On an Isoperimetric Problem with a Competing Nonlocal Term II: The General Case

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Abstract

This paper is the continuation of a previous paper (H. Knüpfer and C. B. Muratov, *Comm. Pure Appl. Math.* **66** (2013), 1129–1162). We investigate the classical isoperimetric problem modified by an addition of a nonlocal repulsive term generated by a kernel given by an inverse power of the distance. In this work, we treat the case of a general space dimension. We obtain basic existence results for minimizers with sufficiently small masses. For certain ranges of the exponent in the kernel, we also obtain nonexistence results for sufficiently large masses, as well as a characterization of minimizers as balls for sufficiently small masses and low spatial dimensionality. The physically important special case of three space dimensions and Coulombic repulsion is included in all the results mentioned above. In particular, our work yields a negative answer to the question if stable atomic nuclei at arbitrarily high atomic numbers can exist in the framework of the classical liquid drop model of nuclear matter. In all cases the minimal energy scales linearly with mass for large masses, even if the infimum of energy cannot be attained. © 2014 Wiley Periodicals, Inc.

1 Introduction

This paper is the second part of [22], in which a nonlocal modification of the classical isoperimetric problem was considered. Namely, we wish to examine minimizers of the energy functional

$$E(u) = \int_{\mathbb{R}^n} |\nabla u| \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x)u(y)}{|x-y|^\alpha} \, dx \, dy, \tag{1.1}$$

in which $u \in BV(\mathbb{R}^n; \{0,1\})$, $n \geq 2$, and $\alpha \in (0,n)$. We assume that the mass associated with u is prescribed, i.e.,

$$\int_{\mathbb{R}^n} u \, dx = m \tag{1.2}$$

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for some $m \in (0, \infty)$. Note that the considered range of values of $\alpha \in (0, n)$ ensures that the nonlocal part of the energy in (1.1) is always well-defined.

The above problem arises in a number of physical contexts [27]. A case that is of particular physical importance is where $n = 3$ and $\alpha = 1$, corresponding to *Coulombic repulsion* (for an overview, see [28]). Perhaps the earliest example where the model in (1.1) and (1.2) appears is the liquid drop model of the atomic nuclei proposed by Gamow in 1928 [20] and then developed by von Weizsäcker [37], Bohr [4, 5], and many other researchers later on. This model was used to explain various properties of nuclear matter and, in particular, the mechanism of nuclear fission (for more recent studies, see [11, 12, 29, 33]). Due to the fundamental nature of Coulombic interaction, the same model (or its diffuse interface analogue) also arises in many other physical situations (see, e.g., [6, 7, 15, 30]) and, in particular, is relevant to a variety of polymer as well as other systems (see, e.g., [13, 21, 23, 26, 31, 32, 35]).

It is well-known that the local part of the energy in (1.1), which leads to the classical isoperimetric problem, is uniquely minimized by balls among all sets of finite perimeter with prescribed mass [14]. The key ingredient in the proof of this celebrated result by De Giorgi is the use of Steiner symmetrization, which lowers the interfacial energy. The effect of the rearrangement in the Steiner symmetrization, however, is quite different for the nonlocal part of the energy in (1.1). Since by the rearrangement the mass is transported closer together, the resulting nonlocal energy actually increases. It is this competition of the cohesive forces due to surface tension and the repulsive long-range forces that makes this variational problem highly nontrivial. In particular, minimizers are no longer expected to be convex or even exist at all for certain ranges of the parameters. To take the particular case of the nuclear drop model, we are not aware of any prior studies establishing existence or nonexistence of minimizers for large masses that would not assume spherical symmetry of the drop. An *ansatz-free* answer to this question, however, is essential in order to develop a basic understanding of the properties of atoms.

In our previous work [22], we have investigated the two-dimensional version of the variational problem associated with (1.1). We investigated existence, nonexistence, and shape of minimizers in the full range of $\alpha \in (0, 2)$ for $n = 2$ and provided a complete characterization of the minimizers for sufficiently small α. The special geometric structure of $\mathbb{R}^2$ simplifies many of the arguments used in that work. In the present work, we extend these results to the general case of $n \geq 2$ space dimensions. We also investigate existence, shape, and regularity of minimizers for prescribed mass $m \in (0, \infty)$. Due to the technical difficulties associated with the transition from the $n = 2$ case to $n \geq 3$, however, we are able to treat in a similar way only certain ranges of α and n. In particular, we need to work within the general framework of sets of finite perimeter.

As in [22], we are able to establish existence of minimizers for sufficiently small masses for all $n \geq 3$ and the full range of $\alpha \in (0, n)$. At the same time, we are only able to prove that balls are the unique minimizers (up to translations) for sufficiently small masses when $n \leq 7$ and $\alpha \in (0, n-1)$. Note that the physically relevant special case of $n = 3$ and $\alpha = 1$ is included. The first restriction, $n \leq 7$, seems to be of a technical nature: In fact, for our arguments we use the result that for $n \leq 7$ quasiminimizers of the perimeter have smooth boundaries (see, e.g., [3,36]). Perhaps one could remove this restriction by using more sophisticated machinery of the regularity theory for quasiminimizers, e.g., following the ideas of the recent work by Figalli and Maggi [17].

The second restriction, $\alpha \in (0, n-1)$, however, is of a more fundamental nature and distinguishes the case of the far-field-dominated regime $\alpha < n-1$ from the near-field-dominated regime $\alpha \geq n-1$ (cf. also with [22]). In the latter case the potential associated with the minimizer is no longer Lipschitz-continuous (cf. (2.8), (4.15)). Hence a different approach is needed in this case to deal with the shape of minimizers; this is necessary even in the regime, in which the perimeter term dominates the nonlocal term.

Similarly, we are only able to prove nonexistence of minimizers for large masses in the case $\alpha < 2$. Observe that, once again, our result covers the physically most relevant case of Coulombic interaction, i.e., $n = 3$ and $\alpha = 1$. We note that for the latter case a nonexistence proof was also very recently obtained by Lu and Otto in their study of the Thomas–Fermi–Dirac–von Weizsäcker model of quantum electron gas, using different arguments [25]. From the point of view of applications, our result provides a basic nonexistence result for uniformly charged drops with sufficiently large masses minimizing the energy in (1.1) and, in particular, for ground states of atomic nuclei with sufficiently large atomic numbers within the charged-drop model of nuclear matter. Our analysis also partially substantiates the picture described in [8, 9] for the Coulombic case in three space dimensions. The nonexistence proof fails in the opposite case of $\alpha \geq 2$, and we do not know whether the result still holds for some $\alpha \in [2, n)$, when the potential has shorter range. What we did show is that independently of α the minimal energy always scales linearly with mass for large masses. Note that this result is consistent both with a minimizing sequence consisting of many isolated balls moving away from each other and with a minimizer in the form of a long "sausage-shaped" drop. Which of these two alternatives occurs for the large mass case and $\alpha \geq 2$ remains to be studied.

Our paper is organized as follows: In Section 2, we introduce the basic notions of geometric measure theory. Here we also reformulate our variational problem in the framework of sets of finite perimeter, provide some technical results that will be used in the analysis, and describe all the notation. In Section 3 we state the main results of the paper and outline the key ideas of their proofs. In Section 4 we prove several technical lemmas that are used throughout the rest of the paper. In Section 5 we establish existence of minimizers (Theorem 3.1) for small masses. In Section 6 we establish the precise shape of the minimizers for small masses in a certain range of the parameters (Theorem 3.2). Finally, in Section 7 we establish nonexistence of minimizers for large masses in a certain range of the parameters

(Theorem 3.3) and the scaling and equipartition of energy for large masses in the whole range of the parameters (Theorem 3.4).

2 Notation and Sets of Finite Perimeter

The variational problem (1.1)–(1.2) is most conveniently addressed in the setting of geometric measure theory. In this section we first introduce some basic measure-theoretic notions; here we refer to [1,2] as references. We then reformulate (1.1) in terms of sets of finite perimeter. We conclude the section by recalling some basic regularity results for minimizers.

