An interplay between dimensionality and topology in thin ferromagnetic films

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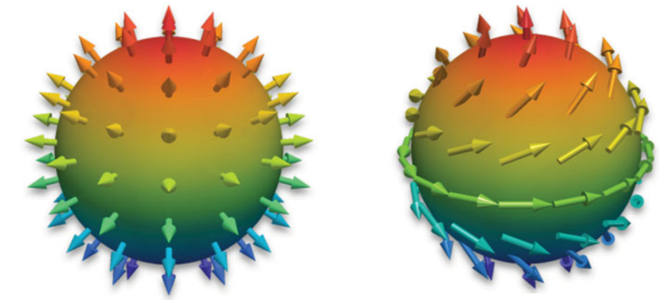
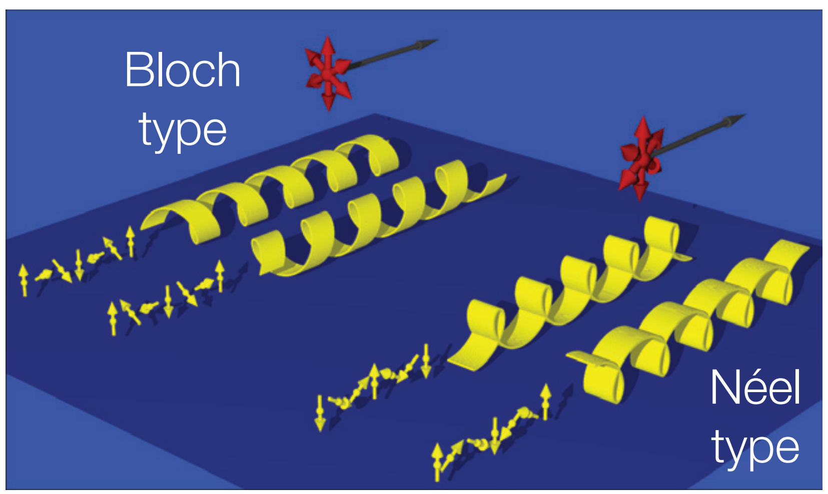
Topological spin textures

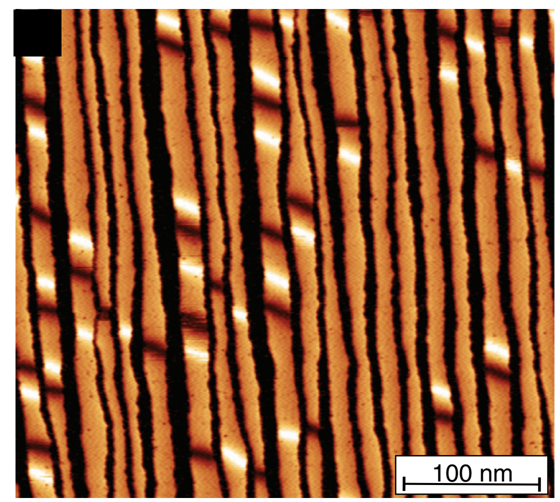


T. Lancaster, *Contemp. Phys.* **60**, 246-261 (2019)

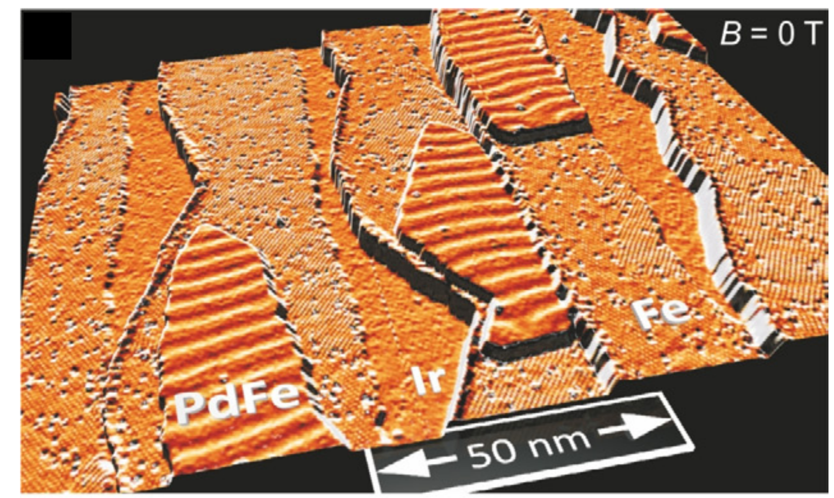
spin spirals and chiral domain walls from **Dzyaloshinskii-Moriya interaction** (DMI):



2ML Fe on W(110)

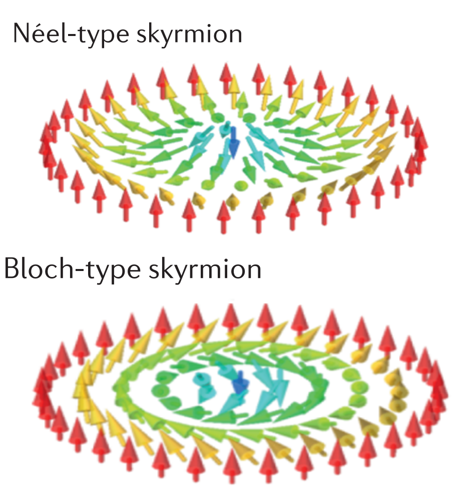


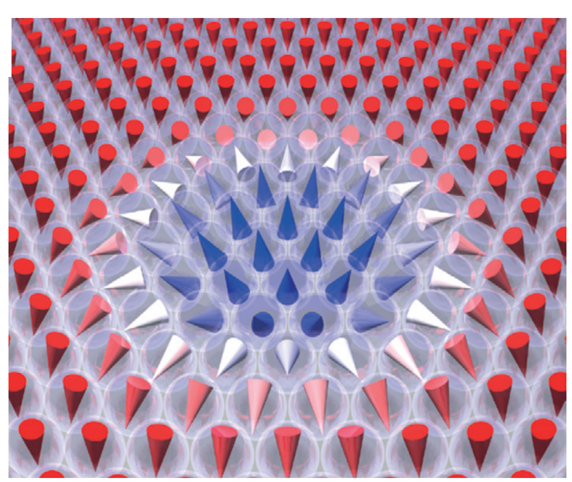
Pd/Fe bilayer on Ir(111)

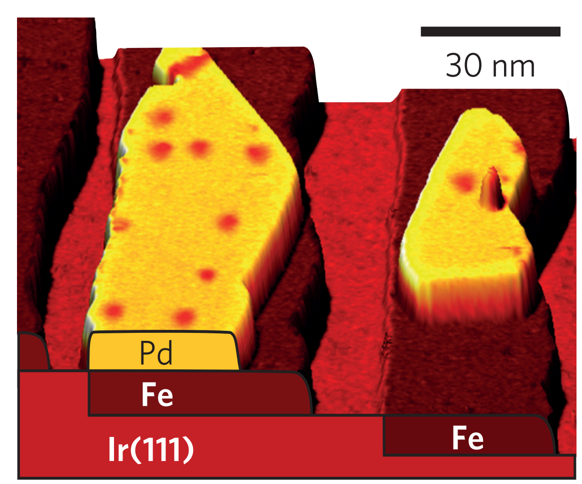


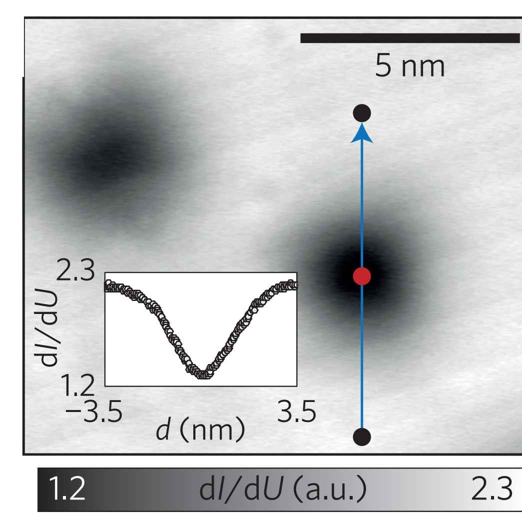
K. von Bergmann et al., J. Phys.: Condens. Matter **26**, 394002 (2014)

magnetic skyrmions:









