One-dimensional domain walls in thin film ferromagnets: an overview

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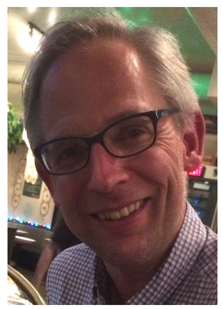
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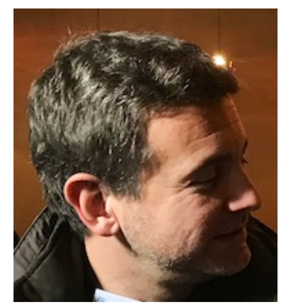
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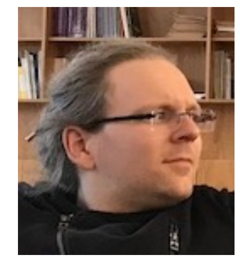
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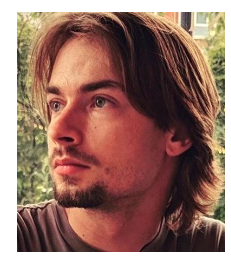
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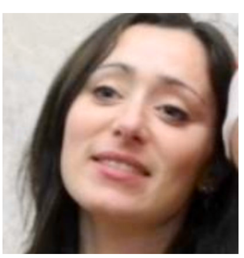
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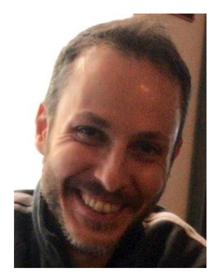
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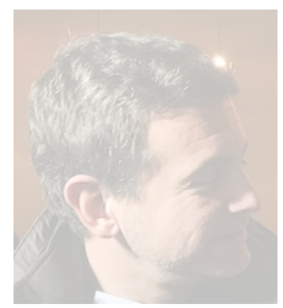
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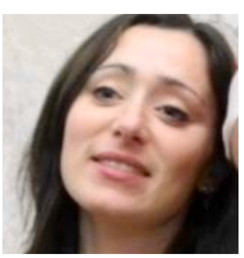
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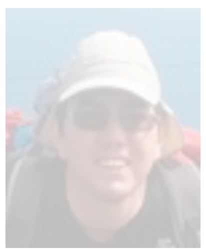
PhD students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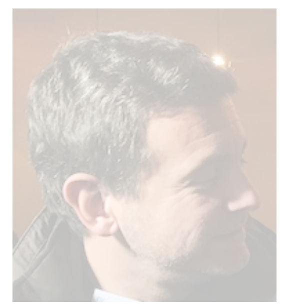
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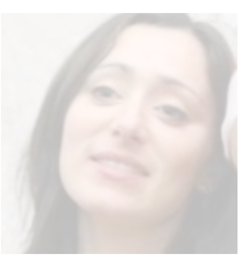
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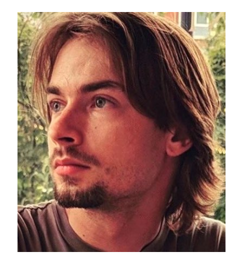
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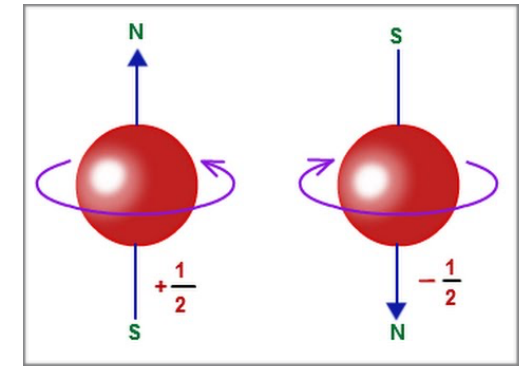
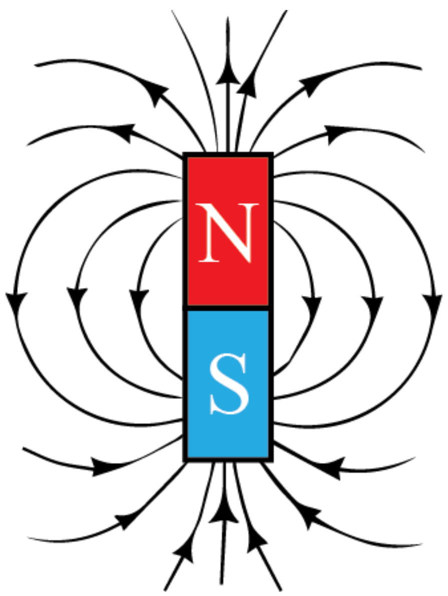
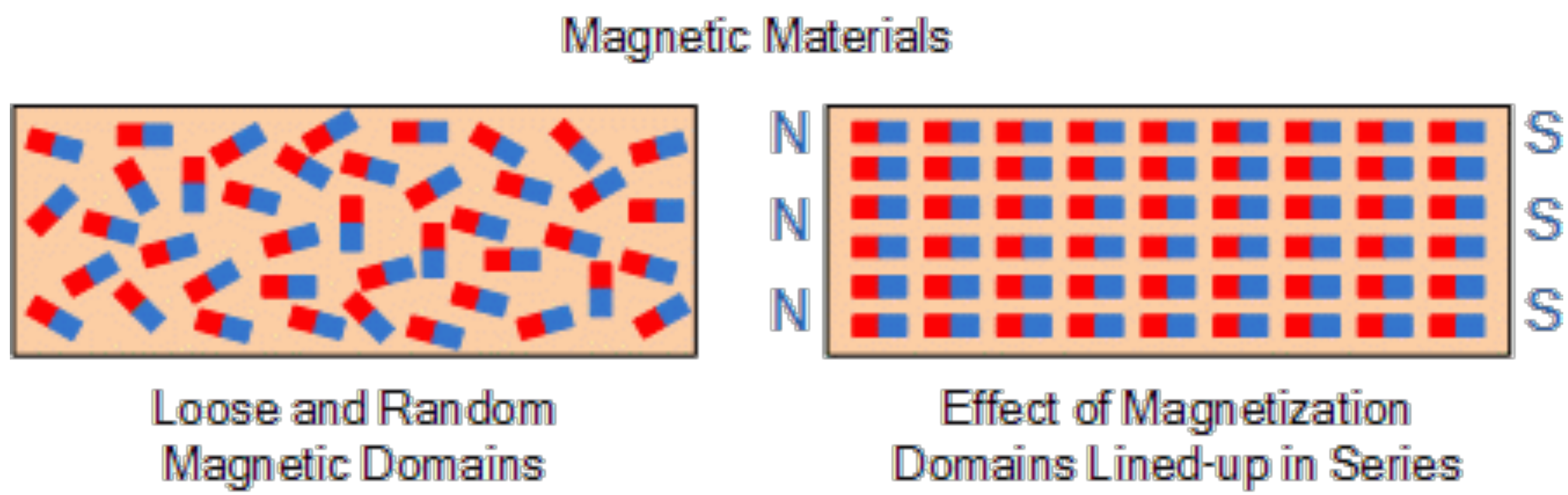
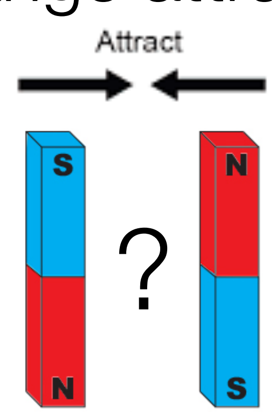
Magnetism and magnets







- spins act as tiny magnetic dipoles
- quantum-mechanical interaction between spins: <u>*exchange*</u>
- in transition metals below the *critical temperature*, exchange results in local spin alignment into the <u>*ferromagnetic state*</u>
- magnetic field mediates long-range attraction/repulsion between magnets



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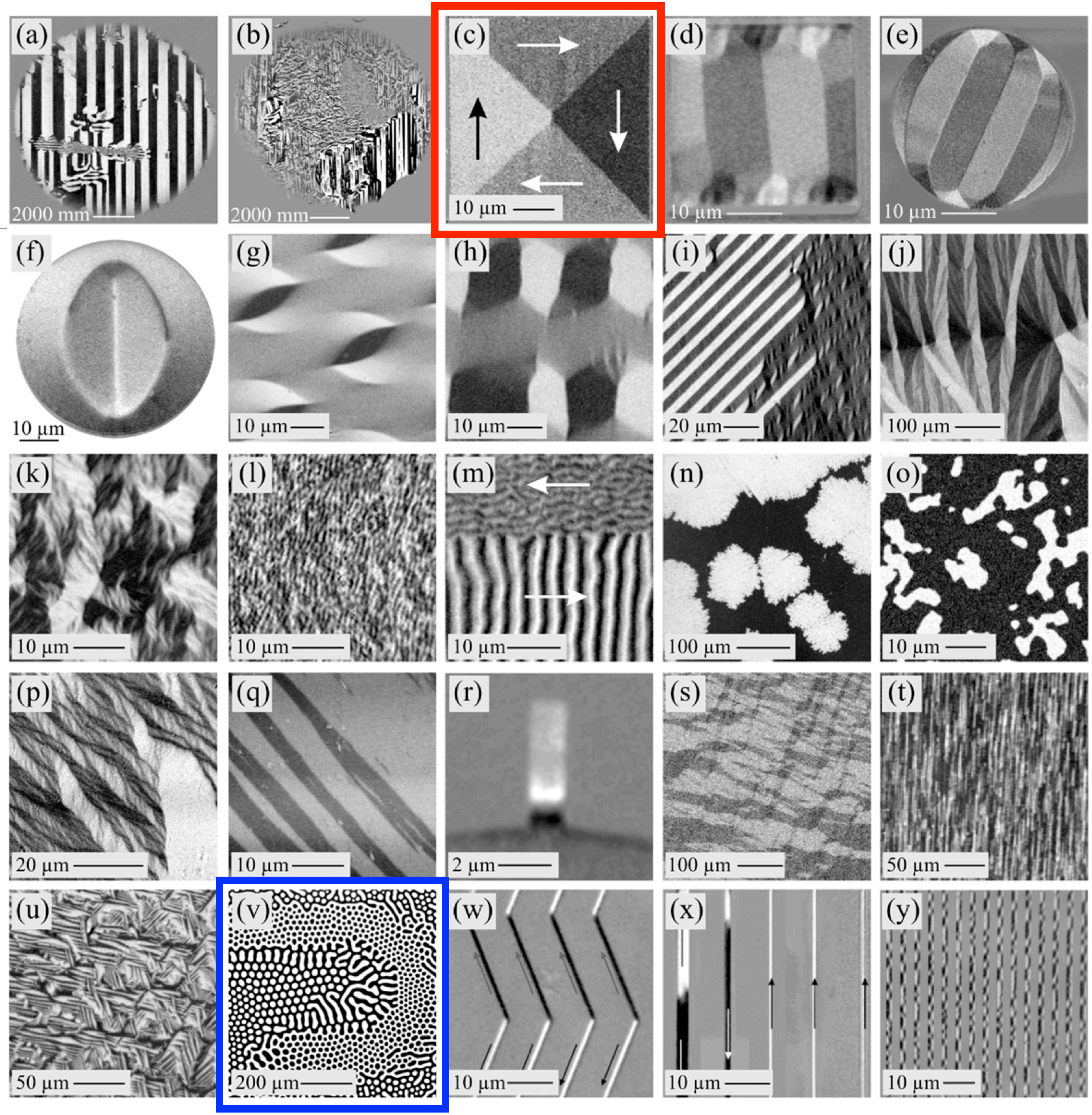
images borrowed from:
Tom Whyntie, (2016), zenodo.com
mammothmemory.net

Magnetic domains

- stray field frustrates the ferromagnetic order
- gives rise to a great variety of spin textures
- principle of pole avoidance
- out-of-plane magnetization: *more complicated*



J. McCord, *J. Phys. D: Appl. Phys.* **48**, 333001 (2015)

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Micromagnetic modeling framework

continuum mesoscopic theory $\Omega \subset \mathbb{R}^3$ $\mathbf{M} : \Omega \to \mathbb{R}^3$ $|\mathbf{M}| = M_s$

$$\boxed{\begin{aligned} E(\mathbf{M}) = {} & \frac{A}{M_s^2} \int_\Omega |\nabla \mathbf{M}|^2 \, \mathrm{d}^3 r + \frac{K}{M_s^2} \int_\Omega (M_1^2 + M_3^2) \mathrm{d}^3 r \\ & - \mu_0 \int_\Omega \mathbf{M} \cdot \mathbf{H} \, \mathrm{d}^3 r + \mu_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{M}(\mathbf{r}) \, \nabla \cdot \mathbf{M}(\mathbf{r}')}{8\pi |\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3 r \, \mathrm{d}^3 r' \end{aligned}}$$