SOME MEASURE-THEORETIC NOTIONS: We say that a function $u \in L^1(\mathbb{R}^n)$ has bounded variation, $u \in BV(\mathbb{R}^n)$, if

$$\int_{\mathbb{R}^n} |\nabla u| dx := \sup_{\|\zeta\|_\infty \le 1} \left\{ \int_{\mathbb{R}^n} u \nabla \cdot \zeta \, dx : \zeta \in C_c^1(\mathbb{R}^n; \mathbb{R}^n) \right\} < \infty. \tag{2.1}$$

For any measurable set $F \subset \mathbb{R}^n$, we denote by $|F|$ its n-dimensional Lebesgue measure. Moreover, F is said to have finite parameter if $\chi_F \in BV(\mathbb{R}^n)$, where χ_F is the characteristic function of F; its perimeter is then defined by $P(F) := \int_{\mathbb{R}^n} |\nabla \chi_F| dx$. The k-dimensional Hausdorff measure with $k \in [0, n]$ is denoted by $\mathcal{H}^k(F)$. We will frequently use Fubini's theorem for measures [2, theorem 1.74] as well as the co-area formula for integration in spherical coordinates [16, sec. 3.4.4, prop. 1].

For any Lebesgue-measurable set F, its upper density at a point $x \in \mathbb{R}^n$ is

$$\overline{D}(F, x) \ := \ \limsup_{r \to 0} \frac{|F \cap B_r(x)|}{|B_r(x)|}, \tag{2.2}$$

where $B_r(x)$ is the open ball with center x and radius r. The *essential interior* $\mathring{F}^M$ of F is then defined as the set of all $x \in \mathbb{R}^n$ for which $\overline{D}(F, x) = 1$, while the *essential closure* $\bar{F}^M$ of F is defined as the set of all $x \in \mathbb{R}^n$ for which $\overline{D}(F, x) > 0$. The *essential boundary* $\partial^M F$ of F is defined as the set of all points where $\overline{D}(F, x) > 0$ and $\overline{D}(\mathbb{R}^n \backslash F, x) > 0$. By a result of Federer, a set has finite perimeter if and only if $\mathcal{H}^{n-1}(\partial^M F) < \infty$. The *reduced boundary* $\partial^* F$ of a set of finite perimeter F is defined as a set of all points $x \in \partial^M F$ such that the measure-theoretic normal exists at x, i.e., if the following limit exists:

$$\nu_F(x) \ := \ \lim_{r \to 0} \frac{\int_{B_r(x)} \nabla \chi_F(y) dy}{\int_{B_r(x)} |\nabla \chi_F(y)| dy} \quad \text{and} \quad |\nu_F(x)| = 1, \tag{2.3}$$

where $\nabla \chi_F$ denotes the vector-valued Radon measure associated with the distributional derivative of χ_F and $|\nabla \chi_F|$ coincides with the $\mathcal{H}^{n-1}$ measure restricted to $\partial^M F$. Again, by a result of Federer we have $\mathcal{H}^{n-1}(\partial^M F \backslash \partial^* F) = 0$ [2, theorem 3.61].

Note that the topological notion of connectedness is not well-defined for sets of finite perimeter, since these sets are only defined up to $\mathcal{H}^n$-negligible sets. However, the following generalization of the notion of connectedness can be defined: We say that a set F of finite perimeter is *decomposable* if there exists a partition (A, B) of F such that $P(F) = P(A) + P(B)$ for two sets A, B with positive Lebesgue measure. Otherwise the set is called *indecomposable*, which is the measure-theoretic equivalent of the notion of a connected set. Similarly, we say that a measurable set F is *essentially bounded* if its essential closure $\bar{F}^M$ is bounded.

NOTATION FOR THE ISOPERIMETRIC PROBLEM: The isoperimetric deficit of a set of finite perimeter $F \subset \mathbb{R}^n$ is defined in this paper by

$$D(F) := \frac{P(F)}{n\omega_n^{1/n}|F|^{(n-1)/n}} - 1, \tag{2.4}$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ denotes the measure of the unit ball in $\mathbb{R}^n$.

A natural notion of the difference of two measurable sets F and G with $|F| = |G|$ is the Fraenkel asymmetry:

$$\Delta(F, G) := \min_{x \in \mathbb{R}^n} \frac{|F \triangle (G + x)|}{|F|}, \tag{2.5}$$

where $F \triangle G := (F \setminus G) \cup (G \setminus F)$ denotes the symmetric difference of the sets F and G. The following quantitative version of the isoperimetric inequality relating the Fraenkel asymmetry (2.5) and the isoperimetric deficit (2.4) was recently established [19]:

$$\Delta(F, B) \le C_n \sqrt{D(F)}, \tag{2.6}$$

where B is a ball with $|B| = |F|$ and C_n is a positive constant that depends only on the dimension n.

Another important notion is the notion of *quasiminimizer of the perimeter* (see, e.g., [3,36]). A set F of finite perimeter is called a quasiminimizer of the perimeter (with prescribed mass) if there exists a constant $C > 0$ such that for all $G \subset \mathbb{R}^n$ with $|G| = |F|$ and $F \triangle G \subset B_r(0)$ for some $r > 0$, one has

$$P(F) \le P(G) + C|F \triangle G|. \tag{2.7}$$

As will be shown below (see Proposition 2.1), minimizers of our variational problem are quasiminimizers of the perimeter in the above sense, and therefore the regularity results of [34, 38] apply to them.

THE VARIATIONAL MODEL: We express (1.1) as a functional on sets of finite perimeter. For any measurable set $F \subset \mathbb{R}^n$, let the potential v_F be given by

$$v_F(x) := \int_F \frac{1}{|x - y|^\alpha} \, dx. \tag{2.8}$$

The nonlocal part of the energy in (1.1) can then be expressed as

$$V(F) := \int_F v_F \, dx = \int_F \int_F \frac{1}{|x-y|^\alpha} \, dx \, dy. \tag{2.9}$$

The energy (1.1) can hence be expressed as

$$E(F) = P(F) + V(F). \tag{2.10}$$

We say that Ω is a minimizer of (2.10) if $E(\Omega) \leq E(F)$ for all sets of finite parameter F with $|F| = |\Omega|$. In the following, we will reserve the symbol Ω to denote minimizers. It can be shown that minimizers of the energy are solutions of an Euler-Lagrange equation in a suitable sense. In this paper, however, we will not use the Euler-Lagrange equation of this variational problem, but instead we will use only the energy to obtain the necessary estimates. In this sense, the methods used in this work are more general than the approach in our related work for the two-dimensional case, where we used the Euler-Lagrange equation [22].

We have the following general result concerning the regularity of minimizers of the considered variational problem:

PROPOSITION 2.1. *Let Ω be a minimizer for* (2.10). *Then the reduced boundary $\partial^*\Omega$ of Ω is a $C^{1,1/2}$ manifold. In addition, $\mathcal{H}^k(\partial^M\Omega \backslash \partial^*\Omega) = 0$ for all $k > n-8$. In particular, for $n \leq 7$ the set Ω is* (*up to a negligible set*) *open with boundary of class $C^{1,1/2}$. The complement of Ω has finitely many connected components.*

PROOF. The proof is an adaptation of the results of [34,38]. In [34] Rigot established the regularity of a class of quasiminimizers of the perimeter with prescribed mass, which includes our notion (2.7) of quasiminimizers. It is hence enough to show that every minimizer Ω of (2.10) is also a quasiminimizer in the sense of definition (2.7). The statement of the theorem then follows from [34, theorem 1.4.9] and [38, theorem 4.5]. Let Ω be a minimizer of (2.10) with prescribed mass, and let F be a set of finite perimeter with $|F| = |\Omega|$ and $F \triangle \Omega \subset B_r(0)$ for some $r > 0$. By the minimizing property of Ω, we have

$$\begin{aligned} P(\Omega) - P(F) \leq V(F) - V(\Omega) \overset{(2.8)}{\leq} \int_{\Omega \triangle F} (v_\Omega + v_F) \, dx \leq \\ 2|\Omega \triangle F| \left(\int_{B_1(0)} \frac{1}{|y|^\alpha} \, dy + m \right) \leq C|\Omega \triangle F|, \end{aligned} \tag{2.11}$$

for some $C > 0$ depending only on n, α, and m. It follows that the minimizers of (2.10) are also quasiminimizers of the perimeter. $\square$

OTHER NOTATIONS: Unless otherwise noted, all constants throughout the proofs are assumed to depend only on n and α. The symbol e_k is reserved for the unit vector in the k^{th} coordinate direction.

3 Main Results

In our first result we show that for sufficiently small masses there exists a minimizer of the considered variational problem.