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C. Hanneken et al., *Nature Nanotechnol.* **10**, 1039–1042 (2015)

Topological what?

topology is a branch of mathematics that deals with properties of objects that are preserved under *continuous deformations* (homotopies)

in the context of ferromagnetism, the object is the magnetization field:

$$\boxed{\mathbf{M} : \Omega \to \mathbb{R}^3}$$

magnetization field is a **map** from a *domain* occupied by the ferromagnet into the space of 3D vectors

a *homotopy invariant* is a characteristic of the magnetization field such that

$$\boxed{\begin{gathered} S(\mathbf{M}_0) \text{ is true} \Rightarrow S(\mathbf{M}_1) \text{ is true} \\ \forall \varphi : [0,1] \times \Omega \to \mathbb{R}^3 \text{ continuous} \\ \mathbf{M}_t(\mathbf{r}) = \varphi(t, \mathbf{r}) \end{gathered}}$$

Topological what?

- need to specify the source manifold Ω where the magnetization **M** lives
- need to specify the target manifold = the range of allowed values of **M**
- need to specify the notion of continuity between different **M** fields

once the above are fixed, need to identify the homotopy invariants *S*(**M**)

all of the above are model-specific!

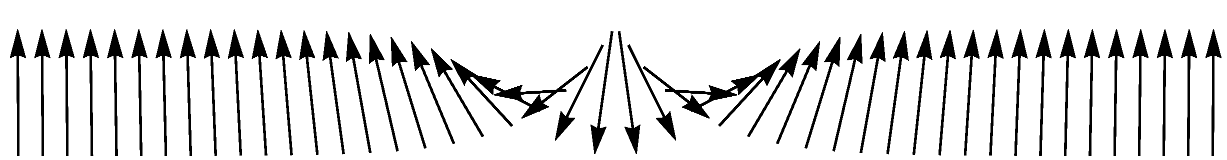
<u>Example:</u> winding Néel wall

$$\Omega = \mathbb{R} \cup \{\infty\} \qquad |\mathbf{M}| = M_s \qquad M_2 = 0$$

continuity:

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ |t - t'| < \delta \text{ and } |x - x'| < \delta \text{ or } |x| > \delta^{-1}, \ x' = \infty \ \Rightarrow \ |\mathbf{M}_t(x) - \mathbf{M}_{t'}(x')| < \varepsilon$$

homotopy invariant = winding number (topological degree) of **M**

S(**M**) = { deg(**M**) = 1 } : 

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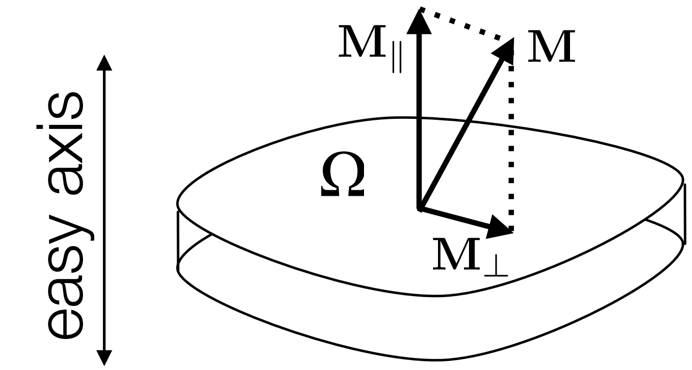
Micromagnetics of thin films (3D)



statics: $\Omega = D \times (0,d), \;\; D \subset \mathbb{R}^2$ $\qquad \overline{\mathbf{M}}(x,y) = \mathbf{M}(x,y,0)$

$$\boxed{\begin{aligned} E(\mathbf{M}) = {} & \frac{A}{M_s^2}\int_\Omega |\nabla \mathbf{M}|^2 \, d^3 r + \frac{Kd}{M_s^2}\int_D |\overline{\mathbf{M}}_\perp|^2 d^2 r - \mu_0 \int_\Omega M_3 \, H \, d^3 r \\ & + \mu_0 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla\cdot\mathbf{M}(\mathbf{r}) \, \nabla\cdot\mathbf{M}(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|} \, d^3 r \, d^3 r' + \frac{Dd}{M_s^2}\int_D \left(\overline{M}_\| \nabla\cdot\overline{\mathbf{M}}_\perp - \overline{\mathbf{M}}_\perp\cdot\nabla\overline{M}_\|\right) d^2 r \end{aligned}}$$

Landau and Lifshitz, 1935; Brown, 1963; Néel, 1954; Crépieux and Lacroix, 1998; M, Slastikov, 2016

Here $\mathbf{M} = (\mathbf{M}_\perp, M_\|)$, $\quad \mathbf{M}_\perp \in \mathbb{R}^2 \quad M_\| \in \mathbb{R} \quad |\mathbf{M}| = M_s$ in Ω, $\quad H \geq 0$

Parameters and their representative values:

- exchange constant $A = 10^{-11} \mathrm{J/m}$
- anisotropy constant $K = 1.25 \times 10^6 \; \mathrm{J/m^3}$
- saturation magnetization $M_s = 1.09 \times 10^6 \; \mathrm{A/m}$
- DMI strength $D = 1 \; \mathrm{mJ/m^2}$ $\qquad$ applied field strength $\mu_0 H = 100 \; \mathrm{mT}$

film thickness $d = 0.5 - 5\,\mathrm{nm}$

lateral dimension:

$L = 50 - 500\,\mathrm{nm}$

$\boxed{\text{exchange length } \ell_{ex} = 3.66 \text{ nm}}$

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Dimension reduction (3D to 2D)

$$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$$

<u>assume</u> the magnetization $\mathbf{m} = \mathbf{M}/M_s$ does not vary significantly across the film thickness, measure lengths in the units of ℓ_{ex}, scale energy by Ad

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa\, \mathbf{m}_\perp \cdot \nabla m_\parallel - 2h(m_\parallel - 1) \right\} d^2 r$$

$$+ \frac{1}{2\pi\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + \delta^2}} - 2\pi \delta^{(2)}(\mathbf{r} - \mathbf{r}')\delta \right) m_\parallel(\mathbf{r}) m_\parallel(\mathbf{r}')\, d^2 r\, d^2 r'$$

$$+ \delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_\delta(|\mathbf{r} - \mathbf{r}'|) \nabla \cdot \mathbf{m}_\perp(\mathbf{r})\, \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')\, d^2 r\, d^2 r'$$