Landau and Lifshitz, 1935; Néel, 1944; Kittel, 1949; Brown, 1963; Hubert and Schaefer, 1998
De Simone, Kohn, Müller and Otto, 2006

statics:
the observed magnetization patterns are local or global energy minimizers

dynamics:
the Landau-Lifshitz-Gilbert equation (in the Landau-Lifshitz form)

$$(1 + \alpha^2) \frac{\partial \mathbf{M}}{\partial t} = -\gamma_0 \left(\mathbf{M} \times \mathbf{H}_{\text{eff}} + \frac{\alpha}{M_s} \mathbf{M} \times \mathbf{M} \times \mathbf{H}_{\text{eff}} \right), \qquad \mathbf{H}_{\text{eff}} = -\frac{1}{\mu_0} \frac{\delta E}{\delta \mathbf{M}}$$

stochasticity can be added

Stray field energy

definition:

$$\boxed{E_{\mathrm{s}}(\mathbf{M}) = \frac{\mu_0}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\operatorname{div} \mathbf{M}(\mathbf{r}) \operatorname{div} \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3 r \, d^3 r'}$$

equivalent to:

$$E_{\mathrm{s}}(\mathbf{M}) = \frac{\mu_0}{2} \int_{\Omega} \mathbf{M} \cdot \nabla U_{\mathrm{d}} \, d^3 r = \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla U_{\mathrm{d}}|^2 \, d^3 r \qquad \Delta U_{\mathrm{d}} = \operatorname{div} \mathbf{M}$$

leading to the static Maxwell's equations for $\mathbf{H}_{\mathrm{d}} = -\nabla U_{\mathrm{d}}$

$$\operatorname{div} \mathbf{B}_{\mathrm{d}} = 0, \qquad \operatorname{curl} \mathbf{H}_{\mathrm{d}} = \mathbf{0},$$

coupled through the magnetization of the material:

$$\mathbf{B}_{\mathrm{d}} = \mu_0 (\mathbf{H}_{\mathrm{d}} + \mathbf{M}).$$

Remark: can also use the vector potential: $\mathbf{B} = \operatorname{curl} \mathbf{A}$, in the Coulomb gauge:

$$\operatorname{curl} (\operatorname{curl} \mathbf{A}_{\mathrm{d}}) = -\Delta \mathbf{A}_{\mathrm{d}} = \mu_0 \operatorname{curl} \mathbf{M} \qquad \operatorname{div} \mathbf{A}_{\mathrm{d}} = 0$$

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Variational principles for magnetostatic energy

a minimax principle:

Brown, 1963
James and Kinderlehrer, 1990

$$E_s(\mathbf{M}) = \max_{U \in \mathring{H}^1(\mathbb{R}^3)} \mu_0 \int_{\mathbb{R}^3} \left(\mathbf{M} \cdot \nabla U - \frac{1}{2} |\nabla U|^2 \right) d^3 r$$

maximize in U at fixed $\mathbf{M}$, then minimize in $\mathbf{M}$

another representation:

$$E_s(\mathbf{M}) = \frac{1}{2} \int_\Omega \left(\mu_0 |\mathbf{M}|^2 - \mathbf{M} \cdot \operatorname{curl} \mathbf{A}_d \right) d^3 r = \frac{1}{2\mu_0} \int_{\mathbb{R}^3} |\operatorname{curl} \mathbf{A}_d - \mu_0 \mathbf{M}|^2 \, d^3 r$$

leads to

Asselin and Thiele, 1986

$$E_s(\mathbf{M}) = \min_{\substack{\mathbf{A} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3) \\ \operatorname{div} \mathbf{A} = 0}} \frac{1}{2\mu_0} \int_{\mathbb{R}^3} |\operatorname{curl} \mathbf{A} - \mu_0 \mathbf{M}|^2 \, d^3 r$$

new minimization principle:

see also Coulomb, 1981

$$E_s(\mathbf{M}) = \frac{1}{2} \mu_0 M_s^2 V + \min_{\mathbf{A} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(\frac{1}{2\mu_0} |\nabla \mathbf{A}|^2 - \mathbf{M} \cdot \operatorname{curl} \mathbf{A} \right) d^3 r.$$

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Di Fratta, M, Rybakov and Slastikov, 2020

Bloch wall solution

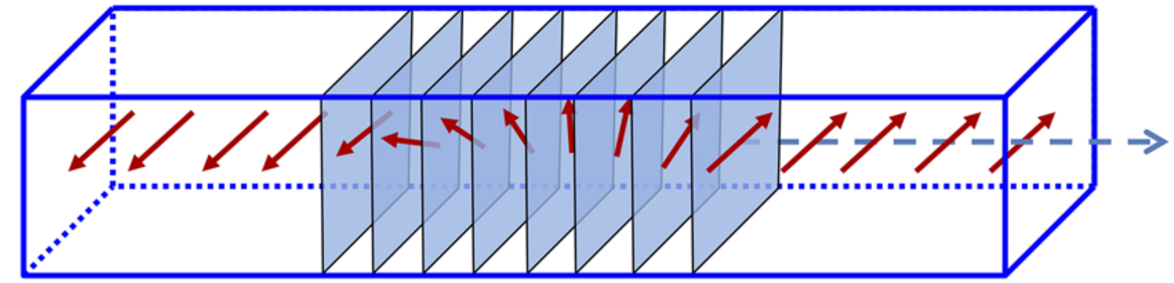

figure from Bhattia et al., 2019

one-dimensional transition profile in a bulk material

Bloch, 1932

$$E_{1\mathrm{d}}(\mathbf{M}) = \int_{-\infty}^{\infty} \left(\frac{A}{M_s^2} |\mathbf{M}'|^2 + \frac{K}{M_s^2}(M_1^2 + M_3^2) + \frac{\mu_0}{2} |\mathbf{H}_\mathrm{d}|^2 \right) dx$$

ansatz:

Landau and Lifshitz, 1935

$$\mathbf{M}_0(x) = M_s(0, \cos\theta(x), \sin\theta(x)) \qquad \nabla \cdot \mathbf{M}_0(x) = 0 \quad \Rightarrow \quad \mathbf{H}_\mathrm{d} = 0$$

Euler-Lagrange equation:

$$A\theta'' - K \sin\theta \cos\theta = 0 \qquad \theta(-\infty) = \pi \qquad \theta(+\infty) = 0$$

solution:

$$\boxed{\begin{gathered} \theta = \arccos(\tanh(x/L)) \qquad L = \sqrt{\frac{K}{A}} \\ E_{1\mathrm{d}}(\mathbf{M}_0) = 4\sqrt{AK} \end{gathered}}$$

<u>Remark</u>:

$$E_s(\mathbf{M}) = \frac{\mu_0}{2} \int_{-\infty}^{\infty} |M_1|^2 dx$$

Néel, 1944

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Bloch wall profile is energy minimizing

consider $\mathbf{M} = M_s \mathbf{m}$ $\quad \mathbf{m} : \mathbb{R} \to \mathbb{S}^2$

$$|\mathbf{m}|^2 = 1 \quad \Rightarrow \quad m_2^2 |m_2'|^2 = (m_1 m_1' + m_3 m_3')^2 \leq (m_1^2 + m_3^2)(|m_1'|^2 + |m_3'|^2)$$

together with the Modica-Mortola trick this yields

$$\begin{aligned} E_{1\mathrm{d}}(\mathbf{M}) &\geq \int_{\mathbb{R}} \big(A|\nabla \mathbf{m}|^2 + K(m_1^2 + m_3^2)\big) dx \\ &\geq \int_{\mathbb{R}} \left(A \frac{|m_2'|^2}{1 - m_2^2} + K(1 - m_2^2) \right) dx \\ &\geq 2\sqrt{AK} \int_{\mathbb{R}} |m_2'| dx \\ &\geq 4\sqrt{AK} = E_{1\mathrm{d}}(\mathbf{M}_0) \end{aligned}$$

the one-dimensional profile is globally optimal Garcia-Cervera, Ph.D. thesis (1999)

open question: is the Bloch profile the unique wall solution in 3D?

an example of a ***magnetic De Giorgi conjecture***

Bloch to Néel transition

thick films: *thin films:*

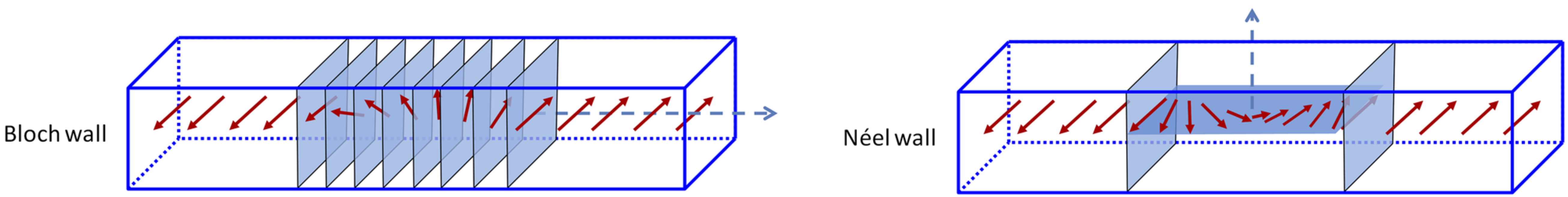


figure from Bhattia et al., 2019

surface charges penalize out of plane magnetization, forcing in-plane rotation in sufficiently thin film

Néel, 1955

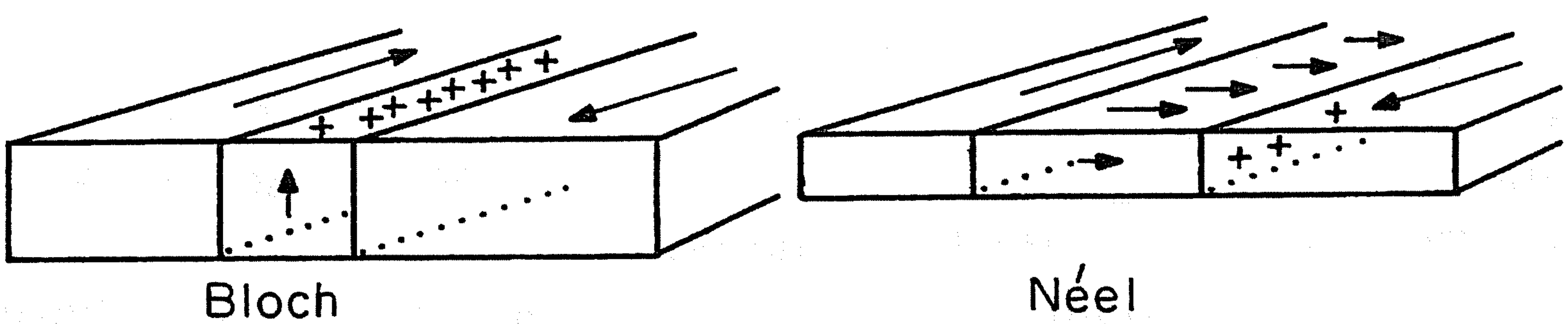
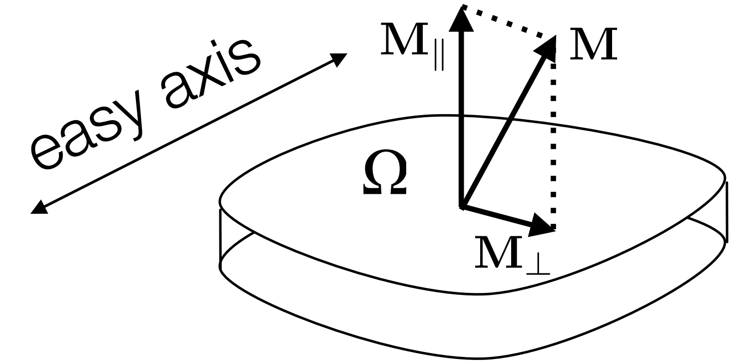


figure from R. O'Handley, *Modern Magnetic Materials* (1999)

Micromagnetics of thin films



consider a ferromagnetic film

$\Omega \subseteq \mathbb{R}^2$ - film shape

$$\boxed{\begin{aligned} E(\mathbf{M}) = {} & \frac{A}{M_s^2}\int_{\Omega\times(0,d)} |\nabla \mathbf{M}|^2\, d^3r + \frac{K}{M_s^2}\int_{\Omega\times(0,d)} \Phi(\mathbf{M}_\perp)\, d^3r \\ & - \mu_0 \int_{\Omega\times(0,d)} \mathbf{M}\cdot\mathbf{H}\, d^3r + \mu_0 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla\cdot\mathbf{M}(\mathbf{r})\,\nabla\cdot\mathbf{M}(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|}\, d^3r\, d^3r' \end{aligned}}$$

Here $\mathbf{M} = (\mathbf{M}_\perp, M_\|)$, $\quad \mathbf{M}_\perp \in \mathbb{R}^2 \quad M_\| \in \mathbb{R} \quad |\mathbf{M}| = M_s$ in $\Omega \times (0,d) \subset \mathbb{R}^3$

Parameters and their representative values:

- exchange constant $A = 1.4 \times 10^{-11}$ J/m
- anisotropy constant $K = 1 \times 10^5$ J/m^3
- saturation magnetization $M_s = 1.4 \times 10^6$ A/m
- applied field strength $\mu_0 H = 100$ mT

film thickness $d = 0.5 \div 5\,\mathrm{nm}$
lateral dimension:
$L = 50 \div 500\,\mathrm{nm}$

exchange length $\ell_{ex} = 3.37$ nm

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Need reduced micromagnetic models

analytically, the full 3D problem poses a **formidable challenge**:

- *vectorial*
- *nonlinear*
- *nonlocal*
- *multiscale*
- *topological constraints*

need a simplified model which is valid for the relevant parameter range and still captures quantitatively the physical features of the system

Solution: introduce reduced thin film models that are amenable to analysis

Use the tools from *rigorous asymptotic analysis* of calculus of variations

Dimension reduction

$$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$$

<u>assume</u> the magnetization $\mathbf{m} = \mathbf{M}/M_s$ does not vary significantly across the film thickness, measure lengths in the units of ℓ_{ex}, scale energy by Ad

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \Big(|\nabla \mathbf{m}|^2 + Q(\mathbf{e}_1 \cdot \mathbf{m}_\perp)^2 + (1+Q) m_\parallel^2 - 2\mathbf{h} \cdot \mathbf{m}_\perp + c(\mathbf{h}) \Big) d^2 r$$
$$+ \frac{1}{2\pi\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + \delta^2}} - 2\pi \delta^{(2)}(\mathbf{r} - \mathbf{r}')\delta \right) m_\parallel(\mathbf{r}) m_\parallel(\mathbf{r}') \, d^2 r \, d^2 r'$$
$$+ \delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_\delta(|\mathbf{r} - \mathbf{r}'|) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \, \nabla \cdot \mathbf{m}_\perp(\mathbf{r}') \, d^2 r \, d^2 r'$$

Here:

$$\boxed{Q = \frac{2K}{\mu_0 M_s^2} \qquad \mathbf{h} = \frac{\mathbf{H}}{M_s} \qquad \ell_{ex} = \sqrt{\frac{2A}{\mu_0 M_s^2}} \qquad \delta = \frac{d}{\ell_{ex}}}$$

$$K_\delta(r) = \frac{1}{2\pi\delta} \left\{ \ln \left(\frac{\delta + \sqrt{\delta^2 + r^2}}{r} \right) - \sqrt{1 + \frac{r^2}{\delta^2}} + \frac{r}{\delta} \right\} \simeq \frac{1}{4\pi r} \qquad \delta \ll 1$$

Garcia-Cervera, Ph.D. thesis (1999)

Reduced thin film energy

$$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$$

regime $\delta \ll 1$: Taylor-expand in Fourier space

$$\boxed{\begin{aligned} E(\mathbf{m}) \simeq & \int_{\mathbb{R}^2} \Big(|\nabla \mathbf{m}|^2 + Q(\mathbf{e}_1 \cdot \mathbf{m}_\perp)^2 + (1+Q) m_\parallel^2 - 2\mathbf{h} \cdot \mathbf{m}_\perp + c(\mathbf{h}) \Big) d^2 r \\ & + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r' \end{aligned}}$$

M, Slastikov, 2016

the expression for the stray field energy is rigorously justified via Γ-expansion

Knüpfer, M, Nolte, 2019

for bounded 2D samples, extra boundary terms appear

Di Fratta, M, Slastikov (in preparation)

proper definition of the non-local terms is via Fourier:

$$\frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{k}| \, |\widehat{m}_\parallel(\mathbf{k})|^2 \frac{d^2 k}{(2\pi)^2} = \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r', \quad \Big\} \text{ surface charges}$$

$$\frac{1}{2} \int_{\mathbb{R}^2} \frac{|\mathbf{k} \cdot \widehat{\mathbf{m}}_\perp(\mathbf{k})|^2}{|\mathbf{k}|} \frac{d^2 k}{(2\pi)^2} = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \, \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r'. \quad \Big\} \text{ volume charges}$$

Asymptotic development

$$\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$$

regime $\delta \ll 1$:

define $\overline{\mathbf{m}} = \frac{1}{\delta}\int_0^\delta \mathbf{m}(\cdot, z)dz$

$$\mathcal{E}_s(\mathbf{m}) = \frac{1}{\delta}\int_{\mathbb{T}_\ell \times (0,\delta)} |m_\parallel|^2 d^3r - \frac{\delta}{8\pi}\int_{\mathbb{T}_\ell}\int_{\mathbb{R}^2} \frac{(\overline{m}_\parallel(\mathbf{r}) - \overline{m}_\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r'$$
$$+ \frac{\delta}{4\pi}\int_{\mathbb{T}_\ell}\int_{\mathbb{R}^2} \frac{\nabla\cdot\overline{\mathbf{m}}_\perp(\mathbf{r})\nabla\cdot\overline{\mathbf{m}}_\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r\, d^2r'$$

Theorem *There exists a universal constant $C > 0$ such that*

$$|\mathcal{E}_s(\mathbf{m}) - E_s(\mathbf{m})| \le C\delta \int_{\mathbb{T}_\ell \times (0,\delta)} |\nabla \mathbf{m}|^2 d^3r$$

the difference between the energies is <u>lower order</u> in $\delta \ll 1$ $\quad \sim \mathcal{E}(\mathbf{m})\delta^2$

Knüpfer, M, Nolte, 2019

Soft thin films

thin film parameter

$$\nu = \frac{\mu_0 M_s^2 d}{2\sqrt{AK}}$$

regime $\delta \ll 1,\ Q = O(\delta^2),\ |\mathbf{h}| = O(\delta^2)$: $\qquad \mathbf{m} = (m_1, m_2, 0)$

$$E(\mathbf{m}) \simeq \frac{1}{2}\int_\Omega \left(|\nabla \mathbf{m}|^2 + m_1^2 - 2\mathbf{h}\cdot\mathbf{m}\right) d^2r + \frac{\nu}{8\pi}\int_\Omega\int_\Omega \frac{\nabla\cdot\mathbf{m}(\mathbf{r})\,\nabla\cdot\mathbf{m}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\, d^2r\, d^2r'$$
$$-\frac{\nu}{4\pi}\int_\Omega\int_{\partial\Omega} \frac{\nabla\cdot\mathbf{m}(\mathbf{r})(\mathbf{m}(\mathbf{r}')\cdot\mathbf{n}(\mathbf{r}'))}{|\mathbf{r}-\mathbf{r}'|}\, d\mathcal{H}^1(\mathbf{r}')\, d^2r$$
$$+\frac{\nu \ln \delta^{-1}}{4\pi}\int_{\partial\Omega} (\mathbf{m}(\mathbf{r})\cdot\mathbf{n}(\mathbf{r}))^2\, d\mathcal{H}^1(\mathbf{r})$$

lengths in units of L:

$$L = \sqrt{\frac{K}{A}}$$

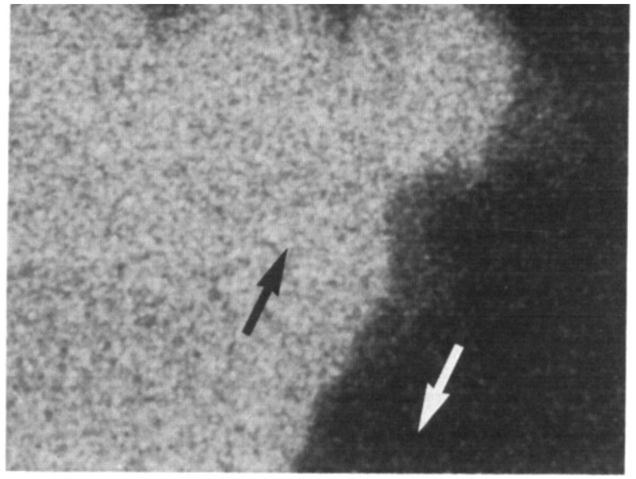
Lund, M, Slastikov, 2018

M and Osipov, 2006; Capella, Melcher and Otto, 2007; Kohn and Slastikov, 2005

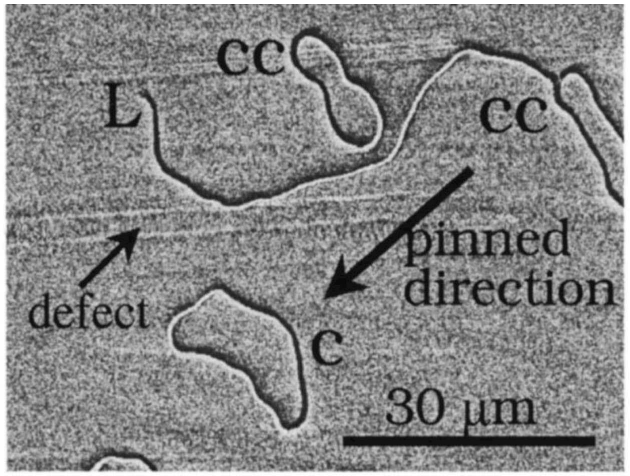
in the whole plane: $\qquad \mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^1$

$$E(\mathbf{m}) \simeq \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla \mathbf{m}|^2 + (m_1 - h_1)^2 + 2h_2(1-m_2)\Big) d^2r$$
$$+\frac{\nu}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\cdot\mathbf{m}(\mathbf{r})\,\nabla\cdot\mathbf{m}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\, d^2r\, d^2r'$$

Uncharged Néel walls

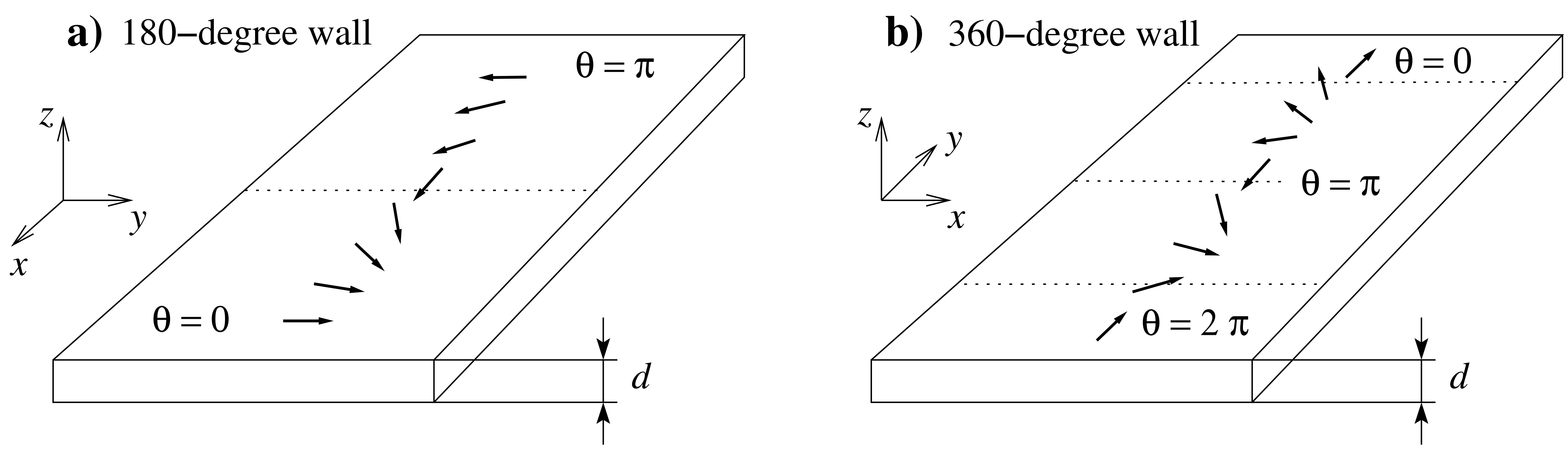


Oepen, 1991



Cho et al., 1999

two basic *charge-free* wall types with zero applied field:



<u>Note</u>: other wall types are possible with non-zero in-plane field

Ignat and Moser, 2017

$$\mathbf{m} = (-\sin\theta, \cos\theta)$$

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No stray field: De Giorgi conjecture