THEOREM 3.1 (Existence of Minimizers). *For all $n \geq 3$ and for all $\alpha \in (0,n)$ there is a mass $m_1 = m_1(\alpha,n) > 0$ such that for all $m \leq m_1$, the energy in* (2.10) *has a minimizer $\Omega \subset \mathbb{R}^n$ with $|\Omega| = m$. The minimizer Ω is essentially bounded and indecomposable.*

The proof of this theorem follows by the direct method of calculus of variations once we show that for sufficiently small mass every minimizing sequence of the energy may be replaced by another minimizing sequence where all sets have uniformly bounded essential diameter. By the regularity result in Proposition 2.1, we also obtain certain regularity of the minimizer's boundary. We note that our result improves an existence result of [8, 9] for the Coulombic case $n = 3$ and $\alpha = 1$, demonstrating that there exists an *interval* of masses near the origin for which the minimizers indeed exist.

By analogy with the two-dimensional case, it is natural to expect that if the mass is sufficiently small, the minimizer of the considered variational problem is precisely a ball [22]. Our next result shows that this is indeed the case at least in a certain range of values for α.

THEOREM 3.2 (Ball Is the Minimizer). *For all $3 \leq n \leq 7$ and for all $\alpha \in (0, n-1)$ there is a mass $m_0 = m_0(\alpha,n) > 0$ such that for all $m \leq m_0$, the unique* (*up to translation*) *minimizer $\Omega \subset \mathbb{R}^n$ of* (1.1) *with $|\Omega| = m$ is given by a ball.*

For the proof of Theorem 3.2, which applies to the regime where the perimeter is the dominant term in the energy, we make use both of the quantitative isoperimetric inequality in (2.6) and regularity estimates for the minimizer as given by Rigot [34]. We first show that the minimizer is close to a ball in the C^1-sense (cf. with [18]). By the Lipschitz continuity of the nonlocal potential, we then deduce that the minimum energy can in fact only be achieved by precisely a ball. Let us note that the assumption $n \leq 7$ in Theorem 3.2 seems to be of only a technical nature and it may be possible to adapt recent results of [17] to extend the statement of Theorem 3.2 to all $n \geq 8$ and $\alpha < n - 1$. However, our method of proof for the $\alpha < n-1$ case does not extend straightforwardly to the case of $\alpha \geq n-1$. Indeed, the technical difficulties encountered in the latter case become rather substantial even for $n = 2$ [22]. Let us mention that some related recent results for the n-dimensional Coulombic case on bounded domains were obtained in [10].

On the contrary, for large masses the repulsive interaction dominates and the variational problem does not admit a minimizer:

THEOREM 3.3 (Nonexistence of Minimizers). *For all $n \geq 3$ and for all $\alpha \in (0, 2)$, there is $m_2 = m_2(\alpha,n)$ such that for all $m \geq m_2$, the energy in* (1.1) *does not admit a minimizer $\Omega \subset \mathbb{R}^n$ with $|\Omega| = m$.*

For the proof, we first show that minimizers must be indecomposable. On the other hand, we can cut any indecomposable set with sufficiently large mass by a hyperplane into two large pieces. We move the two pieces far apart from each other and compare the energy of the new set with the original configuration. The resulting inequality can be expressed as a differential inequality for the mass of cross-sections of the minimizer by different hyperplanes. The proof is concluded by a contradiction argument, whereby the resulting estimate of the total mass is too high.

Our result is restricted to the case $\alpha \in (0,2)$. Note that the above result does, in particular, apply to the physically important special case of Coulomb interaction, i.e., $n = 3$ and $\alpha = 1$ (see also [25]). It is an interesting open question if the nonexistence result extends to arbitrary $\alpha \in [2,n)$. In particular, for $\alpha \to n$ the nonlocal energy is dominated by short-range interactions.

THEOREM 3.4 (Scaling and Equipartition of Energy). *For all $n \geq 3$ and for all $\alpha \in (0,n)$ there exist two constants $C, c > 0$ depending only on n and α such that for the energy in* (1.1) *we have*

$$c \max\{m^{\frac{n-1}{n}}, m\} \leq \inf_{|\Omega|=m} E(\Omega) \leq C \max\{m^{\frac{n-1}{n}}, m\}. \tag{3.1}$$

Furthermore, for $m \geq 1$ we have an equipartition of energy in the sense that for every set of finite perimeter $\Omega \subset \mathbb{R}^n$ satisfying $|\Omega| = m$ and $E(\Omega) \leq \beta m$ with some $\beta > 0$ we have

$$c_\beta m \leq \min\{P(\Omega), V(\Omega)\} \leq \max\{P(\Omega), V(\Omega)\} \leq \beta m \tag{3.2}$$

for some $c_\beta > 0$ depending only on α, n, and β, but not on m.

At the core of the proof of this theorem is the proof of the lower bound of the energy. This estimate follows from an interpolation inequality that connects interfacial and nonlocal parts of the energy.

4 Some Basic Estimates

We start our analysis with a few auxiliary lemmas that will be useful in what follows. By a simple argument using the regularity result in Proposition 2.1, it follows that minimizers are essentially bounded and indecomposable:

LEMMA 4.1 (Boundedness and Connectedness of Minimizers). *Let $\Omega \subset \mathbb{R}^n$ be a minimizer of* (2.10) *with $|\Omega| = m$. Then Ω is essentially bounded and indecomposable.*

PROOF. As was shown in the proof of Proposition 2.1, we can apply [34, lemma 2.1.3] guaranteeing that there exists $r > 0$ and $c > 0$ such that for every $x \in \overline{\Omega}^M$ we have $|\Omega \cap B_r(x)| \geq c r^n$. If Ω is not essentially bounded, then there exists a sequence $(x_k) \in \overline{\Omega}^M$ such that $x_k \to \infty$ and $|x_k - x_{k'}| > 2r$ for all k, k'. Then clearly

$$|\Omega| \geq \sum_k |\Omega \cap B_r(x_k)| \geq \sum_k c r^n = \infty, \tag{4.1}$$

contradicting the fact that $|\Omega| = m < \infty$.

To prove that the minimizers are indecomposable, suppose the opposite is true and that there exist two sets of finite perimeter Ω_1 and Ω_2 such that $\Omega_1 \cap \Omega_2 = \varnothing$ and $\Omega = \Omega_1 \cup \Omega_2$, with $P(\Omega) = P(\Omega_1) + P(\Omega_2)$. Since Ω and hence Ω_1 and Ω_2 are essentially bounded, defining $\Omega_R := \Omega_1 \cup (\Omega_2 + e_1 R)$, we have $|\Omega_R| = m$ and $P(\Omega_R) = P(\Omega)$ for $R > 0$ sufficiently large. At the same time, the nonlocal energy decreases:

$$\begin{aligned}
&\liminf_{R \to \infty} E(\Omega_R) \\
&\quad = P(\Omega_R) + V(\Omega_1) + V(\Omega_2) \\
&\quad < P(\Omega) + V(\Omega_1) + V(\Omega_2) + 2 \int_{\Omega_1} \int_{\Omega_2} \frac{1}{|x-y|^\alpha} \, dx \, dy = E(\Omega).
\end{aligned} \tag{4.2}$$

Thus, choosing R sufficiently large, we obtain $E(\Omega_R) < E(\Omega)$, contradicting the minimizing property of Ω. □

Our next lemma yields a general criterion on a set of finite perimeter being energetically unfavorable that will be helpful for several of our proofs.

LEMMA 4.2 (Nonoptimality Criterion). *Let $F \subset \mathbb{R}^n$ be a set of finite perimeter. Suppose there is a partition of F into two disjoint sets of finite perimeter F_1 and F_2 with positive measures such that*

$$\Sigma := P(F_1) + P(F_2) - P(F) \leq \frac{1}{2} E(F_2). \tag{4.3}$$

Then there is $\varepsilon > 0$ depending only on n and α such that if

$$|F_2| \leq \varepsilon \min\{1, |F_1|\}, \tag{4.4}$$

there exists a set $G \subset \mathbb{R}^n$ such that $|G| = |F|$ and $E(G) < E(F)$.

PROOF. Let $m_1 := |F_1|$ and $m_2 := |F_2|$, and let $\gamma := \frac{m_2}{m_1} \leq \varepsilon$. We will compare F with the following two sets:

- The set $\widetilde{F}$ given by $\widetilde{F} = \ell F_1$, where $\ell := \sqrt[n]{1+\gamma}$; in particular, $|\widetilde{F}| = |F|$.
- The set $\widehat{F}$ given by a collection of $N \geq 1$ balls of equal size and with centers located at $x = jRe_1$, $j = 1, \ldots, N$, with R large enough. The number N is chosen to be the smallest integer for which the mass of each ball does not exceed 1 and such that $|\widehat{F}| = |F|$.