Here:

$$\boxed{Q = \frac{2K}{\mu_0 M_s^2} \qquad \kappa = D\sqrt{\frac{2}{\mu_0 M_s^2 A}} \qquad h = \frac{H}{M_s} \qquad \ell_{ex} = \sqrt{\frac{2A}{\mu_0 M_s^2}} \qquad \delta = \frac{d}{\ell_{ex}}}$$

$$K_\delta(r) = \frac{1}{2\pi\delta} \left\{ \ln\left(\frac{\delta + \sqrt{\delta^2 + r^2}}{r} \right) - \sqrt{1 + \frac{r^2}{\delta^2}} + \frac{r}{\delta} \right\} \simeq \frac{1}{4\pi r} \qquad \delta \ll 1$$

C. Garcia-Cervera, Ph.D. thesis (1999)

Reduced thin film energy (2D)

$$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$$

regime $\delta \ll 1$: Taylor-expand in Fourier space

$$\boxed{\begin{aligned} E(\mathbf{m}) \simeq & \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel - 2h(m_\parallel - 1) \right\} d^2 r \\ & + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r' \end{aligned}}$$

M, Slastikov, 2016

the expression for the stray field energy is rigorously justified via a Γ-expansion

Knüpfer, M, Nolte, 2019

for bounded 2D samples, extra boundary terms appear

Di Fratta, M, Slastikov, 2021

proper definition of the non-local terms via Fourier:

$$\left.\frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{k}| \, |\widehat{m}_\parallel(\mathbf{k})|^2 \frac{d^2 k}{(2\pi)^2} = \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r', \quad \right\} \text{surface charges}$$

$$\left.\frac{1}{2} \int_{\mathbb{R}^2} \frac{|\mathbf{k} \cdot \widehat{\mathbf{m}}_\perp(\mathbf{k})|^2}{|\mathbf{k}|} \frac{d^2 k}{(2\pi)^2} = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \, \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r'. \quad \right\} \text{volume charges}$$

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Reduced thin film energy (1D)

energy per unit length: $\mathbf{m} = (m_1, m_2, m_3)$

$$\boxed{\begin{aligned} E(\mathbf{m}) \simeq & \int_{\mathbb{R}} \left\{ |m_1'|^2 + |m_2'|^2 + |m_3'|^2 + (Q-1)(m_1^2 + m_2^2) - 2\kappa\, m_1 m_3' - 2h(m_3 - 1) \right\} dx \\ & + \frac{\delta}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(m_1(x) - m_1(x'))^2}{(x-x')^2}\, dx\, dx' - \frac{\delta}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(m_3(x) - m_3(x'))^2}{(x-x')^2}\, dx\, dx' \end{aligned}}$$

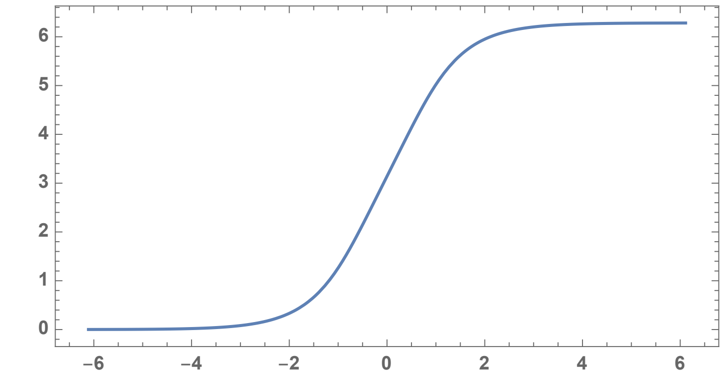
setting δ = 0 and a Néel profile $\mathbf{m} = (m_1, 0, m_3), \qquad (m_1, m_3) \in \mathbb{S}^1$

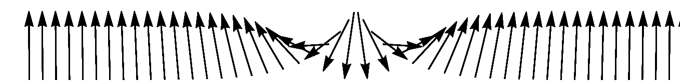
$$E(\mathbf{m}) = \int_{\mathbb{R}} \left(\frac{|m_3'|^2}{1 - m_3^2} + (Q-1)(1 - m_3^2) - 2\kappa m_1 m_3' - 2h(m_3 - 1) \right) dx$$

winding number

$$\deg(\mathbf{m}) = \frac{1}{2\pi} \int_{\mathbb{R}} (m_1 m_3' - m_3 m_1')\, dx = \frac{1}{\pi} \int_{\mathbb{R}} m_1 m_3'\, dx \in \mathbb{Z}$$

360-degree wall





minimize in the homotopy class $\{\ \deg(\mathbf{m}) = \pm 1\ \}$

choosing $\quad m_1 = -\sin\theta \qquad m_3 = \cos\theta \quad$ we get

$$E(\mathbf{m}) = \int_{\mathbb{R}} \left\{ |\theta'|^2 + (Q-1)\sin^2\theta + 2h(1-\cos\theta) \right\} dx - 2\pi\kappa \deg(\mathbf{m})$$

where

$$\deg(\mathbf{m}) = \frac{1}{2\pi}\int_{\mathbb{R}} \left(m_1 m_3' - m_3 m_1' \right)\, dx = \frac{1}{2\pi}\int_{\mathbb{R}} \theta'\, dx \in \mathbb{Z}$$

since $\qquad m_3(\infty) = 1 \quad \Rightarrow \quad \theta(\pm\infty) \in 2\pi\mathbb{Z}$

minimizer exists iff $\qquad \deg(\mathbf{m}) = \dfrac{1}{2\pi}[\theta(+\infty) - \theta(-\infty)] = \pm 1\,, \qquad Q > 1, h > 0$

implicitly $\quad x = -\dfrac{\sqrt{2}\sin\left(\frac{\theta}{2}\right)\sqrt{2h + (Q-1)\cos\theta + Q - 1}\,\tanh^{-1}\left(\dfrac{\sqrt{2(h+Q-1)}\cos\left(\frac{\theta}{2}\right)}{\sqrt{2h+(Q-1)\cos\theta+Q-1}}\right)}{\sqrt{(h+Q-1)(2h(1-\cos\theta) + (Q-1)\sin^2\theta)}}$

Is the 360-degree wall "topologically protected"?

actually, that depends on what you mean!

Topological fact: the obtained Néel profile $\mathbf{m} : \mathbb{R} \to \mathbb{S}^1, \quad \mathbf{m} = (-\sin\theta, \cos\theta)$

cannot be continuously deformed into the ferromagnetic state $\mathbf{m} = (0, 1)$

But why *continuously*?