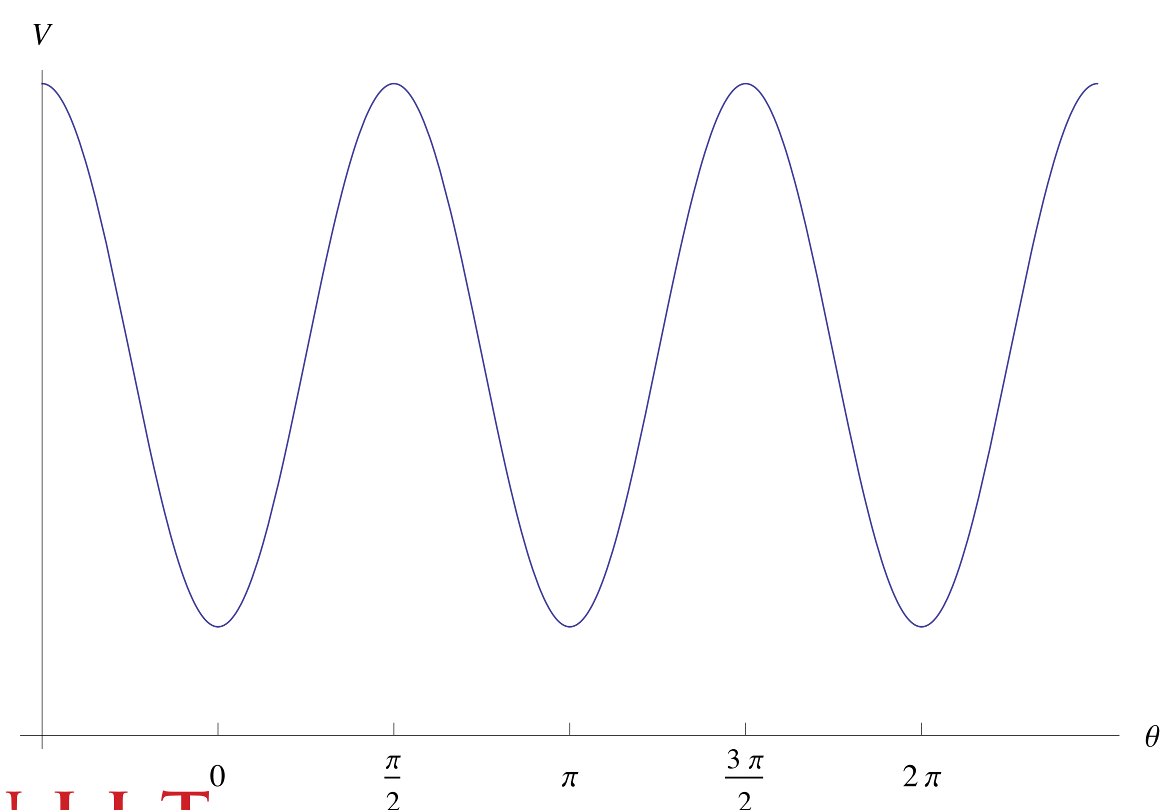
Set $\nu = 0$ (ultra-thin, not so soft materials, e.g. Co films 1 nm thick):

$$\Delta\theta = V'(\theta) \qquad V(\theta) = \frac{1}{2}\sin^2\theta + h_1 \sin\theta - h_2\cos\theta \quad \text{in} \quad \mathbb{R}^2$$

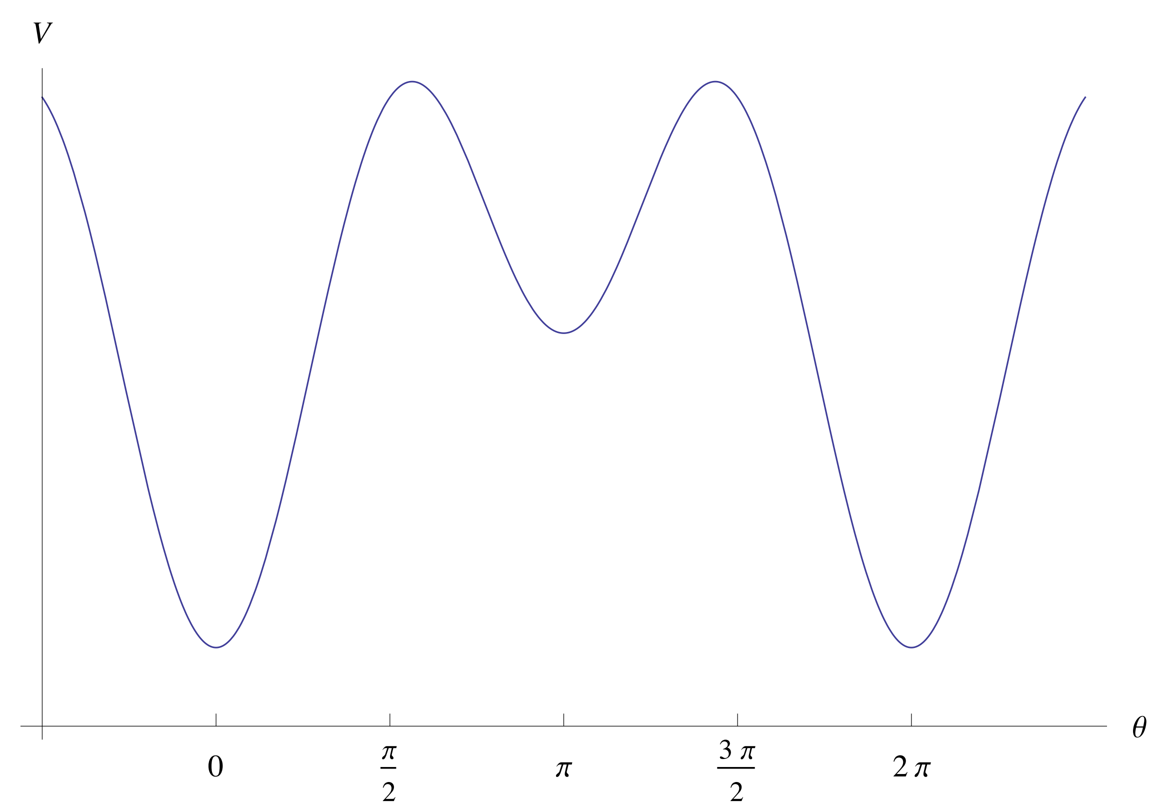
Ghoussoub and Gui, 1998
Ambrosio and Cabre, 2000
Savin, 2009

all bounded monotone solutions are one-dimensional

$h_1 = 0$, $h_2 = 0$: only $180°$-walls



$h_1 = 0$, $h_2 > 0$: only $360°$-walls

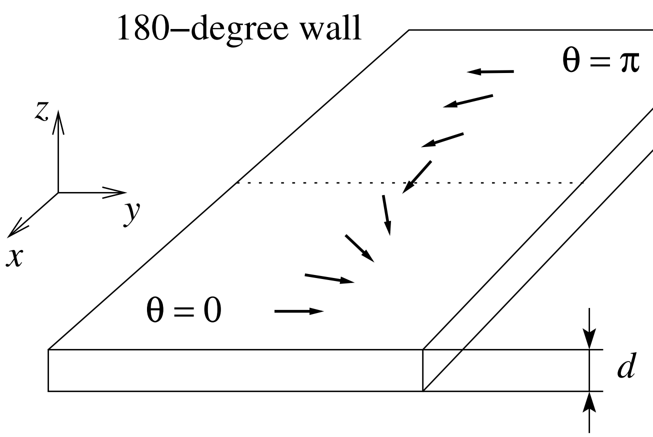


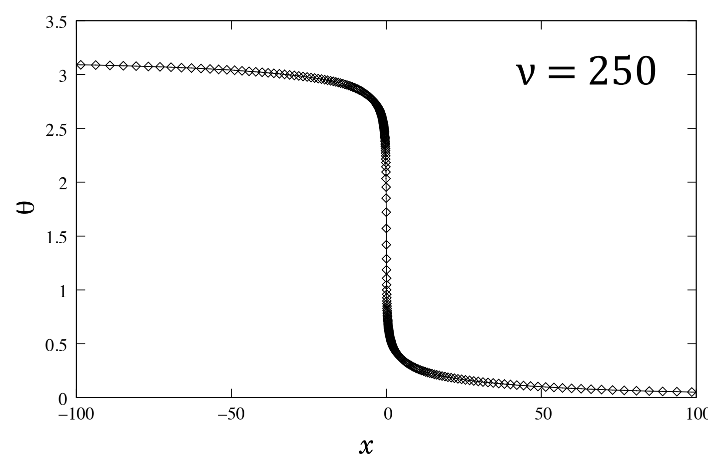
Uncharged 180-degree walls

$$\theta_h := \arcsin h \in \left[0, \frac{\pi}{2}\right)$$

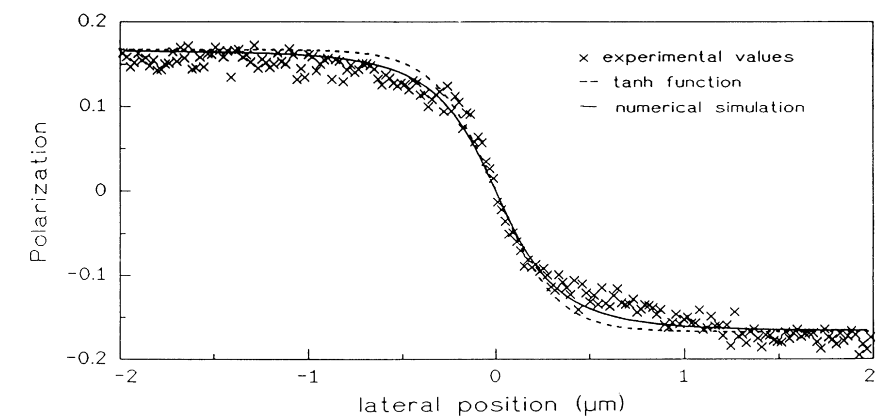
at zero field, 1D minimizers first studied by Melcher in a closely related model

Garcia-Cervera, 1999; Melcher, 2003; see also Garcia-Cervera, 2004; Capella, Melcher and Otto, 2007





M and Osipov, 2006



Berger and Oepen, 1992

variational formulation:

$$\boldsymbol{m}(x) = (-\sin\vartheta(x), \cos\vartheta(x)) \in \mathbb{S}^1$$

$$\boxed{E(\vartheta;\mathbb{R}) = \frac{1}{2}\int_{\mathbb{R}}\left(|\vartheta_x|^2 + (\sin\vartheta - h)^2\right)\mathrm{d}x + \frac{\nu}{8\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\left(\sin\vartheta(x) - \sin\vartheta(y)\right)^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y}$$

minimize in $\mathcal{A} := \{\vartheta \in H^1_{\mathrm{loc}}(\mathbb{R}) : \vartheta - \eta_h \in H^1(\mathbb{R})\}$ $\qquad \eta_h \in C^\infty(\mathbb{R}; [0,\pi])$

$$\eta_h = \pi - \theta_h \text{ in } (-\infty, -1) \text{ and } \eta_h = \theta_h \text{ in } (1, +\infty)$$

Chermisi and M, 2013

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Uncharged 180-degree walls

$$\boxed{\begin{aligned} E(\vartheta;\mathbb{R}) &= \frac{1}{2}\int_{\mathbb{R}}\left\{|\vartheta'|^2 + (\sin\vartheta - h)^2\right\}dx \\ &+ \frac{\nu}{8\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\sin\vartheta(x)-\sin\vartheta(y))^2}{(x-y)^2}dxdy \end{aligned}}$$

Euler-Lagrange equation

$$\lim_{x\to+\infty}\vartheta(x) = \theta_h, \qquad \lim_{x\to-\infty}\vartheta(x) = \pi - \theta_h.$$

$$\boxed{-\vartheta_{xx} + \cos\vartheta\sin\vartheta - h\cos\vartheta + \frac{\nu}{2}\cos\vartheta\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\sin\vartheta = 0}$$

compare with Palatucci, Savin and Valdinoci, 2013

Theorem (existence, uniqueness, regularity, strict monotonicity and decay) *For every $\nu > 0$ and every $h \in [0,1)$ there exists a minimizer of $E(\vartheta;\mathbb{R})$ over $\mathcal{A}$, which is unique (up to translations), strictly decreasing with range equal to $(\vartheta_h, \pi - \vartheta_h)$ and is a smooth solution of the Euler-Lagrange equation that satisfies the limit conditions. Moreover, if $\vartheta^{(0)}$ is the minimizer of E in the class $\mathcal{A}$ satisfying $\vartheta^{(0)}(0) = \frac{\pi}{2}$, then $\vartheta^{(0)}(x) = \pi - \vartheta^{(0)}(-x)$, and there exists a constant $c > 0$ such that $\vartheta^{(0)}(x) \simeq cx^{-2}$ as $x \to +\infty$.*

Chermisi and M, 2013

uniqueness in the class of monotone decreasing profiles — M and Yan, 2016

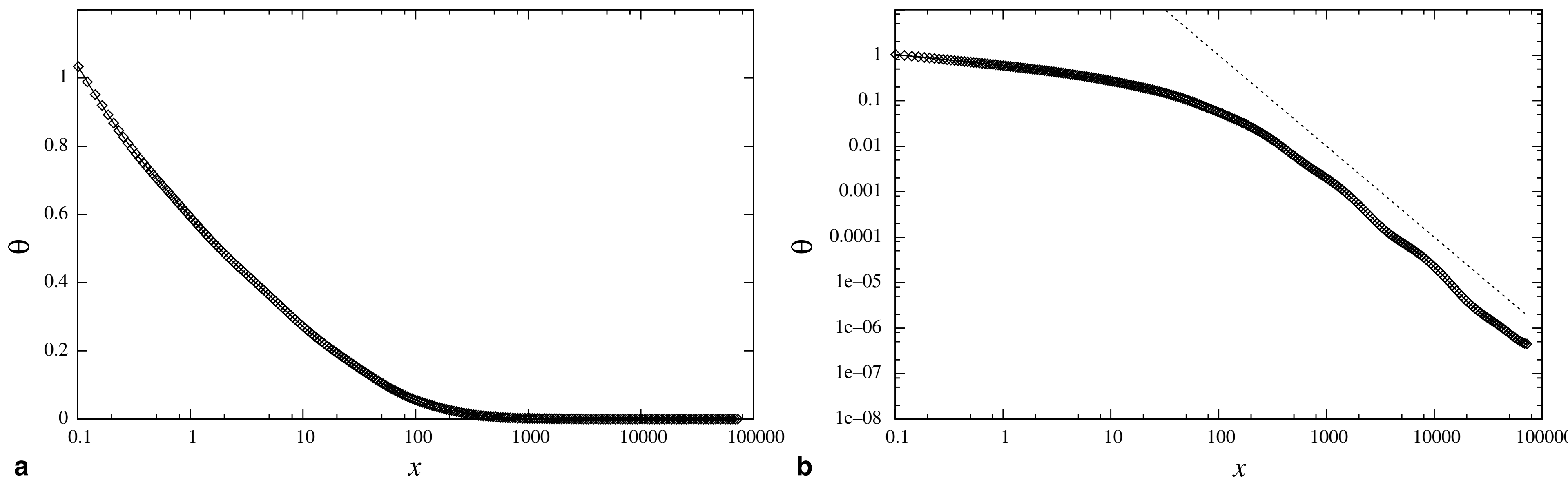
<u>open question</u>: does uniqueness hold in 2D (even in a periodic setting)?

