H^1_c(\Sigma; \mathbb{R}^m)$, tacitly assuming that the equilibrium point is properly subtracted from u.

Let us point out that if one sets $u_2 = 0$, then the problem becomes scalar, and existence of traveling waves connecting P_+, P_-, and O is well-known (see, for example, [2,50]). These are the heteroclinic orbits $P_- O$, $P_+ O$, and $P_+ P_-$, respectively, and there exists a continuous family of solutions monotonically connecting P_+ and P_- with O, and a unique solution going monotonically from P_+ to P_-. Furthermore, an exact solution for the traveling wave $P_+ P_-$ can be found [4], giving $c = 0.393419$ for this wave. These are natural candidates for the solutions of the variational problems under consideration, so, in particular, we need to see whether we can discriminate between them and the solutions of the vector problem.

We start by studying the case of the waves connecting to O. To begin, we compute the value of ν_0, which in all considered cases is simply the smallest eigenvalue of the Hessian at the equilibrium approached at $z = +\infty$. A straightforward calculation shows that at O we have $\nu_0 = -2.94841 < 0$. So, in order to be able to apply Theorem 1.1, we need to find a trial function that makes the functional Φ_c non-positive for $c > c_0 = 2\sqrt{-\nu_0} = 3.43419$. We were not able to find such a trial function.

On the other hand, at O we can estimate the value of μ_- to be slightly greater than -3. By Proposition 6.1, there are no variational traveling waves for $c \geqq c_1 = 3.4641$. Therefore, our method can give solutions only in a narrow range of $3.43419 < c < 3.4641$, if any. Since also for $c < 2\sqrt{-\nu_1} = 3.40401$ the solution will approach O in an oscillatory fashion (see the discussion in [37, Section 3]), it will not be expected to lie in $H_c^1(\mathbb{R})$, either. This suggests that there are no variational traveling waves that connect to O. In fact, we can prove that there are no variational traveling waves satisfying hypothesis (H3) that lie entirely to the left of O (that is, for which $u_1 \leqq p_0$). Indeed, applying the Taylor formula, we have

$$V(u_1, u_2) = V(p_0, 0) + \tfrac{1}{2}\left\{\left(-3 + 3\tilde{u}_1^2 + \tfrac{3}{2}\tilde{u}_2^2\right)(u_1 - p_0)^2 \right.$$
$$\left. +6\tilde{u}_1\tilde{u}_2(u_1 - p_0)u_2 + \left(-3 + \tfrac{3}{2}\tilde{u}_1^2 + 3\tilde{u}_2^2\right)u_2^2\right\},$$

where $u_1 \leqq \tilde{u}_1 \leqq p_0 < 0$ and $\tilde{u}_2$ lies between 0 and u_2. Clearly, the coefficients of the first and the third terms in the curly brackets are greater or equal to $-\nu_0 = -3 + \frac{3}{2}p_0^2$. Furthermore, since $(u_1 - p_0)\tilde{u}_1 \geqq 0$ and $\tilde{u}_2 u_2 \geqq 0$, we then have $V(u_1, u_2) \geqq V(p_0, 0) + \frac{1}{2}\nu_0((u_1 - p_0)^2 + u_2^2)$, which implies that $\Phi_c[u] = 0$ if and only if $u = (p_0, 0)$ for all $u \in H_c^1(\mathbb{R})$ with $c^2 + 4\nu_0 > 0$, so Theorem 1.1 cannot be applied. Then, in view of Proposition 3.5, this means non-existence of variational traveling waves with these speeds. Note also that this argument can be strengthened to show that all the solutions $P_- O$ with $u_2 = 0$ are not variational traveling waves (see also [32]). This is not unusual for the traveling waves invading an unstable equilibrium.

Let us now consider the waves that connect to P_-. Here we obtain $\nu_0 = 0.994441 > 0$, and we know from the case $u_2 = 0$ that problem (P) has a solution. The question is whether this solution is in fact one-dimensional, and what the bounds on the speed are. To begin, we first find that for P_- the value of μ_- is slightly greater than -0.34. Again, by Proposition 6.1 this means that the variational traveling waves connecting to P_- may exist only for $c < c_1 = 2\sqrt{-\mu_-} < 1.1662$. To see whether there are variational traveling waves that move *faster* than in the case $u_2 = 0$, we construct the trial function $u = (u_1, u_2)$ defined as

$$u_1(z, a, b) := p_- + \tfrac{1}{2}(p_+ - p_-)(1 - \tanh az),$$
$$u_2(z, a, b) := b \operatorname{sech}^2 az.$$

Next we evaluate Φ_c on u and minimize with respect to a and b. We then find a (large enough) value of c at which the minimum value of Φ_c is still negative. We found that the choice of $a = 0.5876$, $b = 1.6301$ works with $c = 0.5240$. So now, applying Theorem 1.1, we can conclude that there exists a traveling wave solution connecting to P_- that lies in $\mathcal{K}$ and has speed $0.5240 < c < 1.1662$. Observe that this speed is higher than that of the scalar solution obtained earlier, so the latter is in fact not a solution of problem (P). Also, by Corollary 6.8 the solution is a heteroclinic orbit from P_- to either $Q_\pm$, $S_\pm$, or P_+ (the equilibria O and $R_\pm$ have higher potential than P_-). Let us point out that our arguments can be made rigorous (with a slightly smaller value of c) by performing a linear interpolation of the above trial function, over finitely many intervals, then rationalizing the values of u at the interpolation nodes, and then carrying out some simple, albeit tedious, analysis.

Finally, we turn to the solutions that connect to S_+ and lie in $\mathcal{K}_+$. For S_+, we obtain $\nu_0 = -0.588022$, so in order for hypothesis (H3) to be satisfied, we need to find a trial function for which $\Phi_c < 0$ with $c > 1.53365$. We use the following trial function $u = (u_1, u_2)$:

$$u_1(z, a, b) := p_3 + \frac{p_+ - p_3}{1 + e^{az}},$$
$$u_2(z, a, b) := q_3 - \frac{q_3}{(1 + e^{bz})^{3/2}}.$$

Once again, we fix c and minimize $\Phi_c[u]$ with respect to a and b. As a result, we find that the functional is negative for $a = 1.1536$, $b = 0.8778$, and $c = 1.61 > 1.53365$. Therefore, the assumptions of Theorem 1.1 are satisfied in $\mathcal{K}_+$, and we obtain a traveling wave solution connecting to S_+ that lies in $\mathcal{K}_+$. On the other hand, we find μ_- to be slightly greater than -0.91, implying an upper bound for the speed of the traveling wave to be $c < 2\sqrt{-\mu_-} < 1.91$. Thus, the obtained solution will have speed $1.61 < c < 1.91$. Again, by Corollary 6.8 this is a heteroclinic orbit from S_+ to P_+.

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