In general, solutions of LLG may fail to be continuous — finite time blowup

However, in 1D a discontinuity formation costs infinite energy:

$$|\mathbf{m}(x_1) - \mathbf{m}(x_2)| \leq \int_{x_1}^{x_2} |\mathbf{m}'(x)| dx \leq \left(\int_{x_1}^{x_2} dx \int_{x_1}^{x_2} |\mathbf{m}'(x)|^2 dx \right)^{1/2}$$

$$\leq \sqrt{x_2 - x_1} \left(\int_{\mathbb{R}} |\mathbf{m}'|^2 dx \right)^{1/2}$$

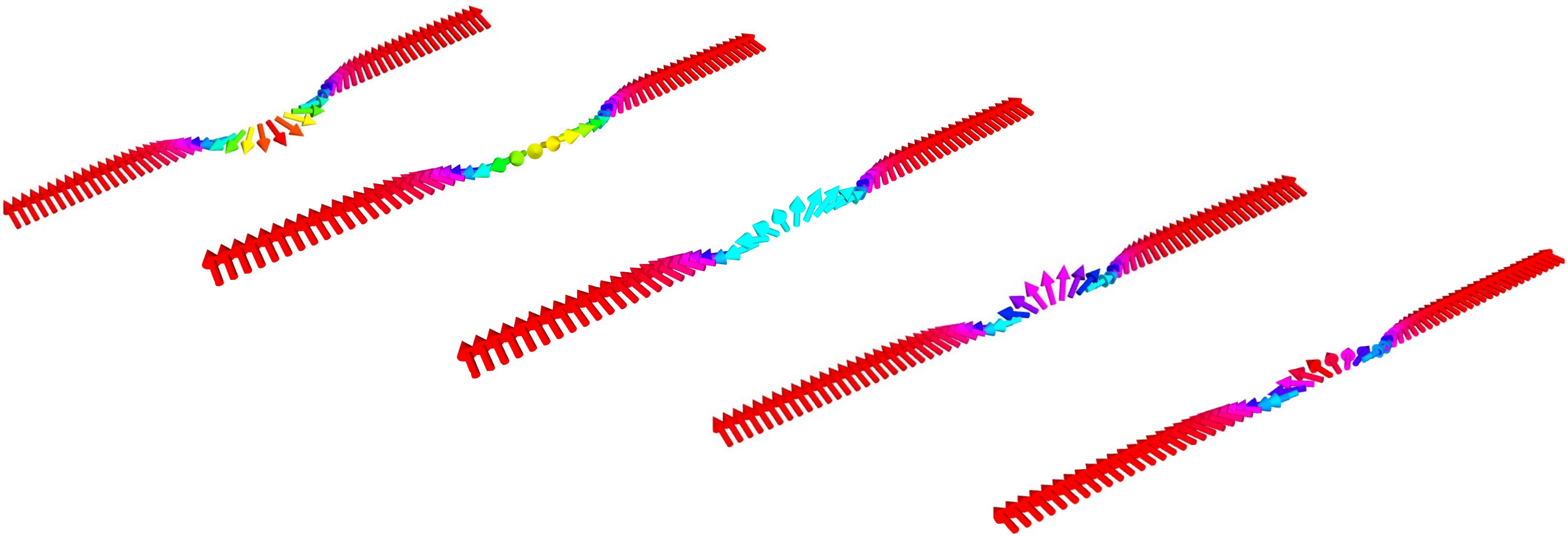
=> here topology is controlled by energy <=> infinite energy barrier

Is the 360-degree wall "topologically protected"?

but wait a minute! The profile is actually a map

$$\mathbf{m} : \mathbb{R} \to \mathbb{S}^2, \quad \mathbf{m} = (-\sin\theta, 0, \cos\theta)$$

this map ***can*** be continuously deformed into the ferromagnetic state:



this transformation passes over a **finite** energy barrier

What is the “topological degree”?

actually, there are several notions of topological degree (or topological charge)

- the one just introduced is an example of the *Kronecker index*:

$$\deg(\mathbf{m}) = \frac{1}{2\pi}\int_{\mathbb{R}} (m_1 m_3' - m_3 m_1')\, dx$$

- a deeper notion in topology is that of the *Brouwer degree*:

given $F : \Omega \to M$ differentiable, Ω, *M* compact oriented *n*-dim manifolds

then if $p \in M$ is a regular point of *F*, i.e., if $\det DF \neq 0 \quad \forall x \in F^{-1}(p)$

$$\deg(F, \Omega, p) = \sum_{x \in F^{-1}(p)} \mathrm{sign}(\det DF(x))$$

where *DF* is the Jacobi matrix of *F* in an oriented coordinate chart containing *x*

<u>Topological fact</u>: if *M* is connected, the Brouwer degree does not depend on *p.*

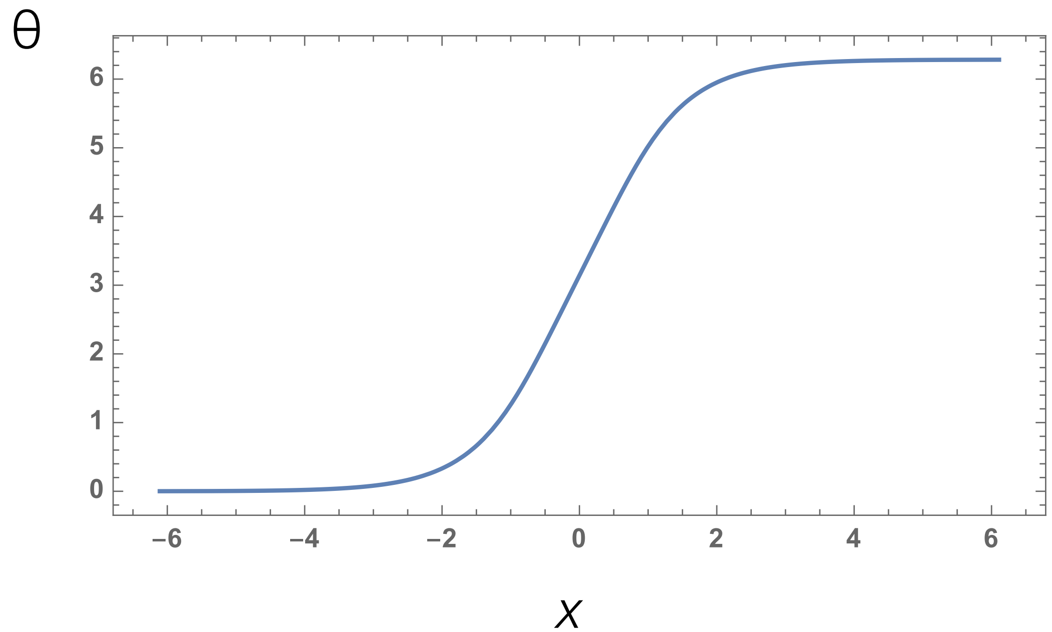
Brouwer degree is a unique homotopy invariant index (integer)

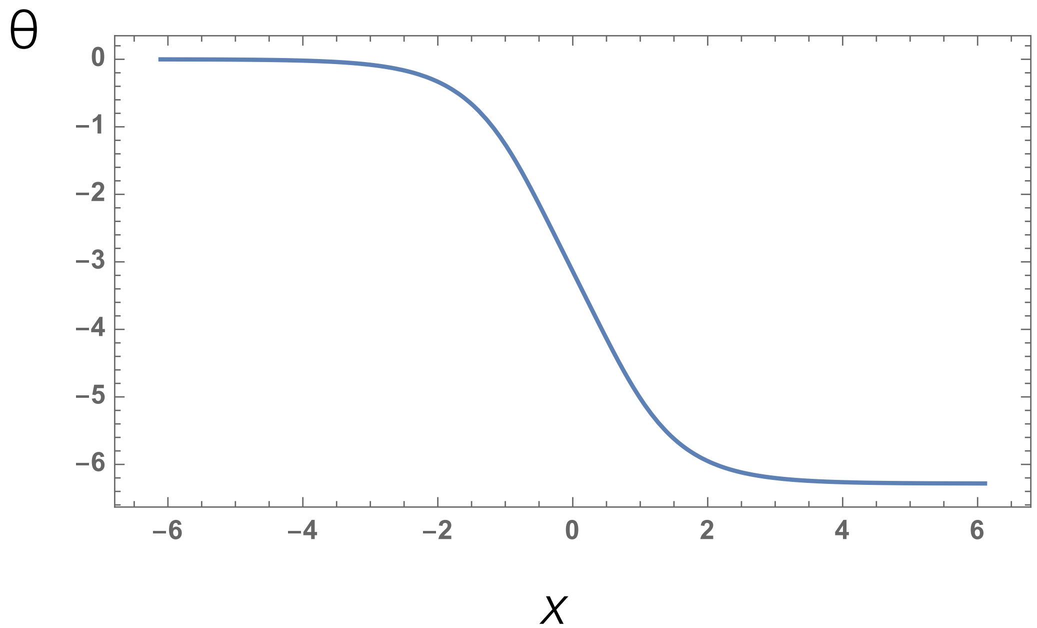
Example: one-to-one maps

if *F* is one-to-one, then it is either orientation preserving or orientation reversing