for a result in this direction, see De Simone, Knüpfer and Otto, 2006

Multiscale structure of the tail

$$\rho(x) := \begin{cases} \vartheta(x) & \text{if } \vartheta(x) \in \left[\theta_h, \frac{\pi}{2}\right] \\ \pi - \vartheta(x) & \text{if } \vartheta(x) \in \left(\frac{\pi}{2}, \pi - \theta_h\right] \end{cases}$$

M and Osipov, 2006



$\nu = 250$

20nm thick permalloy

Riedel and Seeger, 1971
Garcia-Cervera, 1999
Melcher, 2003

for soft materials ($\nu \gg 1$) the wall exhibits logarithmic tails

delicate analysis of decay: write the ELE as $L(\rho(x) - \theta_h) = 2|\vartheta'(0)|\delta(x) + f(x)$

where $L := -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + \dfrac{1}{2}\nu\cos^2\theta_h\left(-\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} + \cos^2\theta_h$, then

$$\boxed{\rho - \theta_h = L^{-1}f(x) = 2|\vartheta'(0)|G(x) + \int_{\mathbb{R}} G(x-y)f(y)dy.} \qquad G(x) \overset{\nu\gg 1}{\simeq} \frac{2}{\pi\nu\cos^2\theta_h}\ln\left(\frac{\nu}{|x|}\right)$$

$$G(x) = \frac{2\nu}{\pi}\int_0^\infty \frac{t e^{-t|x|\cos\theta_h}}{\nu^2 t^2\cos^2\theta_h + 4(t^2-1)^2}\,\mathrm{d}t \qquad = \frac{\nu}{2\pi\cos^2\theta_h}|x|^{-2} + O(|x|^{-4})$$

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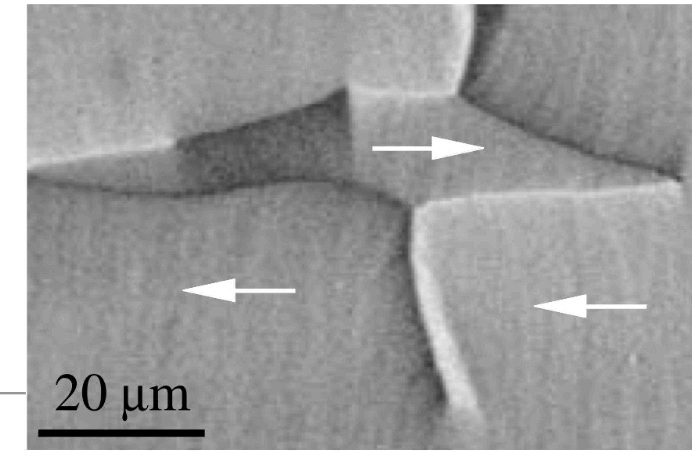
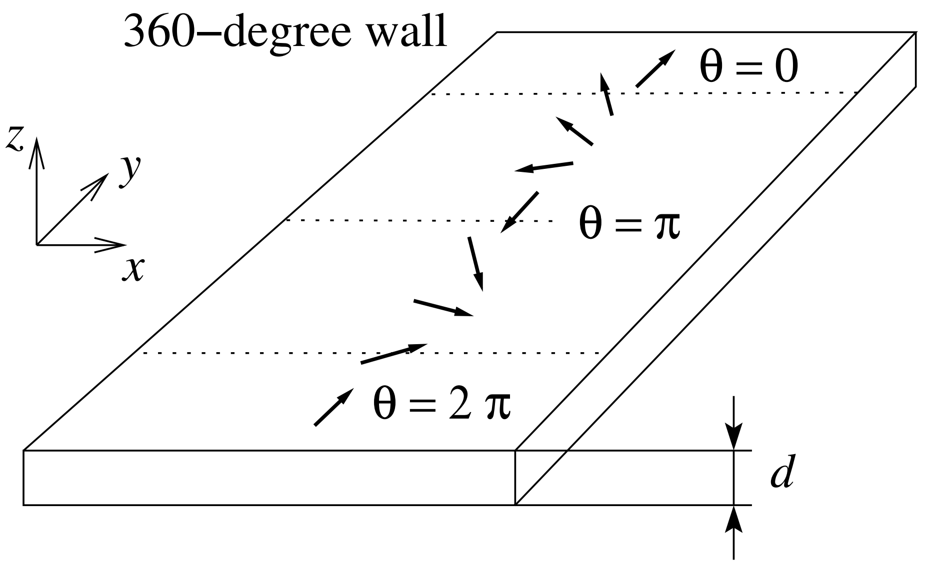
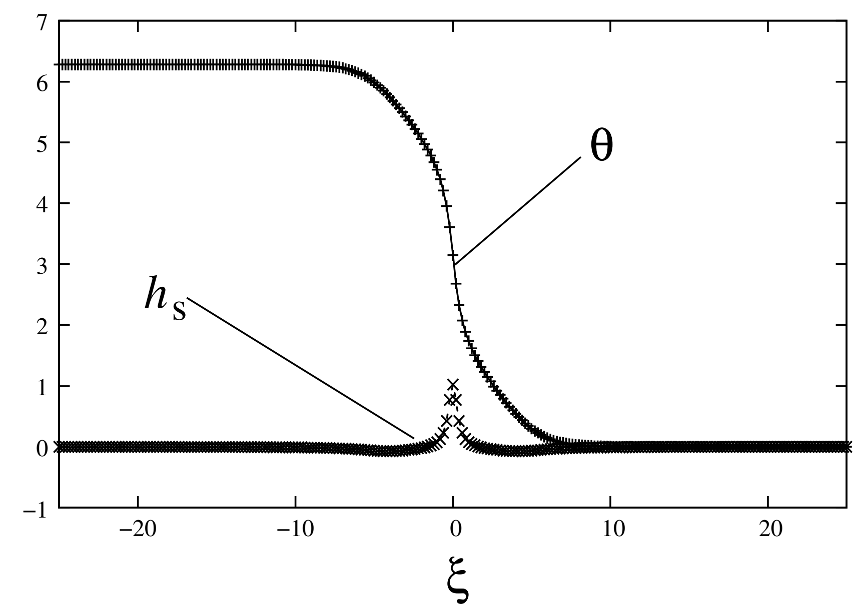
360-degree walls



from Hubert and Schäfer, 1999





$\nu = 10$
$\beta = \pi/2$

M and Osipov, 2008

minimize

$$\boxed{E_\beta(\vartheta) = \frac{1}{2}\int_{-\infty}^{\infty} \left(|\vartheta'|^2 + \sin^2\vartheta\right) dx + \frac{\nu}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\sin(\vartheta(x)-\beta) - \sin(\vartheta(y)-\beta))^2}{(x-y)^2}\, dx\, dy}$$

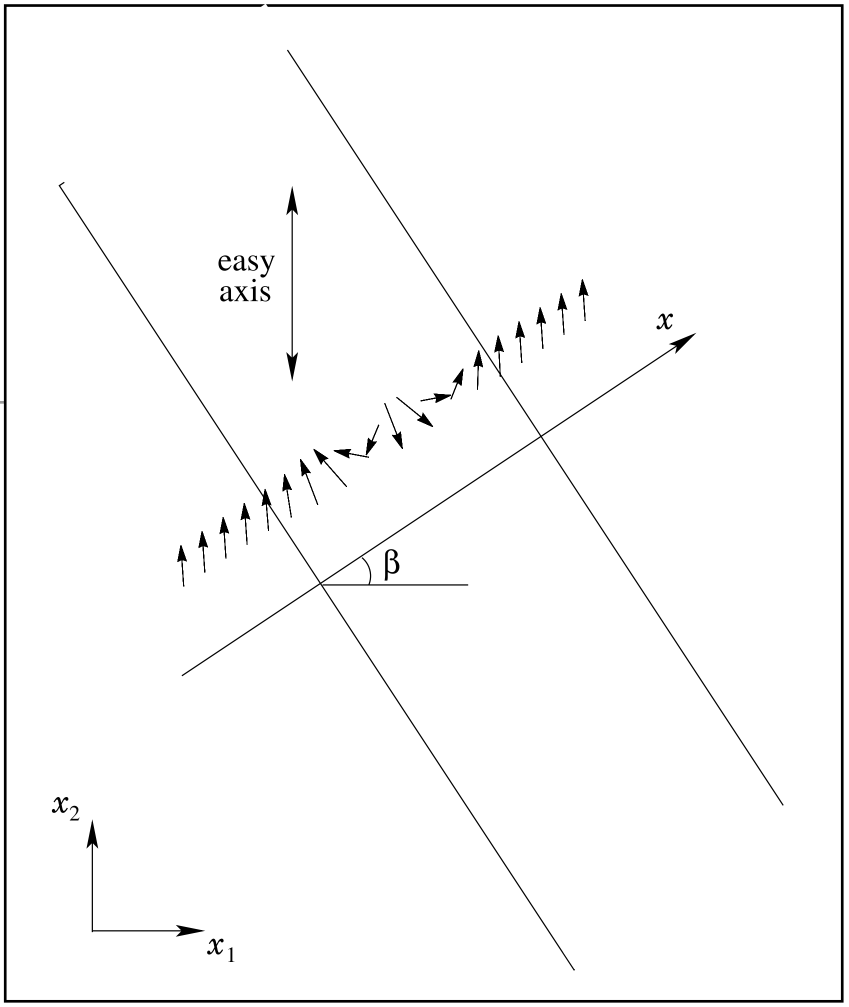
over

$$\mathcal{A} := \{\vartheta \in H^1_{loc}(\mathbb{R}) : \vartheta - \eta \in H^1(\mathbb{R})\}, \quad \eta(x) = 0,\ x > 1, \quad \eta(x) = 2\pi,\ x < -1.$$

ϑ varies in the direction $(\cos\beta, \sin\beta)$

<u>topologically nontrivial!</u>

360-degree walls



existence not a priori clear: no solutions for $\nu = 0$!

need to exclude splitting into two 180-degree walls

for β = π/2 symmetric decreasing rearrangement of 1 - cos ϑ reduces energy => monotonicity of $\vartheta(x)$

for x_1, x_2 such that $\vartheta(x_1) = \dfrac{3\pi}{4}$ and $\vartheta(x_2) = \dfrac{\pi}{4}$

$$E(\vartheta) \geq \frac{\nu}{2\pi} \int_0^{x_1} \int_{x_2}^{\infty} \frac{(\cos\vartheta(x) - \cos\vartheta(y))^2}{(x-y)^2} dx\, dy \geq \frac{\nu}{2\pi} \int_0^{x_1} \int_{x_2}^{\infty} \frac{dx\, dy}{(x-y)^2} = \frac{\nu}{\pi} \ln\left(\frac{x_2}{x_2 - x_1}\right)$$

Theorem (existence of 360° domain wall minimizers). *Let $\beta \in [0, \frac{\pi}{2}]$. Then the following holds:*

(i) If $\beta = 0$, then the infimum of E_β is not attained in $\mathcal{A}$.