If $E(\widehat{F}) < E(F)$, then the assertion of the lemma holds true and the proof is concluded. Therefore, in the following we may assume

$$E(F) \leq E(\widehat{F}) \leq C \max\{m, m^{\frac{n-1}{n}}\} \tag{4.5}$$

for some $C > 0$, where the last inequality is obtained by direct computation. Recall that by convention constants may depend both on α and n.

It hence remains to show that under the assumption (4.5) and for sufficiently small ε, we have $E(\widetilde{F}) < E(F)$. We first note that in view of the scaling of interfacial and nonlocal energies, we have

$$\begin{aligned} E(\widetilde{F}) &= \ell^{n-1} P(F_1) + \ell^{2n-\alpha} V(F_1) \\ &\leq E(F_1) + ((\ell^{n-1} - 1) + (\ell^{2n-\alpha} - 1))E(F_1). \end{aligned} \tag{4.6}$$

Choosing $\varepsilon \leq 1$, we have $1 \leq \ell \leq (1+\varepsilon)^{1/n} \leq 2^{1/n}$, and therefore by Taylor's formula we obtain $\ell^{n-1} - 1 \leq K(\ell - 1)$ and $\ell^{2n-\alpha} - 1 \leq K(\ell - 1)$ for some $K > 0$ independent of ℓ. Furthermore, since $\ell - 1 \leq \gamma$ and by (4.6), we arrive at $E(\widetilde{F}) - E(F_1) \leq 2\gamma K E(F_1)$. By the definition of Σ and with (2.10), this implies

$$\begin{aligned} E(\widetilde{F}) - E(F) &\leq V(F_1) + V(F_2) - V(F) + \Sigma - E(F_2) + 2\gamma K E(F_1) \\ &\overset{(4.3)}{<} -\frac{1}{2} E(F_2) + 2\gamma K E(F_1), \end{aligned} \tag{4.7}$$

where for the second estimate we also used the fact that $V(F_1) + V(F_2) < V(F)$. By positivity of V and the isoperimetric inequality, we have $E(F_2) > P(F_2) \geq c m_2^{(n-1)/n}$ for some $c > 0$. Furthermore, a straightforward calculation using (4.3) and $V(F) > V(F_1) + V(F_2)$ yields $E(F_1) < E(F)$, so that (4.7) turns into

$$\begin{aligned} E(\widetilde{F}) - E(F) &\overset{(4.7)}{\leq} -c m_2^{(n-1)/n} + C\gamma E(F) \\ &\overset{(4.5)}{\leq} -c m_2^{(n-1)/n} + C \max\{m_2, \varepsilon^{1/n} m_2^{(n-1)/n}\} \end{aligned} \tag{4.8}$$

for some $C, c > 0$, where we also used that $\gamma m \leq 2m_2$ and that $\gamma \leq \varepsilon$ by (4.4). Then, since $m_2 \leq \varepsilon$ by (4.4) as well, the assertion of the lemma follows for ε sufficiently small. □

Our next lemma is an improvement of the standard density estimate for quasi-minimizers of the perimeter to a uniform estimate independent of Ω.

LEMMA 4.3 (Uniform Density Bound). *Let $\Omega \subset \mathbb{R}^n$ be a minimizer of* (2.10) *with $|\Omega| = m$. Then for every $x \in \overline{\Omega}^M$ we have for some $c = c(\alpha, n) > 0$,*

$$|\Omega \cap B_1(x)| \geq c \min\{1, m\}. \tag{4.9}$$

PROOF. For given $r > 0$ and $x \in \overline{\Omega}^M$, define the sets $F_2^r := \Omega \cap B_r(x)$ and $F_1^r := \Omega \backslash B_r(x)$. Since $|F_1^r| + |F_2^r| = m$ and $|F_2^r| \leq \omega_n r^n$, there exists $C > 0$ depending only on α and n such that assumption (4.4) of Lemma 4.2 is satisfied for all $r \le r_0 := C \min(1, m^{1/n})$. Since Ω is a minimizer, (4.3) cannot be satisfied for any $r \le r_0$. That is, for all $r \le r_0$ we have

$$\Sigma^r := P(F_1^r) + P(F_2^r) - P(\Omega) > \frac{1}{2} E(F_2^r) > \frac{1}{2} P(F_2^r). \tag{4.10}$$

On the other hand, by [1, prop. 1] and [2, theorem 3.61], we have

$$\Sigma^r = 2\mathcal{H}^{n-1}(\partial^* F_1^r \cap \partial^* F_2^r).$$

In fact, since all the points belonging to $\partial^* F_1^r$ and $\partial^* F_2^r$ are supported on $\partial B_r(x)$ and have density $\frac{1}{2}$ by [2, theorem 3.61], we have $\partial^* F_1^r \cap \partial^* F_2^r = \mathring{\Omega}^M \cap \partial B_r(x)$. Together with (4.10) this yields

$$\begin{aligned} 2\mathcal{H}^{n-1}(\mathring{\Omega}^M \cap \partial B_r(x)) > \frac{1}{2}(&\mathcal{H}^{n-1}(\partial^* \Omega \cap B_r(x)) \\ &+ \mathcal{H}^{n-1}(\mathring{\Omega}^M \cap \partial B_r(x)). \end{aligned} \tag{4.11}$$

We now rearrange terms in (4.11) and apply the relative isoperimetric inequality to the right-hand side, noting that if $|\Omega \cap B_r(x)| \ge \frac{1}{2}\omega_n r^n$ for some $r < r_0$, the conclusion still holds. This results in

$$\mathcal{H}^{n-1}(\mathring{\Omega}^M \cap \partial B_r(x)) \ge c|\Omega \cap B_r(x)|^{\frac{n-1}{n}} \tag{4.12}$$

for some $c > 0$ depending only on n. Finally, denoting $U(r) := |\Omega \cap B_r(x)|$ and since by Fubini's theorem $\frac{dU(r)}{dr} = \mathcal{H}^{n-1}(\mathring{\Omega}^M \cap \partial B_r(x))$ for a.e. $r < r_0$ by the co-area formula, we arrive at the differential inequality

$$\frac{dU(r)}{dr} \ge cU^{\frac{n-1}{n}}(r) \quad \text{for a.e. } r < r_0. \tag{4.13}$$

Together with the fact that $x \in \overline{\Omega}^M$, we have $U(r) > 0$ for all $r > 0$, which implies that $U(r) \ge cr^n$ for some $c > 0$ depending only on n for all $r \le r_0$. The statement of the lemma then follows by choosing $r = r_0$. □

We next establish a basic regularity result for the potential v defined in (2.8) for the range of α used in Theorem 3.2.

LEMMA 4.4. *Let F be a measurable set with $|F| \le m$ for some $m > 0$. Then $\|v_F\|_{L^\infty(\mathbb{R}^n)} \le C$ for some $C > 0$ depending only on α, n, and m. If, in addition, $\alpha \in (0, n-1)$, then also $\|v_F\|_{W^{1,\infty}(\mathbb{R}^n)} \le C'$ for some $C' > 0$ depending only on α, n, and m.*

PROOF. Boundedness of v_F follows by the same argument as in the proof of Proposition 2.1. Differentiating (2.8) in x, we obtain

$$\begin{aligned} |\nabla v_F(x)| &\le \alpha \int_F \frac{1}{|x-y|^{\alpha+1}}\, dy \\ &\le \alpha \int_{B_1(x)} \frac{1}{|x-y|^{\alpha+1}}\, dy + \alpha|F| \le C, \end{aligned} \tag{4.14}$$

whenever $\alpha \in (0, n-1)$. □

Let us point out that if F also has a sufficiently smooth boundary, then the potential v_F may be estimated precisely near the boundary. In particular, for the potential of the ball v_B it is not difficult to see that for $r := |x| - 1$ and for $v_0 := v_B\big|_{r=0}$ the leading-order behavior of v_B near the boundary is given by

$$v_0 - v^B(x) \sim \begin{cases} r & \text{if } \alpha < n-1, \\ r \ln|r| & \text{if } \alpha = n-1, \\ r^{n-\alpha} & \text{if } \alpha > n-1. \end{cases} \tag{4.15}$$

In particular, for $\alpha > n-1$ the statement of Lemma 4.4 is false, even for a ball.