$$\deg(F, \Omega, p) = 1 \qquad \text{or} \qquad \deg(F, \Omega, p) = -1$$

<u>Example</u>: 360-degree Néel wall, $\Omega = \mathbb{R} \cup \{\infty\}, \quad M = \mathbb{S}^1$





$$\deg\left(\mathbf{m}, \mathbb{R} \cup \{\infty\}, p = \mathbf{m}(x)\right) = \operatorname{sign} \theta'(x) = \pm 1$$

Note: deg(**m**) ≠ 0 => the map **m** is <u>onto</u> <=> the image of **m** is all of *M*

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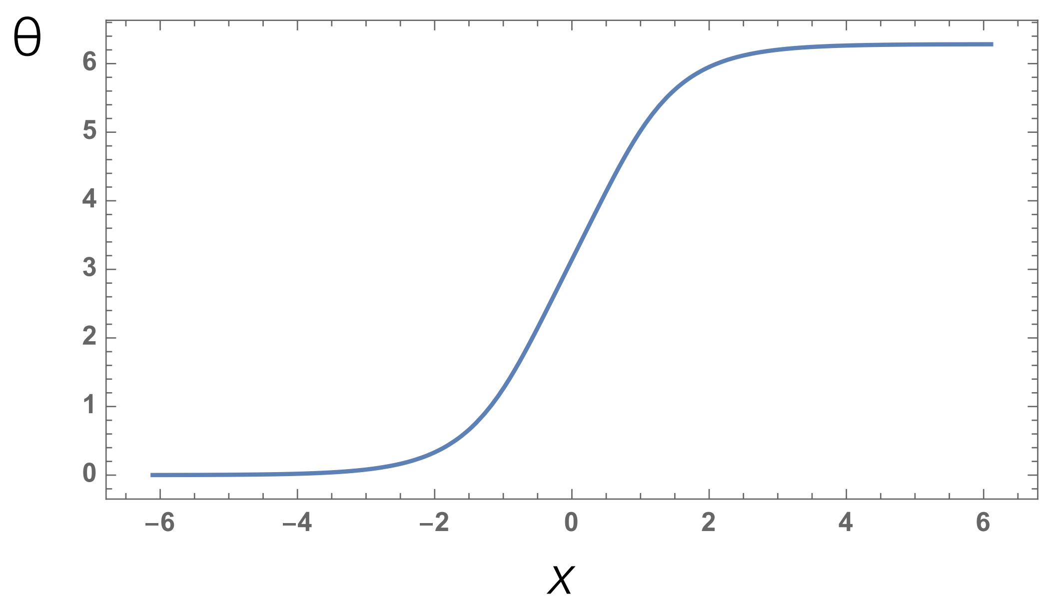
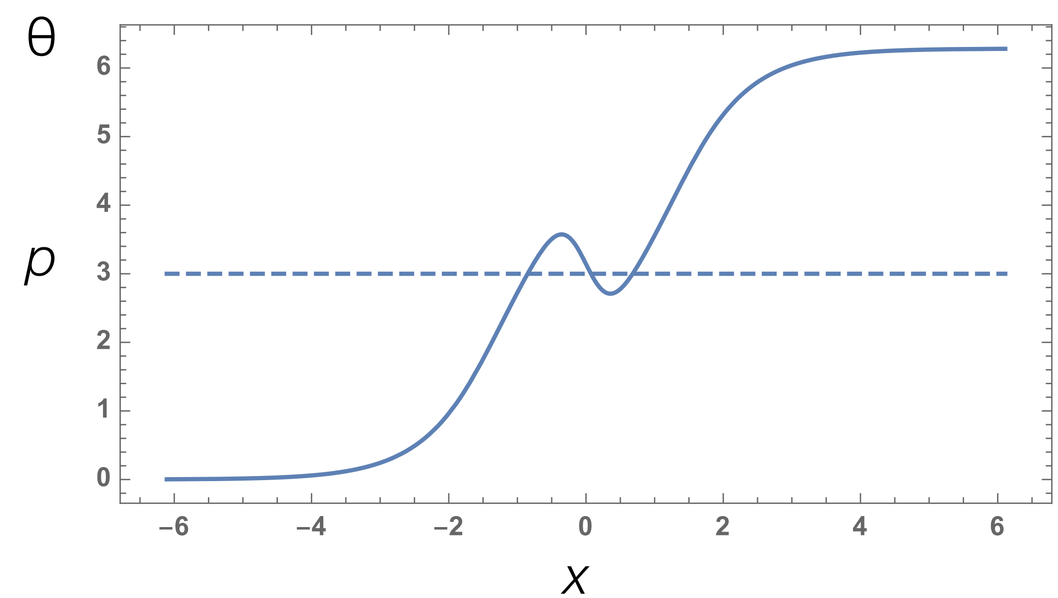
Brouwer degree for $\mathbf{m} : \mathbb{R} \cup \{\infty\} \to \mathbb{S}^1$

degree 1 maps:





$$\deg\left(\mathbf{m}, \mathbb{R} \cup \{\infty\}, p\right) = \sum_{x\,:\,\mathbf{m}(x)=p} \operatorname{sign} \theta'(x)$$

winding number: (by Area formula)

$$\deg(\mathbf{m}) = \frac{1}{2\pi} \int_{\mathbb{R}} \theta' \, dx = \frac{1}{2\pi} \int_0^{2\pi} \deg(\mathbf{m}, \mathbb{R} \cup \{\infty\}, p(\theta)) d\theta = 1$$

Warning: the degree of a map $\mathbf{m} : \mathbb{R} \to \mathbb{S}^2$ is ***not*** defined!