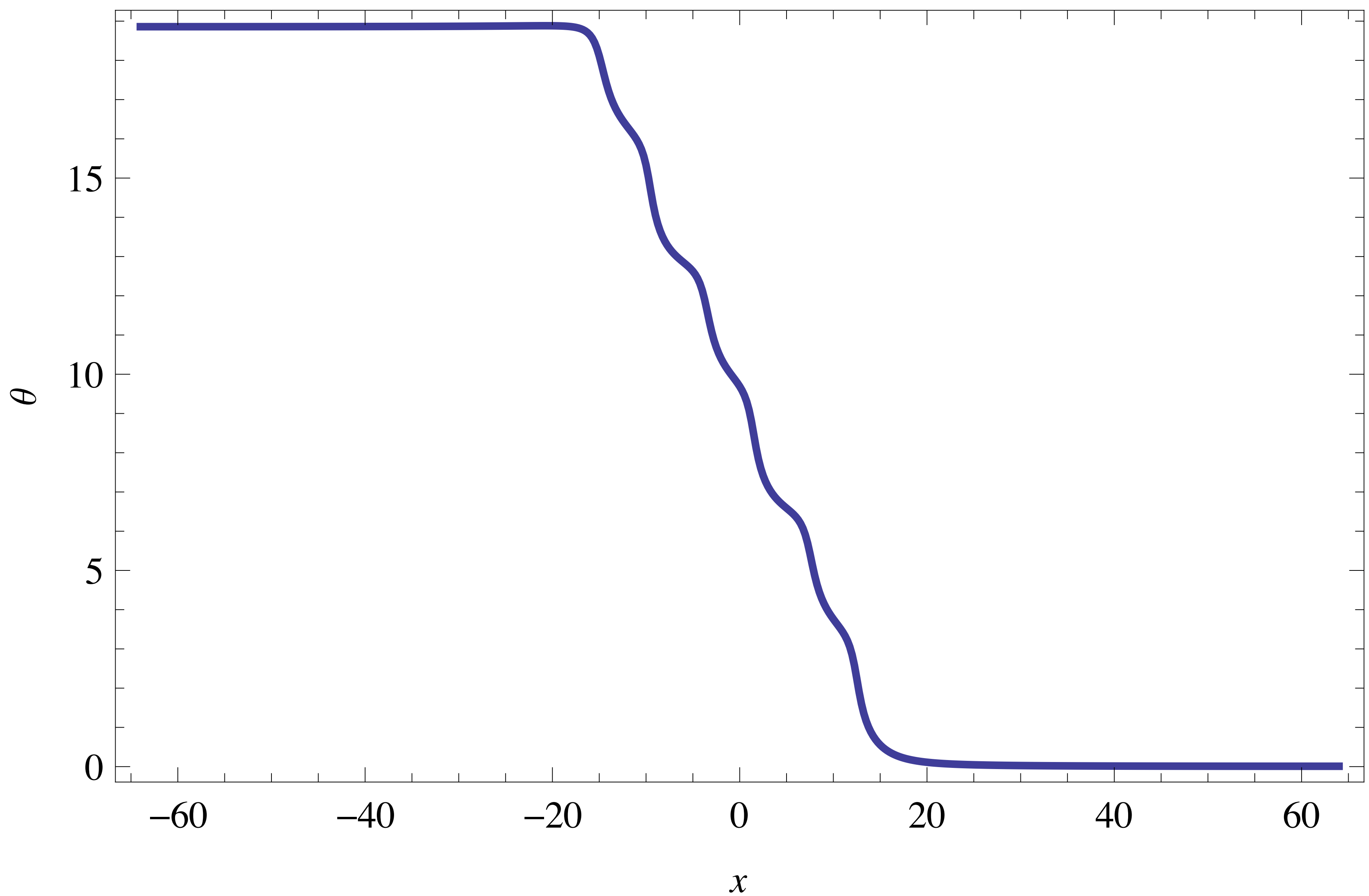
(ii) If $\beta \in (0, \frac{\pi}{2})$, then there is $\nu_0 = \nu_0(\beta) > 0$ such that for every $\nu \in (0, \nu_0)$ there exist a minimizer of E_β over $\mathcal{A}$.

(iii) If $\beta = \frac{\pi}{2}$, then for every $\nu > 0$ there exist a minimizer of E_β over $\mathcal{A}$.

Capella, Knüpfer and M, 2021 see also Ignat and Knüpfer, 2010; Ignat and Moser, 2017

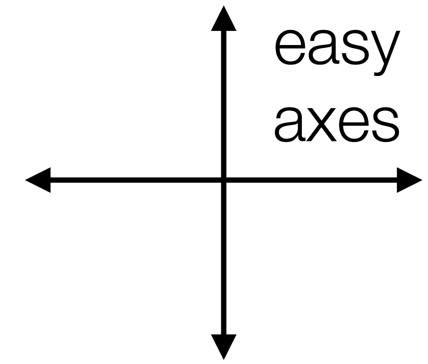
Walls with higher winding

example: 6π-wall, $\beta = \pi/4$, $\nu = 5$



different 2π segments carry like dipoles => attract each other

see also Ignat and Moser, 2017



Four-fold anisotropy

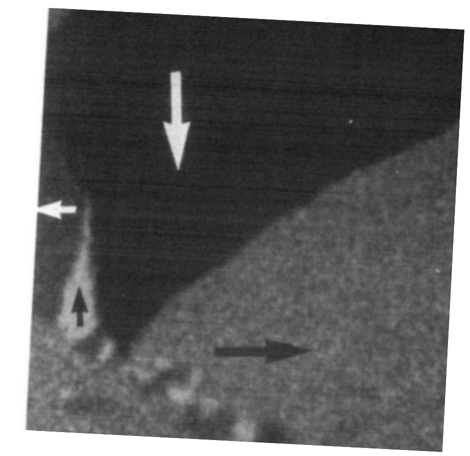
bulk cubic crystals exhibit 4-fold magnetocrystalline anisotropy

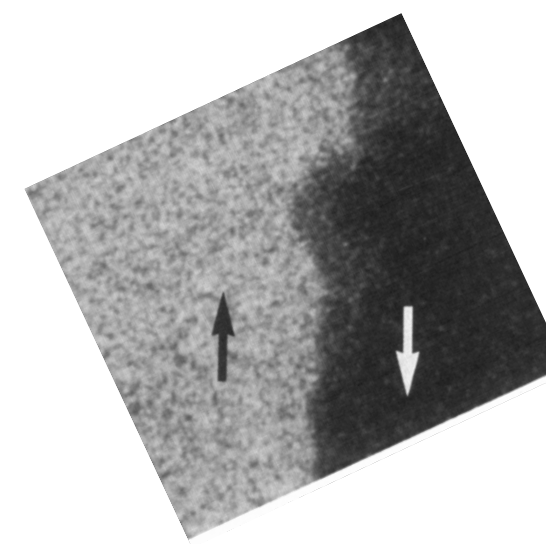
Theorem (90°-walls: existence, uniqueness, regularity and strict monotonicity). *For $\beta = -\pi/4$ and each $\nu > 0$, there exists a minimizer of the energy $E_\beta(\theta)$ over the admissible class $\mathcal{A}_{\pi/2}$. The minimizer is unique (up to translations), strictly decreasing with range equal to $(0, \pi/2)$, and is a smooth solution of the EL equation satisfying the limit conditions*

$$\lim_{x \to +\infty} \theta(x) = 0, \qquad \lim_{x \to -\infty} \theta(x) = \pi/2.$$

Moreover, if $\theta_{min} : \mathbb{R} \to (0, \pi/2)$ is the minimizer of $E_{-\pi/4}(\theta)$ over $\mathcal{A}_{\pi/2}$ satisfying $\theta_{min}(0) = \pi/4$, then $\theta_{min}(x) = \pi/2 - \theta_{min}(-x)$.

Lund and M, 2016

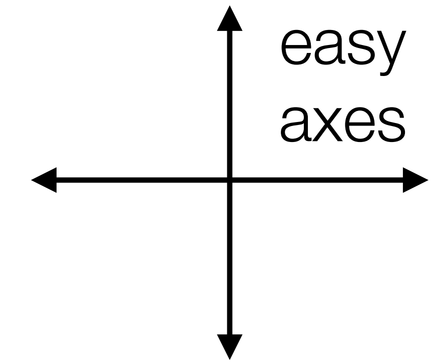




Oepen, 1991

$$\boxed{0 = -\theta_{xx} + \frac{1}{4}\sin 4\theta + \frac{\nu}{2}\cos(\theta - \beta)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\sin(\theta - \beta)}$$

$$\boxed{E_\beta(\vartheta) = \int_{-\infty}^{\infty}\left(\frac{1}{2}|\vartheta'|^2 + \frac{1}{8}\sin^2 2\vartheta\right)dx + \frac{\nu}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\sin(\vartheta(x) - \beta) - \sin(\vartheta(y) - \beta))^2}{(x-y)^2}\,dx\,dy}$$

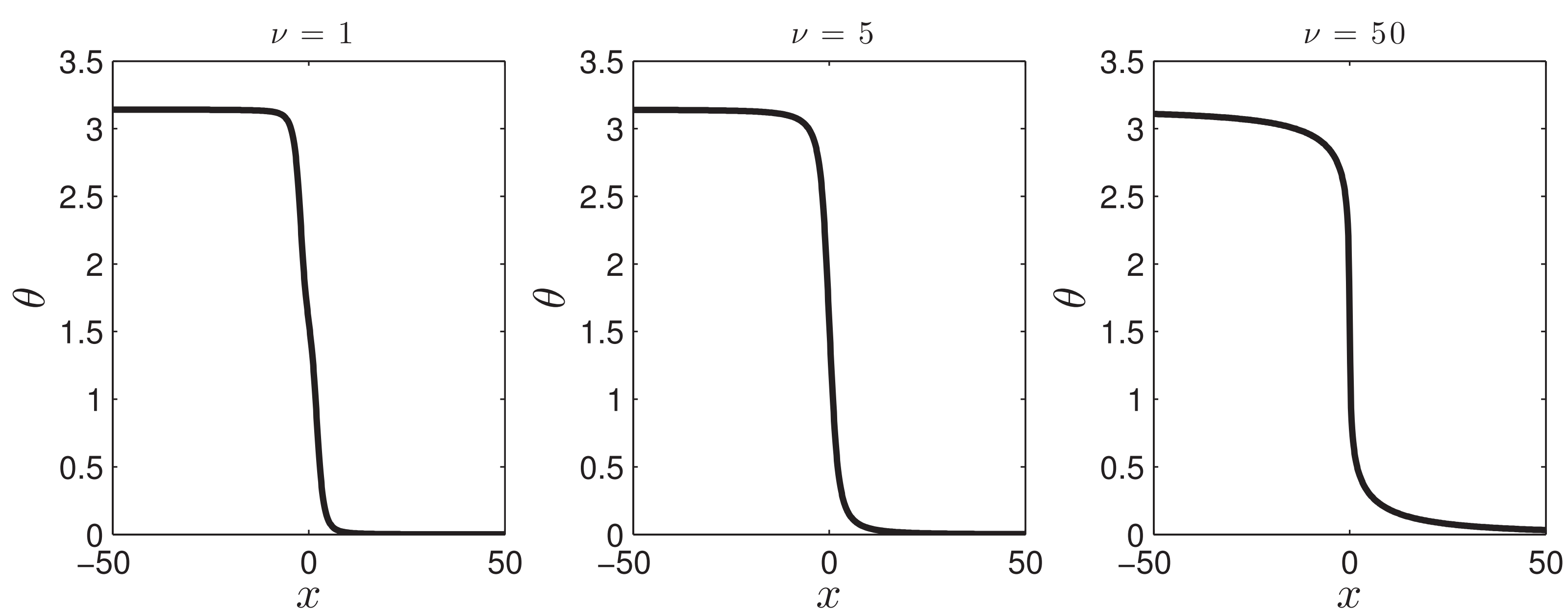


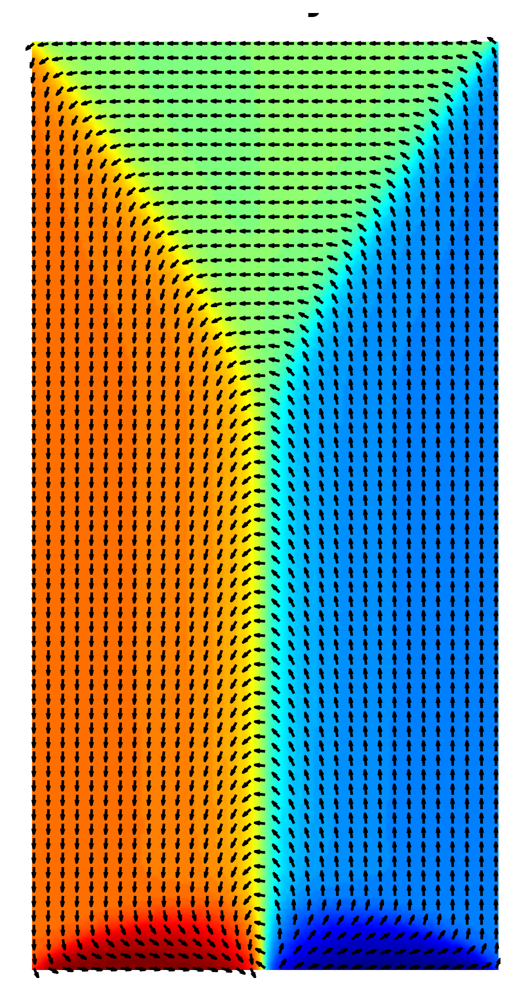
Four-fold anisotropy

Theorem (180°-walls: existence, regularity and strict monotonicity). *For $\beta = 0$ and each $\nu > 0$, there exists a minimizer of the energy $E_\beta(\theta)$ over the admissible class $\mathcal{A}_\pi$. The minimizer is strictly decreasing with range equal to $(0, \pi)$, and is a smooth solution of the EL equation satisfying the limit conditions*

$$\lim_{x \to +\infty} \theta(x) = 0, \qquad \lim_{x \to -\infty} \theta(x) = \pi.$$

Moreover, if $\theta_{min} : \mathbb{R} \to (0, \pi)$ is the minimizer of $E_0(\theta)$ over $\mathcal{A}_\pi$ satisfying $\theta_{min}(0) = \pi/2$, then $\theta_{min}(x) = \pi - \theta_{min}(-x)$.