The next lemma provides a tool to compare the nonlocal part of the energy of two sets in terms of the potential of one of the two sets:

LEMMA 4.5. *Let F and G be measurable subsets of $\mathbb{R}^n$ with $|F| = |G| < \infty$. Let v_F be the potential defined in* (2.8)*. Then for any $c \in R$, we have*

$$V(F) - V(G) \leq 2\bigg(\int\limits_{F\setminus G} (v_F(x) - c)dx - \int\limits_{G\setminus F} (v_F(x) - c)dx \bigg). \tag{4.16}$$

PROOF. Let χ_F and χ_G be the characteristic functions of the sets F and G, respectively. After some straightforward algebra one can write

$$\begin{aligned} V(F) - V(G) &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\chi_F(x)\chi_F(y)}{|x-y|^\alpha}\,dx\,dy - \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\chi_G(x)\chi_G(y)}{|x-y|^\alpha}\,dx\,dy \\ &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{(\chi_F(x) + \chi_G(x))(\chi_F(y) - \chi_G(y))}{|x-y|^\alpha}\,dx\,dy \\ &= 2\int\limits_{\mathbb{R}^n} v_F(x)(\chi_F(x) - \chi_G(x))dx \\ &\quad - \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{(\chi_F(x) - \chi_G(x))(\chi_F(y) - \chi_G(y))}{|x-y|^\alpha}\,dx\,dy. \end{aligned} \tag{4.17}$$

In fact, since both χ_F and χ_G belong to $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, one can use the Fourier transform to show that the last integral in (4.17) is positive; see, e.g., [24]. Furthermore, since $|F| = |G|$, the right-hand side of (4.17) does not change if we replace $v_F(x)$ by $v_F(x) + c$ for arbitrary $c \in \mathbb{R}$. The assertion of the lemma follows. □

5 Existence

We prove existence of minimizers of E with prescribed mass by suitably localizing the minimizing sequence, which is possible for sufficiently small mass when the perimeter is the dominant term in the energy.

LEMMA 5.1 (Comparison with Set of Bounded Support). *There exists* $m_1 = m_1(\alpha, n) > 0$ *such that for every* $m \le m_1$ *and every set of finite perimeter* F *with* $|F| = m$ *there exists a set of finite perimeter* G *such that*

$$E(G) \le E(F) \quad \textit{and} \quad G \subset B_1. \tag{5.1}$$

PROOF. Throughout the proof we will use the assumption $m \le 1$.

We may assume that $E(F) \le E(B_r)$, with $|B_r| = m$, since otherwise we can choose $G = B_r$, which yields the assertion of the lemma. In particular,

$$D(F) \overset{(2.4)}{=} \frac{C}{r^{n-1}}(P(F) - P(B_r)) \overset{(2.10)}{\le} \frac{C}{r^{n-1}}(V(B_r) - V(F)) \le C'r \tag{5.2}$$

for some constants $C, C' > 0$ depending only on α and n, where we used Lemma 4.4. By the quantitative isoperimetric estimate (2.6) we therefore get the bound $\Delta(F, B_r) \le Cr^{1/2}$ for the Fraenkel asymmetry. Hence, after a suitable translation, we have $|B_r \Delta F| \le Cr^{n+1/2}$. Since $|B_r| = |F|$, we also have $|B_r \Delta F| = 2|F \setminus B_r|$, which in turn implies

$$|F \setminus B_r| \le Cr^{n+1/2} \tag{5.3}$$

for some $C > 0$ depending only on α and n.

For any $\rho > 0$, let $F_1 = |F \cap B_\rho|$ and $F_2 = |F \backslash B_\rho|$. Note that by (5.3) and for sufficiently small $m > 0$, the condition (4.4) of Lemma 4.2 is satisfied for the two sets F_1 and F_2 for all $\rho > r$. We suppose that furthermore

$$\Sigma := P(F_1) + P(F_2) - P(F) > \frac{1}{2} E(F_2). \tag{5.4}$$

Indeed, if (5.4) is not satisfied, then Lemma 4.2 can be applied. Furthermore, both sets $\widetilde{F}$ and $\widehat{F}$ constructed in the proof of Lemma 4.2 are contained in a ball of radius 1, which concludes the proof. Hence we may assume in the following that (5.4) is satisfied for all $\rho > r$.

In order to conclude the proof, we will apply an argument, similar to the one used in the proof of Lemma 4.3. For this, we define the monotonically decreasing function $U(\rho) = |F \setminus B_\rho|$. Observe that by (5.3), we have $U(\rho) \le C\rho^{n+1/2}$. Furthermore, as in the proof of Lemma 4.3, we have

$$\frac{dU(\rho)}{d\rho} \le -cU^{\frac{n-1}{n}}(\rho) \tag{5.5}$$

for some $c > 0$ depending only on n. For r sufficiently small, it then follows that $U(\rho) = 0$ for $\rho \ge 1$, which concludes the proof. □

PROOF OF THEOREM 3.1. We choose a sequence of sets of finite perimeter F_k with $|F_k| = m$ such that $E(F_k) \to \inf_{|F|=m} E(F)$. By Lemma 5.1, we can choose a minimizing sequence such that these sets are uniformly bounded, i.e., $F_k \subset B_1(0)$. By lower semicontinuity of the perimeter, there is a set of finite perimeter Ω supported in $B_1(0)$ such that for some subsequence we have $\Delta(F_{k_j}, \Omega) \to 0$ and $P(\Omega) \le \liminf_k P(F_k)$. Moreover, $|F_k| \to |\Omega|$ and, furthermore, by Lemma

4.4 the nonlocal part of the energy convergences, i.e., $V(F_k) \to V(\Omega)$. Hence $|\Omega| = m$ and $E(\Omega) = \inf_{|F|=m} E(F)$, which concludes the proof. $\square$

6 Ball as the Minimizer for Small Masses

In this section, we give the proof of Theorem 3.2. For this it is convenient to rescale length in such a way that the rescaled set Ω has the mass ω_n of the unit ball. We set $\lambda = (\frac{m}{\omega_n})^{1/n}$. We also introduce a positive parameter $\varepsilon > 0$ by

$$\varepsilon := \lambda^{n+1-\alpha} = \left(\frac{m}{\omega_n}\right)^{\frac{n+1-\alpha}{n}}. \tag{6.1}$$

Furthermore, we set $E_\varepsilon(\Omega_\varepsilon) := \lambda^{n-1} E(\Omega)$ where $\Omega_\varepsilon := \lambda^{-1}\Omega$. In the rescaled variables, this yields the following energy to be minimized:

$$E_\varepsilon(F) := P(F) + \varepsilon V(F), \quad |F| = \omega_n, \tag{6.2}$$

Note that by Theorem 3.1, the minimizers of E_ε exist for all $\varepsilon \le \varepsilon_1$, where ε_1 is related to m_1 via (6.1). Furthermore, the regularity result in Proposition 2.1 holds for the minimizers of E_ε. In this section, however, we will need to further strengthen the statement of Proposition 2.1 for the minimizers of E_ε to allow for the regularity properties that are uniform in ε. For this we apply a result of Rigot [34, theorem 1.4.9], which we summarize in the following proposition:

PROPOSITION 6.1. *Let $3 \le n \le 7$. Then there exists $\varepsilon_1 = \varepsilon_1(\alpha, n) > 0$ such that for all $\varepsilon \le \varepsilon_1$ there exists a minimizer Ω_ε of E_ε in* (6.2)*. Furthermore, the set Ω_ε is open* (*up to a negligible set*)*, and there exists $r_0 > 0$ depending only on α and n such that if $x \in \partial\Omega_\varepsilon$, then $\Omega_\varepsilon \cap B_{r_0}(x)$ is (up to a rotation) the subgraph of a function of class $C^{1,1/2}$, with the regularity constants depending only on α and n.*

Expressed in terms of the rescaled problem, Theorem 3.2 takes the following form:

PROPOSITION 6.2. *For all $3 \le n \le 7$ and for all $\alpha \in (0, n-1)$ there is $\varepsilon_0 = \varepsilon_0(\alpha, n) > 0$ such that for all $\varepsilon \le \varepsilon_0$ the unique* (*up to translation*) *minimizer of E_ε in* (6.2) *is given by the unit ball.*

We prove this proposition using a sequence of lemmas. As a first step, we note that in the limit $\varepsilon \to 0$, minimizers Ω_ε of E_ε converge in the L^1-norm sense (after a suitable translation) to the unit ball.