dimension mismatch

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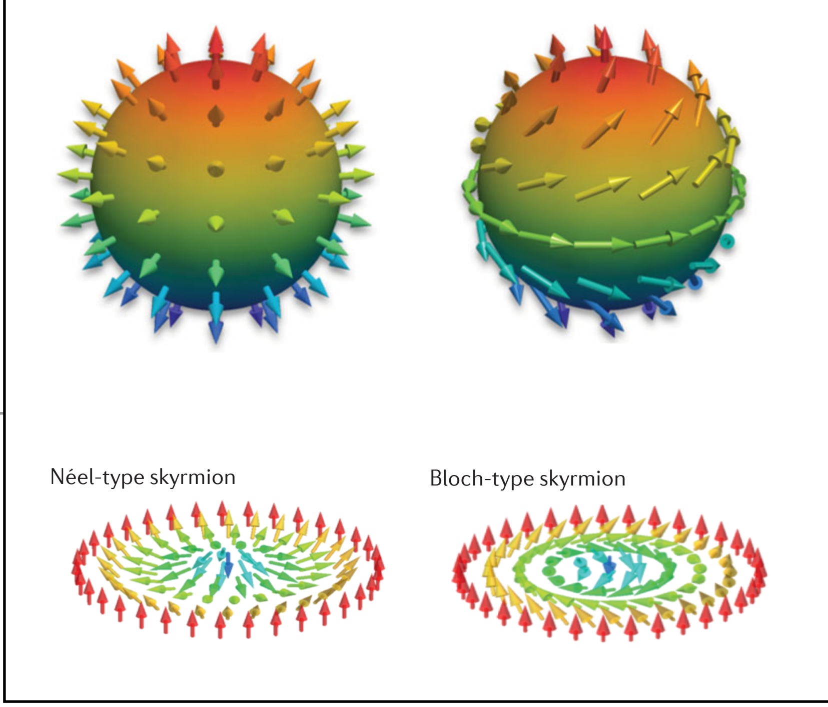
Magnetic skyrmions



T. Lancaster, *Contemp. Phys.* **60**, 246-261 (2019)

maps $\mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^2$ with non-trivial topology

<u>example</u>: harmonic maps

$$\boxed{E(\mathbf{m}) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r}$$

all minimizers with prescribed degree are known — Belavin and Polyakov, 1975

after stereographic projection, reduces to harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$

they are holomorphic or anti-holomorphic maps

Eells and Sampson, 1964
Lemaire, 1978
Wood, 1974
Brezis, Coron, 1985

specifically, all degree 1 minimizing maps are

dilations, rotations and translations of:

$$\boxed{\mathbf{m}(\mathbf{r}) = \left(-\frac{2\mathbf{r}}{1 + |\mathbf{r}|^2}, \frac{1 - |\mathbf{r}|^2}{1 + |\mathbf{r}|^2} \right)}$$

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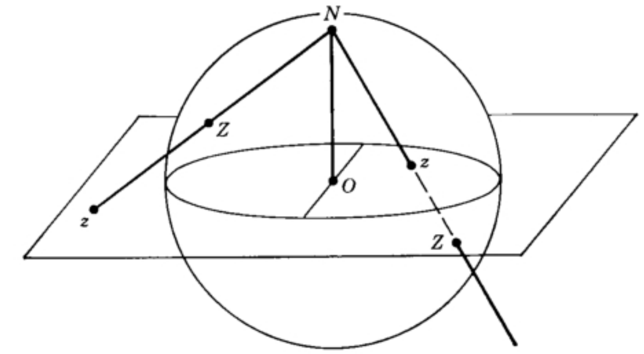
One-to-one maps from $\mathbb{R}^2 \cup \{\infty\}$ to $\mathbb{S}^2$



introduce a stereographic projection $\mathbf{m} \mapsto (u,v) \in \mathbb{R}^2$ $\qquad \mathbf{m}(\infty) = (0,0,-1)$

$$m_1 = -\frac{2u}{1+u^2+v^2} \quad m_2 = -\frac{2v}{1+u^2+v^2} \quad m_3 = \frac{1-u^2-v^2}{1+u^2+v^2}$$

defines one-to-one maps from $\mathbb{R}^2 \cup \{\infty\}$ to $\mathbb{R}^2 \cup \{\infty\}$: $\qquad (x,y) \mapsto (u,v) \in \mathbb{R}^2$

topological charge (Kronecker index): $\qquad$ Brouwer degree = ±1

$$\boxed{\mathcal{N}(\mathbf{m}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \mathbf{m}\cdot\left(\frac{\partial \mathbf{m}}{\partial x}\times\frac{\partial \mathbf{m}}{\partial y}\right) dx\,dy = \operatorname{sign}\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right)\int_{\mathbb{R}^2}\frac{du\,dv}{\pi(1+u^2+v^2)^2} = \pm 1}$$

more generally, an <u>integer</u>: $\quad$ defining $\quad \widetilde{\mathbf{m}}(x,y,z) = \mathbf{m}(x,y)e^{-z}$ $\quad$ we get

$$\mathcal{N}(\mathbf{m}) = -\frac{1}{4\pi}\int_0^\infty\int_{\mathbb{R}^2}\frac{\partial}{\partial z}\left(\widetilde{\mathbf{m}}\cdot\left[\frac{\partial\widetilde{\mathbf{m}}}{\partial x}\times\frac{\partial\widetilde{\mathbf{m}}}{\partial y}\right]\right)dx\,dy\,dz$$

$$= -\frac{3}{4\pi}\int_0^\infty\int_{\mathbb{R}^2}\det\left(\frac{\partial(\widetilde{m}_1,\widetilde{m}_2,\widetilde{m}_3)}{\partial(x,y,z)}\right)d^3r = \frac{3}{4\pi}\deg(\mathbf{m})\int_{B_1(0)} d^3\widetilde{m} = \deg(\mathbf{m})$$

Belavin-Polyakov profiles

$$\mathbf{m}(\mathbf{r}) = \left(-\frac{2\mathbf{r}}{1+|\mathbf{r}|^2}, \frac{1-|\mathbf{r}|^2}{1+|\mathbf{r}|^2}\right)$$

orientation preserving and minimize the exchange energy:

identity map
$(u,v)=(x,y)$
equality

$$\int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r = 8\pi + 4\int_{\mathbb{R}^2} \frac{(\partial_x u - \partial_y v)^2 + (\partial_y u + \partial_x v)^2}{(1+u^2+v^2)^2} d^2 r \geq 8\pi$$

equality achieved => every minimizer satisfies the Cauchy-Riemann equations

$$w = u + iv \qquad z = x + iy \quad \Rightarrow \quad w = f(z) \text{ analytic}$$

f(*z*) is smooth and goes to infinity at infinity => it is a polynomial

f(*z*) is one-to-one => $f(z) = \rho e^{i\theta}(z - z_0)$

Belavin and Polyakov, 1975

=> all sol's are translations, dilations and in-plane rotations of the BP profile

<u>topologically protected</u>? ***yes***: impossible to deform continuously to $\mathbf{m}(\infty)$

<u>energy barrier</u>? ***no***: the minimum is degenerate

can be deformed to $\mathbf{m}(\infty)$ <u>almost everywhere</u> without paying any energy!