$\nu = 10$
Lx=32, Ly=64

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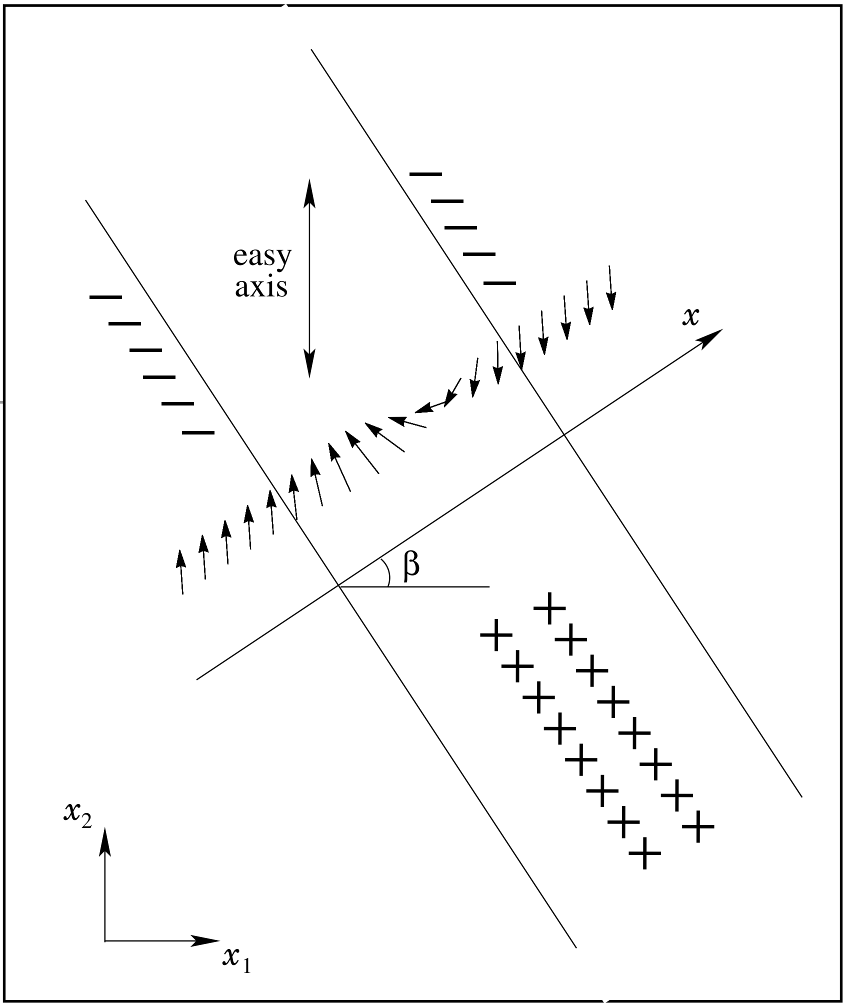
Lund and M, 2016

Charged Néel walls



180-degree wall carries a net line charge

energy **per unit wall length** is infinite:

<u>Consider</u> a *finite* strip of width w

If $m_\beta = m_1 \cos\beta + m_2 \sin\beta$ then

$$E_{\beta,w}(\mathbf{m}) = \frac{1}{2}\int_0^w \left(|m_1'|^2 + |m_2'|^2 + m_1^2\right) \mathrm{d}x + \frac{\nu}{4\pi}\int_0^w \int_0^w \ln|x-y|^{-1} m_\beta'(x) m_\beta'(y)\, \mathrm{d}x\, \mathrm{d}y.$$

at x = 0 and x = w enforce $\mathbf{m} = \pm(-\sin\beta, \cos\beta)$. Then a 180-degree wall

in the middle will have

$$E_{\beta,w}(\mathbf{m}) \gtrsim \frac{\nu}{4\pi}\left(4\sin^2\beta \ln\frac{w}{2} - \sin^2\beta \ln w\right) \to \infty \quad \text{as} \quad w \to \infty$$

a suitable *renormalization* of energy is necessary: **open problem**!

for a semilinear problems of this type, see Chen, M, and Yan, 2017

for a recent asymptotic study, see Knüpfer and Shi, 2020

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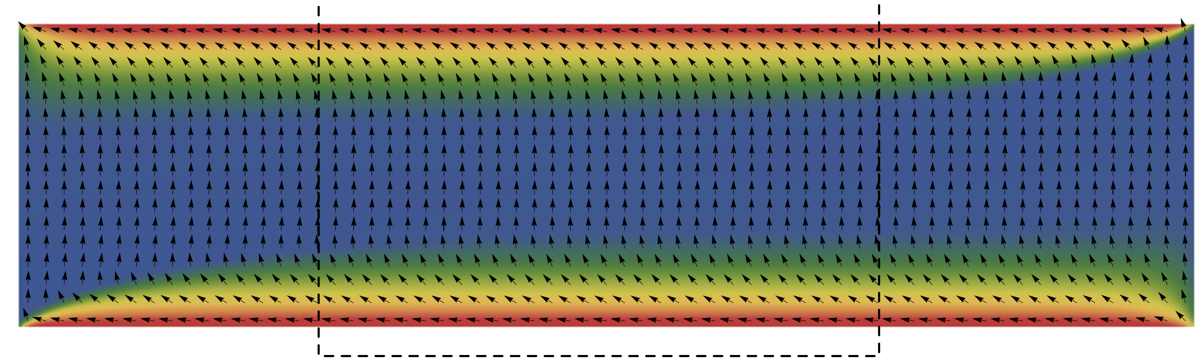
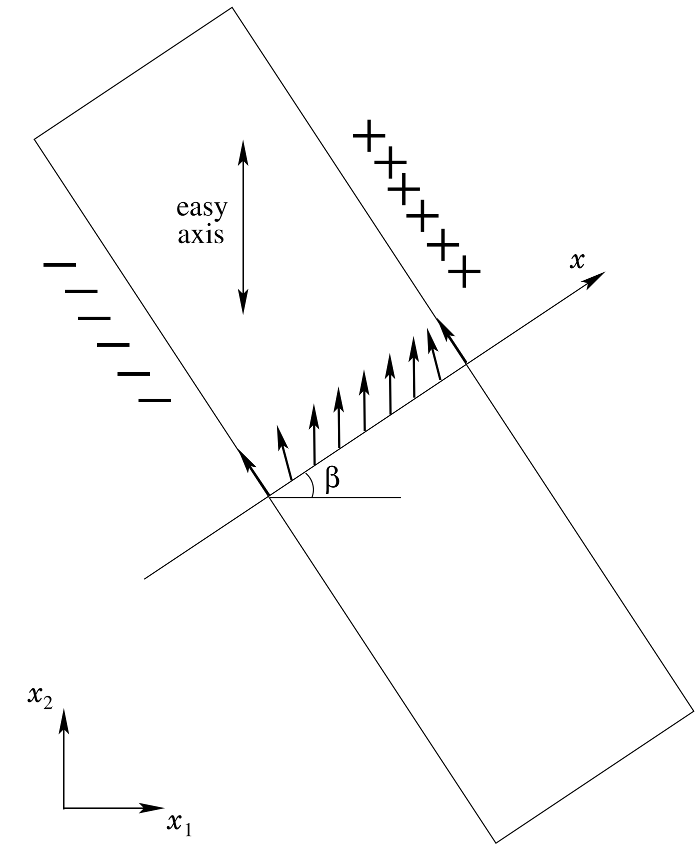
Edge walls



simulations of a 20.7 μm $\times$5.2 μm $\times$4 nm permalloy

the issue persists even without domain walls in the bulk

$\mathbf{m} = (-\sin\theta, \cos\theta)$, set $\theta(x) = \beta$ for all $x \notin (0, w)$



the 1D energy on a strip is

$$E_{\beta,w}(\theta) = \frac{1}{2}\int_0^w \left(|\theta'|^2 + \sin^2\theta\right) dx + \frac{\nu}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\sin(\theta(x)-\beta) - \sin(\theta(y)-\beta))^2}{(x-y)^2}\, dx\, dy$$

Euler-Lagrange equation:

$$0 = \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \sin\theta\cos\theta - \frac{\nu}{2}\cos(\theta-\beta)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\sin(\theta-\beta) \qquad x \in (0, w)$$

with Dirichlet “boundary” conditions

$$\boxed{w \to \infty}$$

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Renormalized energy

<u>problem</u>: the nonlocal term blows up as $w \to \infty$

define

$$\boxed{\begin{aligned} E_\beta(\theta) := & \frac{1}{2}\int_0^\infty \left(|\theta'|^2 + \sin^2\theta\right) dx + \frac{\nu}{8\pi}\times \\ & \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{(\sin(\theta(x)-\beta) - \sin(\theta(y)-\beta))^2 - (\sin(\eta_\beta(x)-\beta) - \sin(\eta_\beta(y)-\beta))^2}{(x-y)^2}\, dx\, dy \end{aligned}}$$

smooth non-increasing cutoff $\eta_\beta : \mathbb{R} \to [0, \beta]$

$$\eta_\beta(x) = \beta \text{ when } x \leqslant 0, \qquad \eta_\beta(x) = 0 \text{ when } x \geqslant 1$$

hence

$$\begin{aligned} E_\beta(\theta) = & \int_0^\infty \left(\frac{1}{2}|\theta'|^2 + \frac{1}{2}\sin^2\theta + \frac{\nu}{4\pi}\cdot \frac{\sin^2(\theta-\beta) - \sin^2(\eta_\beta - \beta)}{x} \right) \mathrm{d}x \\ & + \frac{\nu}{8\pi}\int_0^\infty \int_0^\infty \frac{(\sin(\theta(x)-\beta) - \sin(\theta(y)-\beta))^2}{(x-y)^2}\, \mathrm{d}x\, \mathrm{d}y \\ & - \frac{\nu}{8\pi}\int_0^\infty \int_0^\infty \frac{(\sin(\eta_\beta(x)-\beta) - \sin(\eta_\beta(y)-\beta))^2}{(x-y)^2}\, \mathrm{d}x\, \mathrm{d}y, \end{aligned}$$

Minimizers

admissible class

$$\mathcal{A} := \left\{ \theta \in C\left(\overline{\mathbb{R}^+}\right) : \theta - \beta \in \mathring{H}_0^1(\mathbb{R}^+) \right\}$$

Theorem 1. *For each $\beta \in (0, \pi/2]$ and each $\nu > 0$, there exists $\theta \in \mathcal{A}$ such that $E_\beta(\theta) = \inf_{\tilde{\theta} \in \mathcal{A}} E(\tilde{\theta})$. Furthermore, we have $\theta \in L^\infty(\mathbb{R}^+)$ and $\lim_{x\to\infty} \theta(x) = \theta_\infty$ for some $\theta_\infty \in \pi\mathbb{Z}$.*

Euler-Lagrange equation (in the classical form):

$$\begin{aligned} \theta''(x) &= \sin\theta(x)\cos\theta(x) + \frac{\nu}{2\pi} \cdot \frac{\sin(\theta(x)-\beta)\cos(\theta(x)-\beta)}{x} \\ &\quad + \frac{\nu}{2\pi}\cos(\theta(x)-\beta)\left(\fint_0^\infty \frac{\sin(\theta(x)-\beta) - \sin(\theta(y)-\beta)}{(x-y)^2}\,\mathrm{dy}\right) \qquad \forall x > 0. \end{aligned} \tag{$*$}$$