LEMMA 6.3. *Let Ω_ε be a minimizer for* (6.2)*. Then there is $\varepsilon_0 = \varepsilon_0(\alpha, n) > 0$ such that for some $C = C(\alpha, n) > 0$ and all $\varepsilon \le \varepsilon_0$, we have*

$$\Delta(\Omega_\varepsilon, B_1) \le C\varepsilon^{\frac{1}{2}}. \tag{6.3}$$

PROOF. The proof is based on an application of the quantitative isoperimetric inequality given by (2.6) [19, theorem 1.1]. Since Ω_ε is a minimizer, we have $E_\varepsilon(\Omega_\varepsilon) \leq E_\varepsilon(B_1)$, which yields

$$\begin{aligned} D(\Omega_\varepsilon) &\leq \frac{\varepsilon}{n\omega_n}\left(\int_{B_1(x_0)}\int_{B_1(x_0)} \frac{1}{|x-y|^\alpha}\,dx\,dy - \int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} \frac{1}{|x-y|^\alpha}\,dx\,dy\right) \\ &\leq \frac{\varepsilon}{n}\sup_{x\in\mathbb{R}^n}\int_{B_1(0)} \frac{1}{|x-y|^\alpha}\,dy + \frac{\varepsilon}{n}\sup_{x\in\mathbb{R}^n}\int_{\Omega_\varepsilon} \frac{1}{|x-y|^\alpha}\,dy \\ &\leq \frac{2\varepsilon}{n}\left(\omega_n + \int_{B_1(0)} \frac{1}{|y|^\alpha}\,dy\right) \leq C\varepsilon, \end{aligned} \tag{6.4}$$

where in the last line we split the integration over $\mathbb{R}^n \backslash B_1(x)$ and $B_1(x)$, respectively. Together with (2.6), this concludes the proof. □

In fact, using the regularity result in Proposition 6.1, we can show that for sufficiently small ε every minimizer Ω_ε may be represented by the subgraph of a map $\rho_\varepsilon : \partial B_1(0) \to \partial\Omega$, with $\partial\Omega_\varepsilon$ close to $\partial B_1(0)$ in the C^1-norm on $\partial B_1(0)$:

LEMMA 6.4. *For all* $3 \leq n \leq 7$, $\alpha \in (0, n)$, *and* $\delta > 0$ *there is* $\varepsilon_0 = \varepsilon_0(\alpha, n, \delta) > 0$ *such that for all* $\varepsilon \leq \varepsilon_0$ *every minimizer* Ω_ε *of* (6.2) *is given by* (*up to a negligible set*)

$$\Omega_\varepsilon - x_0 = \{x : |x| < 1 + \rho_\varepsilon(x/|x|)\} \quad \text{for some } \rho_\varepsilon \in C^{1,1/2}(\partial B_1(0)), \tag{6.5}$$

with $\|\rho_\varepsilon\|_{W^{1,\infty}(\partial B_1(0))} \leq \delta$ *and* $x_0 \in \mathbb{R}^n$ *the barycenter of* Ω.

PROOF. By Proposition 6.1, there is $r_0 > 0$ such that for every $x \in \partial\Omega_\varepsilon$ the set $\partial\Omega_\varepsilon \cap B_{r_0}(x)$ is a graph whose $C^{1,1/2}$ norm is uniformly bounded; in particular, it does not depend on ε. This implies that the translated and rescaled set $((\partial\Omega_\varepsilon \cap B_r(x)) - x)/r$ approaches the tangent hyperplane $\Pi_r(x)$ in $B_1(0)$ as $r \to 0$ in the C^1 sense, with the modulus of continuity depending only on α and n.

Let now $x_1 \in \mathbb{R}^n$ be the point such that the ball $B_1(x_1)$ minimizes the Fraenkel asymmetry $\Delta(\Omega_\varepsilon, B_1)$. Then for any $x \in \partial\Omega$ the set $((\partial B_1(x_1) \cap B_r(x)) - x)/r$ is either empty or is close to another hyperplane $\widetilde{\Pi}_r(x)$ in the C^1 sense in $B_1(0)$ and for r sufficiently small. Therefore, for every $c > 0$ sufficiently small and depending only on α and n, there exists $r \in (0, r_0)$ depending only on α, n, and c such that $\Delta(\Omega_\varepsilon, B_1)\,\omega_n = |\Omega_\varepsilon \triangle B_1(x_1)| \geq |(\Omega_\varepsilon \triangle B_1(x_1)) \cap B_r(x)| \geq cr^n$ unless $\partial B_1(x_1) \cap B_r(x) \neq \varnothing$ and $\widetilde{\Pi}_r(x)$ is sufficiently close to $\Pi_r(x)$ in $B_1(0)$ in the Hausdorff sense (with closeness controlled by c). Closeness of $\partial\Omega_\varepsilon$ and

$\partial B_1(x_1)$ in the C^1 sense controlled by ε then follows by Lemma 6.3 for all $\varepsilon \le \varepsilon_0$, with $\varepsilon_0 > 0$ depending only on α and n.

Thus, locally $\partial\Omega_\varepsilon$ may be represented by a C^1-map from an open subset of $\partial B_1(0)$ to $\mathbb{R}^n$. In fact, by the uniform C^1 closeness of $\partial\Omega_\varepsilon$ to $\partial B_1(x_1)$, this map can be extended from the neighborhood of each point $x \in \partial\Omega_\varepsilon$ to a global C^1-map from $\partial B_1(0)$ to $\partial\Omega_\varepsilon$. This implies that $\partial\Omega_\varepsilon$ may be represented by a union of finitely many connected components consisting of nonintersecting graphs of C^1-maps from $\partial B_1(0)$ to $\mathbb{R}^n$, with the degree of closeness controlled by ε and with the perimeter of each component approaching $P(B_1(x_1))$ as $\varepsilon \to 0$. Since by the minimizing property of Ω_ε and positivity of the nonlocal term we have $P(\Omega_\varepsilon) \le E_\varepsilon(\Omega_\varepsilon) \le E_\varepsilon(B_1(x_1)) \le P(B_1(x_1)) + C\varepsilon$ for some $C > 0$, we conclude that for all $\varepsilon \le \varepsilon_0'$ with $\varepsilon_0' > 0$ depending only on α and n the set $\partial\Omega_\varepsilon$ consists of only one connected component, and therefore Ω_ε can be represented by (6.5).

Finally, since Ω_ε is a simply connected open set whose boundary $\partial\Omega_\varepsilon$ is close in the C^1 sense to $\partial B_1(x_1)$, the quantity $|x_0 - x_1|$, where x_0 is the barycenter of Ω_ε, is small, and is controlled by ε as well. The statement of the lemma then follows from the C^1 closeness of $B_1(x_0)$ to $B_1(x_1)$. □

We next use a bound on the isoperimetric deficit of almost spherical sets derived by Fuglede [18, theorem on p. 623]:

LEMMA 6.5. *For all* $3 \le n \le 7$, *all* $\alpha \in (0, n)$, *and all* $\varepsilon \le \varepsilon_0$, *where* $\varepsilon_0 = \varepsilon_0(\alpha, n)$, *the minimizer* Ω_ε *of* (6.2) *satisfies*

$$\|\rho_\varepsilon\|^2_{L^2(\partial B_1(0))} + \|\nabla\rho_\varepsilon\|^2_{L^2(\partial B_1(0))} \le C D(\Omega_\varepsilon), \tag{6.6}$$

where ρ_ε *is as in Lemma* 6.4 *for some universal* $C > 0$.