topological collapse

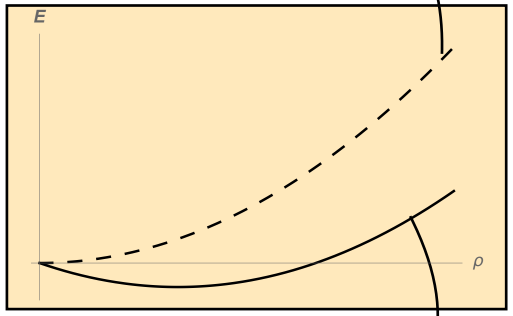
Skyrmions as strict degree 1 energy minimizers

exchange + anisotropy (and/or Zeeman): let $\mathbf{m}_\rho(\mathbf{r}) = \mathbf{m}_0(\rho^{-1}\mathbf{r})$

$$E(\mathbf{m}_\rho) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}_0|^2 d^2r + (Q-1)\rho^2 \int_{\mathbb{R}^2} |\mathbf{m}_{0,\perp}|^2 d^2r + 2h\rho^2 \int_{\mathbb{R}^2} (1 - m_{0,\parallel}) d^2r$$

achieves minimum when $\rho \to 0$ => no minimizer

goes back to Derrick, 1964; Pokhozhaev, 1965; Berestycki and Lions, 1983



exchange + anisotropy/Zeeman + DMI:

$$E(\mathbf{m}_\rho) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}_0|^2 d^2r - 2\kappa\rho \int_{\mathbb{R}^2} \mathbf{m}_{0,\perp} \cdot \nabla m_{0,\parallel} d^2r + 2h\rho^2 \int_{\mathbb{R}^2} (1 - m_{0,\parallel}) d^2r$$

choosing $\mathbf{m}_0$ as the Belavin-Polyakov profile, we find $\min E(\mathbf{m}) < 8\pi$

=> suggests existence of a global energy minimizer with non-trivial degree

Bogdanov and Yablonskii, 1989; Bogdanov, Kudinov and Yablonskii, 1989; Ivanov et al., 1990

rigorous proof for sufficiently small κ yields existence of a global energy minimizer with degree 1, finite energy barrier

Melcher, 2014; Li and Melcher, 2018

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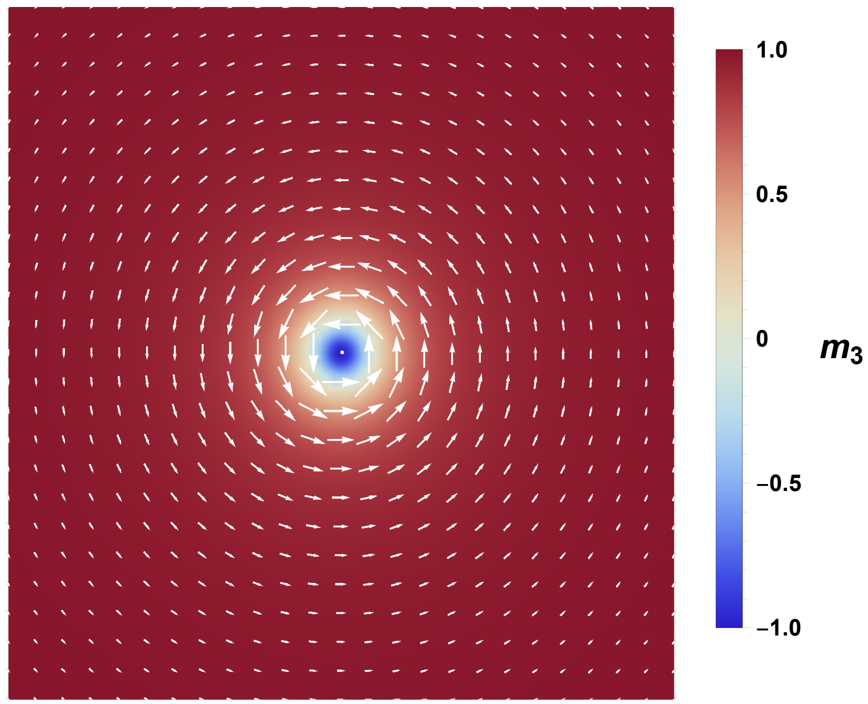
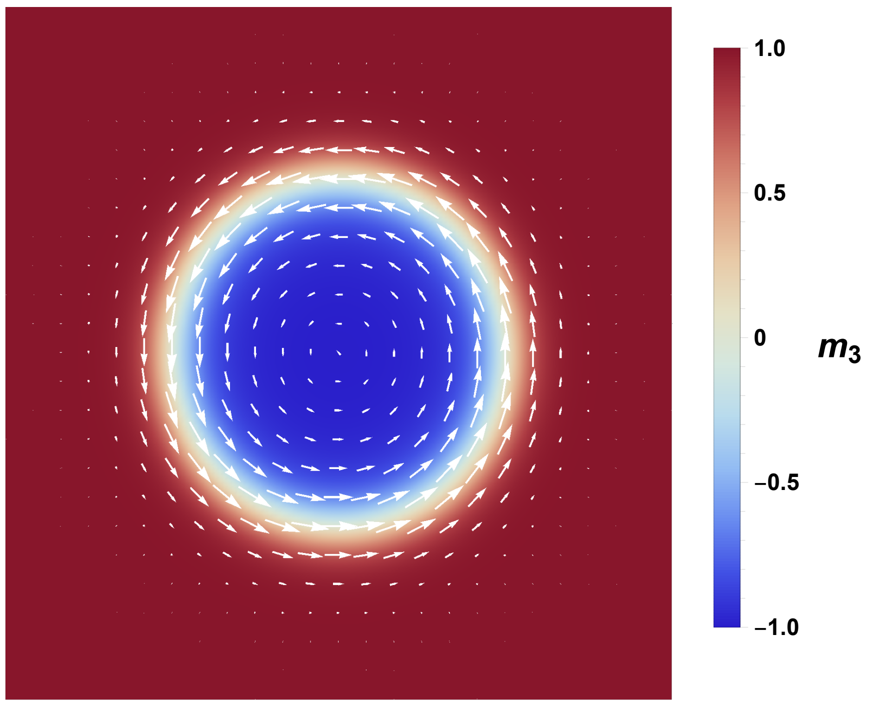
$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \{|\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel\} \, d^2 r + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2 r \, d^2 r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2 r \, d^2 r'$$

Admissible class?

compact skyrmion vs. skyrmionic bubble





for bubble skyrmion, the stray field energy *diverges* with radius:

$$E_s(\mathbf{m}_R) \sim -R \ln R$$

M, Simon, 2019

hence $\mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^2, \ \nabla \mathbf{m} \in L^2, \ \mathbf{m}_\perp \in L^2 \quad \not\Rightarrow \quad E(\mathbf{m}) > -\infty$

no hope to construct solutions as absolute minimizers with prescribed degree

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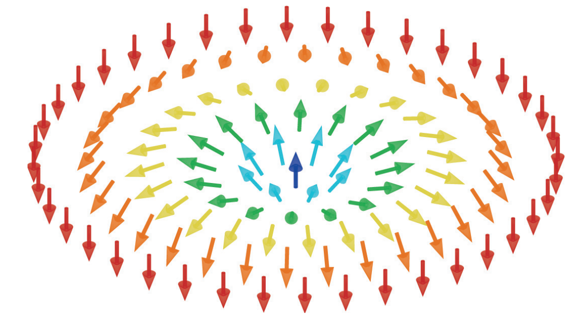
Compact skyrmions as local minimizers



introduce:

$$\boxed{\mathcal{N}(\mathbf{m}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \mathbf{m}\cdot(\partial_1\mathbf{m}\times\partial_2\mathbf{m})\, d^2r}$$

$$\mathcal{A} := \left\{ \mathbf{m} \in \mathring{H}^1(\mathbb{R}^2;\mathbb{S}^2) : \mathcal{N}(\mathbf{m}) = 1,\ \mathbf{m}+\mathbf{e}_3 \in L^2(\mathbb{R}^2),\ \int_{\mathbb{R}^2} |\nabla\mathbf{m}|^2 d^2r < 16\pi \right\}$$

why 16π? Topological lower bound: $\mathbf{m} \in \mathring{H}^1(\mathbb{R}^2;\mathbb{S}^2)$

$$\boxed{\int_{\mathbb{R}^2} |\nabla\mathbf{m}|^2 d^2r \geq 8\pi\, |\mathcal{N}(\mathbf{m})|}$$

$$|\nabla\mathbf{m}|^2 \pm 2\mathbf{m}\cdot(\partial_1\mathbf{m}\times\partial_2\mathbf{m}) = |\partial_1\mathbf{m}\mp\mathbf{m}\times\partial_2\mathbf{m}|^2$$

allows to exclude splitting in the concentration compactness arguments

Theorem 1. *Let $Q > 1$, $\delta > 0$ and $\kappa \in \mathbb{R}$ be such that $(2|\kappa|+\delta)^2 < 2(Q-1)$. Then there exists $\mathbf{m} \in \mathcal{A}$ such that*

$$E(\mathbf{m}) = \inf_{\widetilde{\mathbf{m}}\in\mathcal{A}} E(\widetilde{\mathbf{m}}).$$