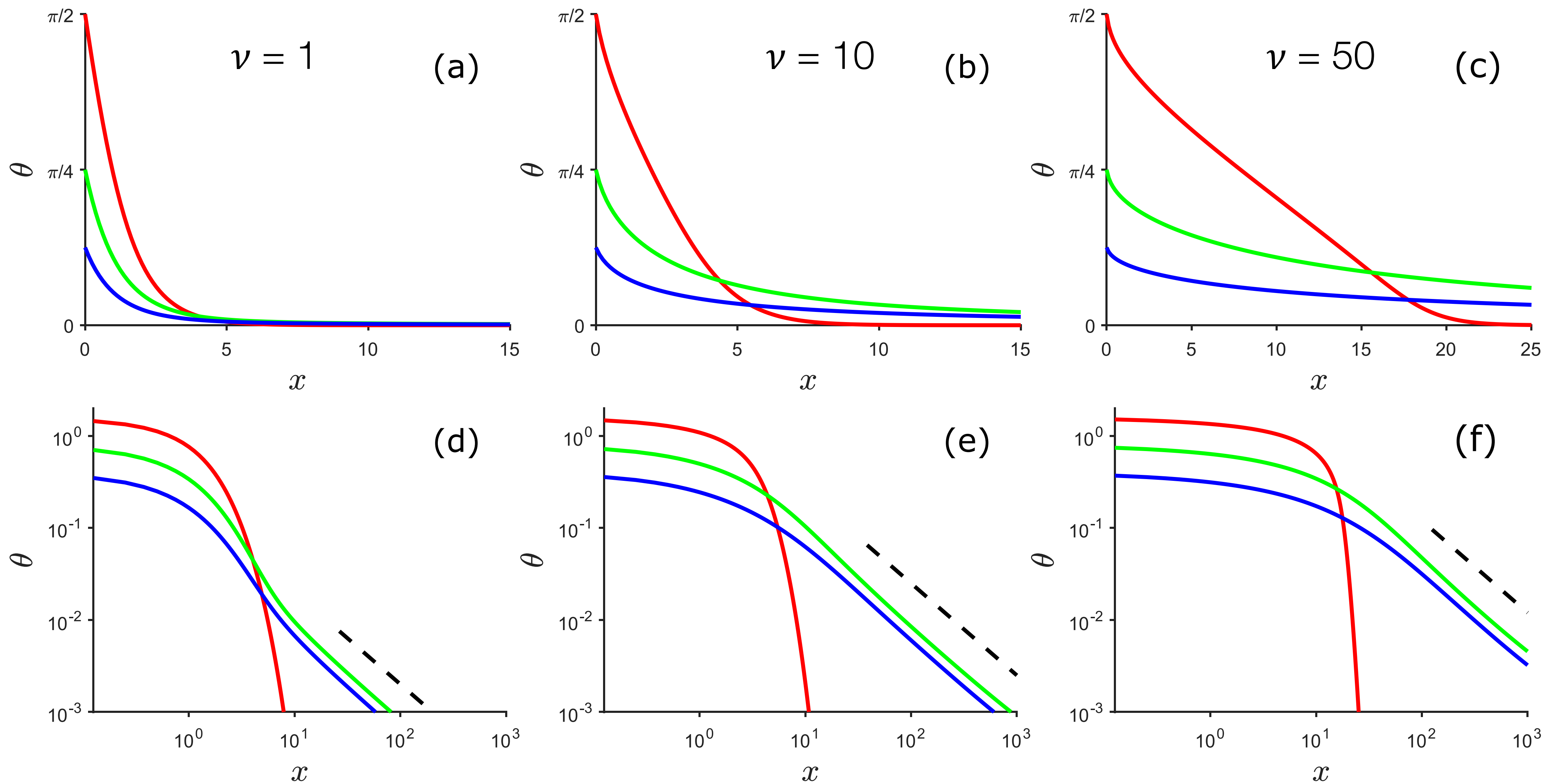
Theorem 2. *For each $\beta \in (0, \pi/2]$ and each $\nu > 0$, let θ be a minimizer from theorem 1. Then $\theta \in C^2(\mathbb{R}^+) \cap C^1(\overline{\mathbb{R}^+}) \cap W^{1,\infty}(\mathbb{R}^+)$ and $(*)$ holds. In addition, we have $|\theta'(0)| = \sin\beta$ and $\lim_{x\to 0^+} |\theta''(x)| = \infty$.*

Lund, M and Slastikov, 2018

Further remarks

given $\nu > 0$, for all $0 < \beta \leq \beta_0(\nu)$ the minimizer is unique and goes to zero

=> <u>no winding</u> for all $\beta \ll 1$



dashed line indicates $1/x$ decay

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Lund, M and Slastikov, 2018

Monotonicity? Winding?

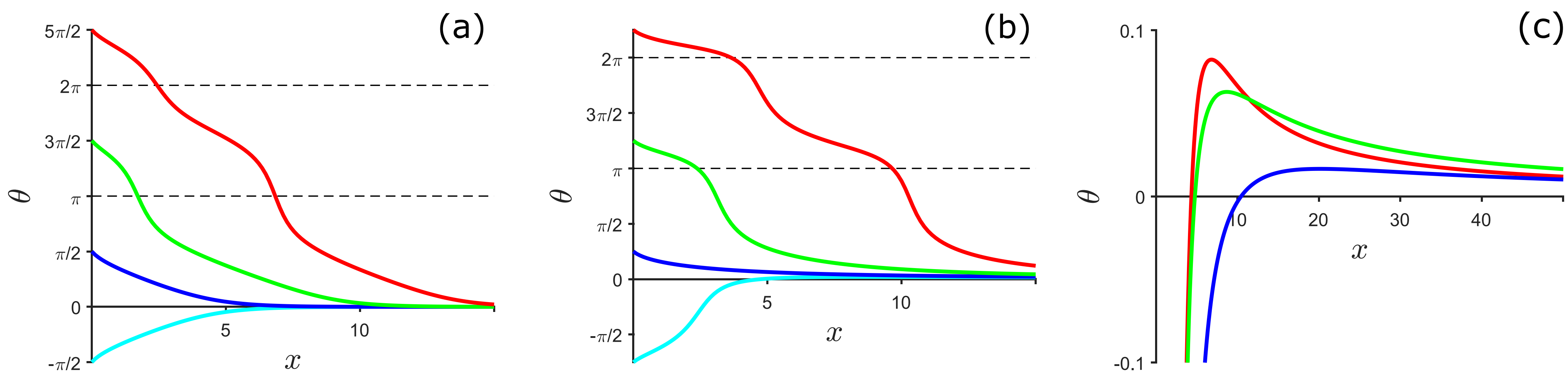

Figure Edge domain walls exhibiting winding and lack of monotonicity obtained by solving (∗) for $\nu = 10$ and different values of β. In (a), $\beta = -\pi/2, \pi/2, 3\pi/2, 5\pi/2$. In (b), $\beta = -3\pi/4, \pi/4, 5\pi/4, 9\pi/4$. In (c), the non-monotone decay in the tails of the solutions for $\beta = -5\pi/8$ (red), $\beta = -3\pi/4$ (green) and $\beta = -7\pi/8$ (blue) at large x is emphasized.

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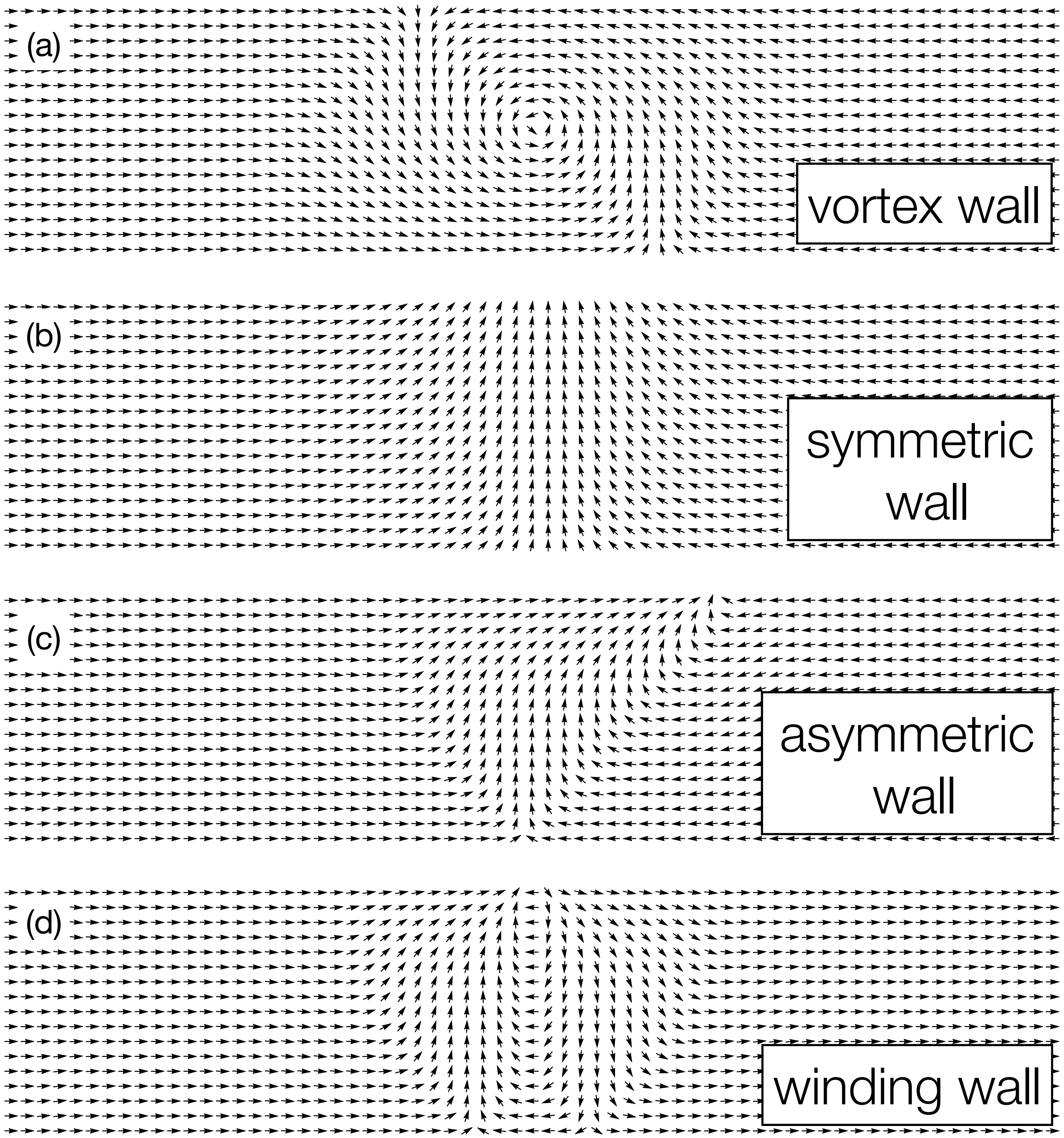
Lund, M and Slastikov, 2018

Charged walls in strips

Figure. Domain wall profiles in the numerical simulations of amorphous cobalt nanostrips: (a) vortex head-to-head wall in a 100 nm wide and 5 nm thick strip; (b) symmetric transverse head-to-head wall in a 50 nm wide and 2 nm thick strip; (c) asymmetric head-to-head wall in a 400 nm wide and 5 nm thick strip; (d) a winding transverse domain wall in a 400 nm wide and 5 nm thick strip. The material parameters are: exchange constant $A = 1.4 \times 10^{-11}$ J/m, saturation magnetization $M_s = 1.4 \times 10^6$ A/m, and zero magnetocrystalline anisotropy or applied magnetic field.



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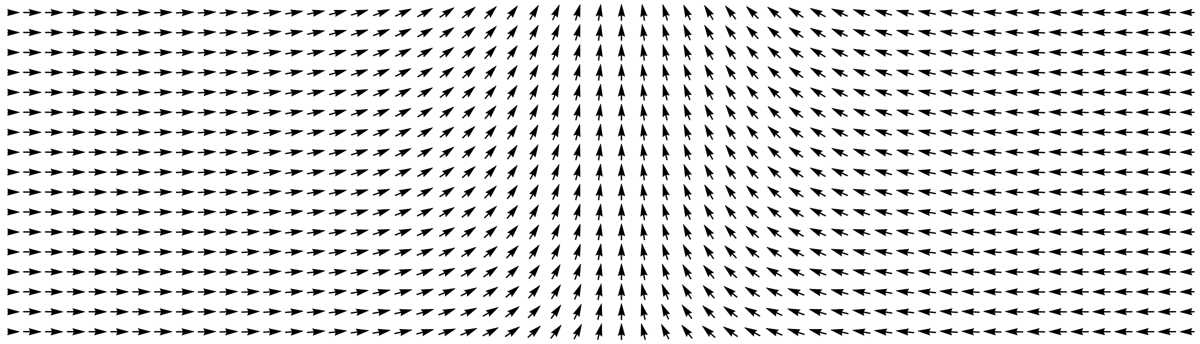
Morini, M, Novaga and Slastikov, 2021

Head-to-head walls



reduced two-dimensional thin film energy

$$E_\varepsilon(\mathbf{m}) = \frac{1}{2}\int_\Sigma |\nabla \mathbf{m}|^2 \, d^2 r + \frac{\gamma}{2|\ln \varepsilon|}\int_\Sigma \int_\Sigma \frac{\nabla \cdot (\eta_\varepsilon \mathbf{m})(\mathbf{r}) \, \nabla \cdot (\eta_\varepsilon \mathbf{m})(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^2 r \, d^2 r'$$

cutoff function at the edge of the unit strip $\Sigma = (-\infty, +\infty)\times(0,1)$:

$$\eta_\varepsilon(\mathbf{r}) = \eta\left(d_\Sigma(\mathbf{r})/\varepsilon\right), \qquad d_\Sigma(\mathbf{r}) := \mathrm{dist}(\mathbf{r}, \mathbb{R}^2 \backslash \Sigma)$$

after a suitable relaxation on

$$\mathfrak{M} := \left\{ \mathbf{m} \in H^1_l(\Sigma; \mathbb{S}^1) \, : \, \nabla \mathbf{m} \in L^2(\Sigma; \mathbb{R}^2), \, m_2 \in L^2(\Sigma) \right\}$$

Theorem (existence of 180° walls). *Let $\gamma > 0$ and $k \in \mathbb{N}$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a minimizer $\mathbf{m} = (\cos\theta, \sin\theta)$ of E_ε over all $\mathbf{m} \in \mathfrak{M}$ such that $\lim_{x\to+\infty}\theta(x,\cdot) = 0$ and $\lim_{x\to-\infty}\theta(x,\cdot) = k\pi$ if and only if $k = 1$.*

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