PROOF. Since Ω_ε is near the ball in the C^1-norm, choosing $\delta > 0$ in Lemma 6.4 sufficiently small, we can apply the result by Fuglede in [18, theorem 1.2] to yield the estimate. □

PROOF OF PROPOSITION 6.2. By Proposition 6.1, there exists a minimizer Ω_ε of E_ε if ε is sufficiently small. Furthermore, the set Ω_ε satisfies the conclusions of Lemma 6.4. Since Ω_ε is a minimizer, we have $E_\varepsilon(\Omega_\varepsilon) \le E_\varepsilon(B_1(x_0))$, where x_0 is the barycenter of Ω_ε, which is equivalent to

$$D(\Omega_\varepsilon) \le \frac{\varepsilon}{n\omega_n} \left(V(B_1(x_0)) - V(\Omega_\varepsilon) \right). \tag{6.7}$$

On the other hand, choosing $c = v_0$ in Lemma 4.5, where v_0 is as in (4.15), and applying Lemma 4.4, we obtain

$$\begin{aligned} V(B_1(x_0)) - V(\Omega_\varepsilon) &\overset{(4.16)}{\leq} 2\Big(\int_{B_1(x_0)\setminus\Omega_\varepsilon} (v^B(x - x_0) - v_0)dx - \int_{\Omega_\varepsilon\setminus B_1(x_0)} (v^B(x - x_0) - v_0)dx \Big) \\ &\leq 2 \int_{\Omega_\varepsilon \triangle B_1(x_0)} |v^B(x - x_0) - v_0| dx \\ &\leq C \int_{\partial B_1(x_0)} \Big(\int_0^{\rho_\varepsilon(x)} t\, dt \Big) d\mathcal{H}^{n-1}(x) \leq C' \|\rho_\varepsilon\|^2_{L^2(\partial B_1(x_0))} \end{aligned} \tag{6.8}$$

for some $C, C' > 0$ depending on α and n and where ρ_ε is the function from (6.6). Combining this inequality with (6.6), (6.7), and (6.8), we get

$$c \|\rho_\varepsilon\|^2_{L^2} \leq D(\Omega_\varepsilon) \leq C\varepsilon \|\rho_\varepsilon\|^2_{L^2} \tag{6.9}$$

for some universal $c > 0$ and some $C > 0$ depending only α and n. Therefore, as long as ε is small enough, we have $D(\Omega_\varepsilon) = 0$. This implies that $\Omega_\varepsilon = B_1(x_0)$, thus concluding the proof of the proposition. □

7 Nonexistence and Equipartition of Energy

We now establish Theorem 3.4. The following interpolation estimate is a generalization of a corresponding two-dimensional result, proved in [22, eq. (5.3)]:

LEMMA 7.1 (Interpolation). *For any $u \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} u^2\, dx \leq C \|u\|^{\frac{n-\alpha}{n+1-\alpha}}_{L^\infty(\mathbb{R}^n)} \Big(\int_{\mathbb{R}^n} |\nabla u| dx \Big)^{\frac{n-\alpha}{n+1-\alpha}} \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)u(y)}{|x-y|^\alpha}\, dx\, dy \Big)^{\frac{1}{n+1-\alpha}} \tag{7.1}$$

for some $C > 0$ depending only on α and n.

PROOF. The proof follows by a straightforward extension of the proof in [22, (5.3)]. □

Using the result in Lemma 7.1, we are ready to give the proof of Theorem 3.4.

PROOF OF THEOREM 3.4. The lower bound is a consequence of Lemma 7.1. Indeed, for any set of finite perimeter F with $|\Omega| = m$, an application of (7.1) with u as the characteristic function of F yields

$$m \leq C P^{\frac{n-\alpha}{n+1-\alpha}}(F) V^{\frac{1}{n+1-\alpha}}(F) \leq C E(F) \tag{7.2}$$

for some constant $C > 0$ depending only on α and n; the lower bound follows. Since the estimate (7.1) is multiplicative, the assertion on an equipartition of energy in (3.2) follows as well. The proof of the upper bound can be shown by explicit construction of a collection of balls sufficiently far apart. The argument proceeds similarly as in [22, lemma 5.1]. □

We now turn to the proof of Theorem 3.3. We proceed by first establishing a lemma about the spatial extent of minimizers for large masses.

LEMMA 7.2. *Let* Ω *be a minimizer of* E *with* $|\Omega| = m$ *and* $m \geq 1$. *Then*

$$cm^{\frac{1}{\alpha}} \leq \operatorname{diam} \overline{\Omega}^{M} \leq Cm \tag{7.3}$$

for some $C, c > 0$ *depending only on* α *and* n.

PROOF. We recall that by Lemma 4.1 the minimizer Ω is essentially bounded and indecomposable; in particular, $d := \operatorname{diam} \overline{\Omega}^{M} < \infty$. By Theorem 3.4 and in view of (2.8), we get for some $C > 0$ depending only on α and n that

$$\frac{m^2}{d^\alpha} \leq V(\Omega) \leq E(\Omega) \leq Cm, \tag{7.4}$$

which implies the lower bound in (7.3).

We next turn to the proof of the upper bound in (7.3). Clearly, we may assume that $d > 5$. Furthermore, without loss of generality we may assume that there exist $x^{(1)}, x^{(2)} \in \overline{\Omega}^{M}$ such that $x^{(1)} \cdot e_1 < 1$ and $x^{(2)} \cdot e_1 > d - 1$. Let N be the largest integer smaller than $\frac{d-2}{3}$. Since Ω is indecomposable, there exist N disjoint balls $B_1(x_j)$, $j = 1, \ldots, N$, with $x_j \in \overline{\Omega}^{M}$ such that $3j - 1 < x_j \cdot e_1 < 3j$. The upper bound now follows from an application of Lemma 4.3:

$$m = |\Omega| \geq \sum_{j=1}^{N} |B_1(x_j) \cap \Omega| \geq cN \geq c'd \tag{7.5}$$

for some universal $c' > 0$. □

As a direct consequence of Lemma 7.2, we get the following:

COROLLARY 7.3. *The statement of Theorem* 3.3 *holds if* $\alpha < 1$.

It remains to give the proof of Theorem 3.3 in the case $\alpha \in [1, 2)$.

PROOF OF THEOREM 3.3. By Corollary 7.3, we may assume that $\alpha \in [1, 2)$. Arguing by contradiction, we assume that for every $m_2 > 0$ there exists a minimizer Ω of E with $|\Omega| = m$ for some $m > m_2$. We define $d := \operatorname{diam} \overline{\Omega}^{M} < \infty$ and

$$U(t) := |\Omega \cap \{x \in \mathbb{R}^n : 0 < x \cdot e_1 < t\}| \quad \forall t > 0. \tag{7.6}$$

According to Lemma 7.2, we may assume that $4 < d < \infty$. By a suitable rotation and translation of the coordinate system, we may further assume that there are points $x^{(1)}, x^{(2)} \in \overline{\Omega}^{M}$ with $x^{(1)} \cdot e_1 < 1$ and $x^{(2)} \cdot e_1 > d - 1$, and

$$U(d) = m \quad \text{and} \quad U(\tfrac{1}{2}d) \leq \tfrac{1}{2}m. \tag{7.7}$$

Let Π_t be the hyperplane $\Pi_t := \{x \in \mathbb{R}^n : x \cdot e_1 = t\}$. For a given $t \in (0, \frac{1}{2}d)$, we cut the set Ω by the hyperplane Π_t into two pieces, which are then moved apart to a large distance $R > 0$, with the new set denoted as Ω_t^R. In general, the perimeter of Ω_t^R is larger than that of Ω, while the nonlocal part of the energy of the new set is smaller.

To make this statement quantitative, as in the proof of Lemma 4.3, we define

$$\rho(t) := \frac{1}{2}\big(P\big(\Omega_t^R\big) - P(\Omega)\big). \tag{7.8}$$

Using the same arguments as in the proof of Lemma 4.3, we have

$$\rho(t) = \mathcal{H}^{n-1}(\mathring{\Omega}^M \cap \Pi_t).$$

Hence, by Fubini's theorem we obtain

$$\begin{aligned} V(\Omega_t^R) - V(\Omega) &\le -\frac{m}{2d^\alpha} U(t) + K(R) \\ &= -\frac{m}{2d^\alpha} \int_0^t \rho(t')dt' + K(R), \end{aligned} \tag{7.9}$$

where $K(R) \to 0$ as $R \to \infty$. Combining (7.8) and (7.9), we obtain

$$E\big(\Omega_t^R\big) - E(\Omega) \le 2\rho(t) - \frac{m}{2d^\alpha} \int_0^t \rho(t')dt' + K(R). \tag{7.10}$$

Since Ω is assumed to be a minimizer, we have in particular

$$2\rho(t) \ge \frac{m}{2d^\alpha} \int_0^t \rho(t')dt' \quad \forall t \in (\tfrac{1}{4}d, \tfrac{1}{2}d). \tag{7.11}$$

From (7.11) and the fact that $U \in C^{0,1}([0, d])$, we infer that

$$\frac{dU(t)}{dt} \ge \frac{m}{4d^\alpha} U(t) \quad \text{for a.e. } t \in (\tfrac{1}{4}d, \tfrac{1}{2}d), \tag{7.12}$$

Integrating this expression yields

$$U(t) \ge U(\tfrac{1}{4}d)e^{m(t-\frac{1}{4}d)/(4d^\alpha)}. \tag{7.13}$$

Note that by Lemma 4.3 we have $U(\frac{1}{4}d) \ge |\Omega \cap B_1(x^{(1)})| \ge c > 0$, which in view of (7.13) implies $U(\frac{1}{2}d) \ge Ce^{\frac{1}{16}md^{1-\alpha}}$. This contradicts the inequality in (7.7), thus concluding the proof. $\square$

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Bibliography

[1] Ambrosio, L.; Caselles, V.; Masnou, S.; Morel, J.-M. Connected components of sets of finite perimeter and applications to image processing. *J. Eur. Math. Soc. (JEMS)* **3** (2001), no. 1, 39–92. doi:10.1007/PL00011302

[2] Ambrosio, L.; Fusco, N.; Pallara, D. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] Ambrosio, L.; Paolini, E. Partial regularity for quasi minimizers of perimeter. *Ricerche Mat.* **48** (1999), suppl., 167–186.