Bernand-Mantel, M and Simon, 2020

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complete asymptotic description in the *conformal limit*

Film of finite thickness — skyrmion tubes?

how to define a skyrmion solution for $\Omega = \mathbb{R}^2 \times [0,\delta] \subset \mathbb{R}^3$?

<u>dimension mismatch!</u> $\mathbf{m} : \mathbb{R}^2 \times [0,\delta] \to \mathbb{S}^2$

in the absence of stray field effects the non-dimensionalized energy is

$$\boxed{\begin{aligned} E(\mathbf{m}) = \int_0^\delta \int_{\mathbb{R}^2} |\nabla \mathbf{m}(x,y,z)|^2 dx\,dy\,dz + \beta \int_0^\delta \int_{\mathbb{R}^2} (1 + m_\parallel(x,y,z))\,dx\,dy\,dz \\ + \alpha \int_{\mathbb{R}^2} |\mathbf{m}_\perp(x,y,0)|^2 dx\,dy - 2\lambda \int_{\mathbb{R}^2} \mathbf{m}_\perp(x,y,0) \cdot \nabla m_\parallel(x,y,0)\,dx\,dy \end{aligned}}$$

can define the degree on slices: $d(z) = \deg(\mathbf{m}(\cdot, z), \mathbb{R}^2)$ $\quad \mathbf{m}(\infty) = (0,0,-1)$

if $d(z) = 1$ and $\alpha, \beta, \lambda = 0$, then the minimizers are the BP profiles indep. of z

what if only $d(0) = 1$ is forced? **No existence!**

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$$\widehat{\mathbf{m}}_{\mathbf{k}}(z) = \int_{\mathbb{R}^2} e^{-i\mathbf{k}\cdot\mathbf{r}} \mathbf{m}(\mathbf{r}, z)\, d^2 r$$

Non-existence of minimizers

- consider test configurations $\mathbf{m}_R$:

$$E(\mathbf{m}_R) = C_1 R + C_2 \beta R^3 + C_3 \alpha R^2 - C_4 \lambda R$$

hence

$$E(\mathbf{m}_R) \to 0 \quad \text{as} \qquad R \to 0$$

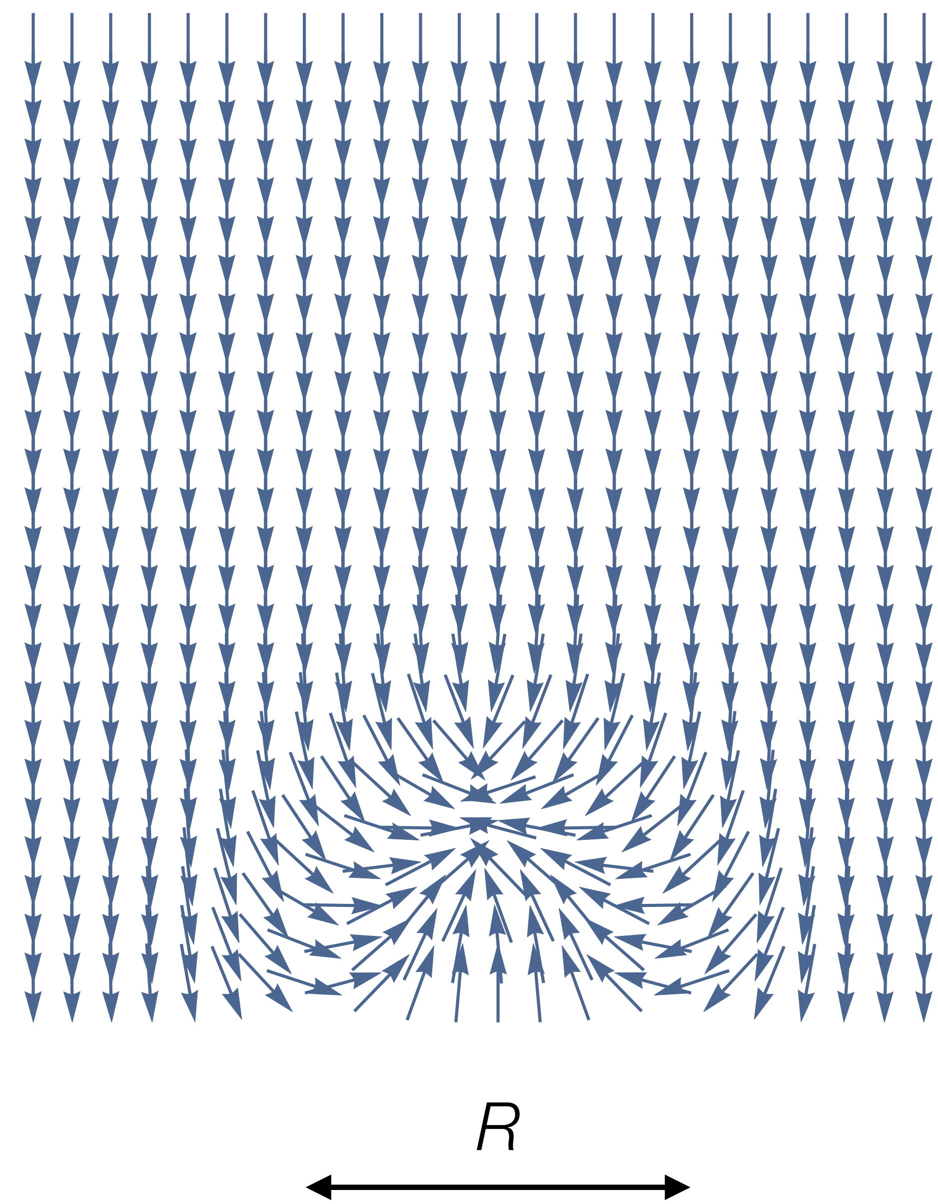
- but dropping |**m**| = 1 for z > 0 we have

$$\boxed{E(\mathbf{m}) \geq \int_{\mathbb{R}^2} |\mathbf{k}| \{ \tanh(\delta |\mathbf{k}|) - |\lambda| \} |\widehat{\mathbf{m}}_{\mathbf{k}}(0)|^2 \frac{d^2 k}{(2\pi)^2} \geq 0}$$

whenever |λ| ≤ 1 (<u>Note</u>: otherwise ill-posed)

=> the minimum is not attained!

$\mathbf{m}_R$



new definition of a skyrmion is needed

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