[4] Bohr, N. Neutron capture and nuclear constitution. *Nature* **137** (1936), 344–348. doi:10.1038/137344a0

[5] Bohr, N.; Wheeler, J. A. The mechanism of nuclear fission. *Phys. Rev.* **56** (1939), no. 5, 426–450. doi:10.1103/PhysRev.56.426

[6] Care, C. M.; March, N. H. Electron crystallization. *Adv. Phys.* **24** (1975), no. 1, 101–116. doi:10.1080/00018737500101381

[7] Chen, L. Q.; Khachaturyan, A. G. Dynamics of simultaneous ordering and phase separation and effect of long-range Coulomb interactions. *Phys. Rev. Lett.* **70** (1993), no. 10, 1477–1480. doi:10.1103/PhysRevLett.70.1477

[8] Choksi, R.; Peletier, M. A. Small volume fraction limit of the diblock copolymer problem: I. Sharp-interface functional. *SIAM J. Math. Anal.* **42** (2010), no. 3, 1334–1370. doi:10.1137/090764888

[9] Choksi, R.; Peletier, M. A. Small volume-fraction limit of the diblock copolymer problem: II. Diffuse-interface functional. *SIAM J. Math. Anal.* **43** (2011), no. 2, 739–763. doi:10.1137/10079330X

[10] Cicalese, M.; Spadaro, E. Droplet minimizers of an isoperimetric problem with long-range interactions. *Comm. Pure Appl. Math.* **66** (2013), no. 8, 1298–1333. doi:10.1002/cpa.21463

[11] Cohen, S.; Plasil, F.; Swiatecki, W. J. Equilibrium configurations of rotating charged or gravitating liquid masses with surface tension. II. *Annals of Physics* **82** (1974), no. 2, 557–596. doi:10.1016/0003-4916(74)90126-2

[12] Cohen, S.; Swiatecki, W. J. The deformation energy of a charged drop: IV. Evidence for a discontinuity in the conventional family of saddle point shapes. *Annals of Physics* **19** (1962), no. 1, 67–164.

[13] de Gennes, P. G. Effect of cross-links on a mixture of polymers. *J. Physique Lett.* **40** (1979), no. 4, 69–72. doi:10.1051/jphyslet:0197900400406900

[14] De Giorgi, E. Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita. *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8)* **5** (1958), 33–44.

[15] Emery, V. J.; Kivelson, S. A. Frustrated electronic phase-separation and high-temperature superconductors. *Physica C* **209** (1993), no. 4, 597–621. doi:10.1016/0921-4534(93)90581-A

[16] Evans, L. C.; Gariepy, R. *Measure theory and fine properties of functions*. CRC Press, Boca Raton, Fla., 1992.

[17] Figalli, A.; Maggi, F. On the shape of liquid drops and crystals in the small mass regime. *Arch. Ration. Mech. Anal.* **201** (2011), no. 1, 143–207. doi:10.1007/s00205-010-0383-x

[18] Fuglede, B. Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbf{R}^n$. *Trans. Amer. Math. Soc.* **314** (1989), no. 2, 619–638. doi:10.2307/2001401

[19] Fusco, N.; Maggi, F.; Pratelli, A. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)* **168** (2008), no. 3, 941–980. doi:10.4007/annals.2008.168.941

[20] Gamow, G. Mass defect curve and nuclear constitution. *Proceedings of the Royal Society of London. Series A* **126** (1930), no. 803, 632–644. doi:10.1098/rspa.1930.0032

[21] Glotzer, S. C.; Di Marzio, E. A.; Muthukumar, M. Reaction-controlled morphology of phase-separating mixtures. *Phys. Rev. Lett.* **74** (1995), no. 11, 2034–2037. doi:10.1103/PhysRevLett.74.2034

[22] Knüpfer, H.; Muratov, C. On an isoperimetric problem with a competing nonlocal term I: The planar case. *Comm. Pure Appl. Math.* **66** (2013), no. 7, 1129–1162. doi:10.1002/cpa.21451

[23] Kovalenko, V. F.; Nagaev, E. L. Photoinduced magnetism. *Sov. Phys. Uspekhi* **29** (1986), no. 4, 297–321. doi:10.1070/PU1986v029n04ABEH003305

[24] Lieb, E. H.; Loss, M. *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, R.I., 2001.

[25] Lu, J.; Otto, F. Nonexistence of minimizer for Thomas–Fermi–Dirac–von Weizsäcker model. *Comm. Pure Appl. Math.*, forthcoming. doi:10.1002/cpa.21477

[26] Mamin, R. F. Domain structure of a new type near a photostimulated phase transition; autosolitons. *JETP Lett.* **60** (1994), 52–56.

[27] Muratov, C. Theory of domain patterns in systems with long-range interactions of Coulombic type. Thesis, Boston University, 1998.

[28] Muratov, C. B. Theory of domain patterns in systems with long-range interactions of Coulomb type. *Phys. Rev. E (3)* **66** (2002), no. 6, 066108, 25 pp. doi:10.1103/PhysRevE.66.066108

[29] Myers, W. D.; Swiatecki, W. J. Nuclear properties according to the Thomas-Fermi model. *Nuclear Physics A* **601** (1996), no. 2, 141–167. doi:10.1016/0375-9474(95)00509-9

[30] Nagaev, E. L. Phase separation in high-temperature superconductors and related magnetic systems. *Phys. Uspekhi* **38** (1995), no. 5, 497–521. doi:10.1070/PU1995v038n05ABEH000085

[31] Nyrkova, I. A.; Khokhlov, A. R.; Doi, M. Microdomain structures in polyelectrolyte systems: calculation of the phase diagrams by direct minimization of the free energy. *Macromolecules* **27** (1994), no. 15, 4220–4230. doi:10.1021/ma00093a025

[32] Ohta, T.; Kawasaki, K. Equilibrium morphology of block copolymer melts. *Macromolecules* **19** (1986), no. 10, 2621–2632. doi:10.1021/ma00164a028

[33] Pelekasis, N. A.; Tsamopoulos, J. A.; Manolis, G. D. Equilibrium shapes and stability of charged and conducting drops. *Phys. Fluids A* **2** (1990), no. 8, 1328–1340. doi:10.1063/1.857583

[34] Rigot, S. Ensembles quasi-minimaux avec contrainte de volume et rectifiabilité uniforme. *Mém. Soc. Math. Fr. (N.S.)* **82** (2000), vi+104 pp.

[35] Stillinger, F. H. Variational model for micelle structure. *J. Chem. Phys.* **78** (1983), no. 7, 4654–4661. doi:10.1063/1.445310

[36] Tamanini, I. Regularity results for almost minimal oriented hypersurfaces in $\mathbb{R}^N$. *Quaderni Dipartimento Mat. Univ. Lecce* **1** (1984), 1–92.

[37] von Weizsäcker, C. F. Zur theorie der Kernmassen. *Zeitschrift für Physik* **96** (1935), no. 7–8, 431–458. doi:10.1007/BF01337700

[38] Xia, Q. Regularity of minimizers of quasi perimeters with a volume constraint. *Interfaces Free Bound* **7** (2005), no. 3, 339–352. doi:10.4171/IFB/128

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