

# Linear vs. Nonlinear Selection for the Propagation Speed of the Solutions of Scalar Reaction-Diffusion Equations Invading an Unstable Equilibrium

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## Abstract

We revisit the classical problem of speed selection for the propagation of disturbances in scalar reaction-diffusion equations with one linearly stable and one linearly unstable equilibrium. For a wide class of initial data this problem reduces to finding the minimal speed of the monotone traveling wave solutions connecting these two equilibria in one space dimension. We introduce a variational characterization of these traveling wave solutions and give a necessary and sufficient condition for linear versus nonlinear selection mechanism. We obtain sufficient conditions for the linear and nonlinear selection mechanisms that are easily verifiable. Our method also allows us to obtain efficient lower and upper bounds for the propagation speed. © 2004 Wiley Periodicals, Inc.

## 1 Introduction

In this paper, we revisit the problem of speed selection for the propagation of disturbances in scalar reaction-diffusion systems. To this end, we consider the following initial value problem:

$$(1.1) \quad u_t = u_{xx} + f(u), \quad u(x, 0) = u_0(x),$$

where  $u = u(x, t)$  is a real-valued function of one spatial variable  $x$  and time  $t$ . For simplicity, we consider the problem on a real line; a straightforward generalization to the cylindrical geometry with Neumann boundary conditions or the entire  $\mathbb{R}^n$  is possible. Furthermore, we will consider nonlinearities  $f$  possessing an unstable equilibrium (which without any loss of generality may be assumed to be zero) and a stable equilibrium of the space-independent dynamics governed by equation (1.1), with no other equilibria in between.

This kind of equation is a prototypical model for a variety of applications in physics, chemistry, and biology (see, for example, [10, 12, 21, 23]). One of the

most notable examples is the Fisher equation with nonlinearity  $f(u) = u(1 - u)$  describing the spread of advantageous genes in a population, which has been studied since the 1930s [14, 18, 23]. Equation (1.1) arises naturally in the context of autocatalytic reactions and combustion systems, for example, in the case of the Arrhenius nonlinearity one gets equation (1.1) with  $f(u) = e^{-a/u}(1 - u)$ ,  $a > 0$  [10, 12, 21]. Another important class of problems that leads to equation (1.1) with an unstable equilibrium arises from the analysis of amplitude equations describing the dynamics of the system near a bifurcation point [10]. As one characteristic example, consider the subcritical quintic Ginzburg-Landau equation

$$(1.2) \quad u_t = u_{xx} + \mu u + u^3 - u^5,$$

which has an unstable equilibrium  $u = 0$  for  $\mu > 0$ .

We are interested in the process of invasion of the unstable equilibrium by a stable one in reaction-diffusion systems described by equation (1.1). So, we will consider the solutions of equation (1.1) with the initial data decaying exponentially as  $x \rightarrow +\infty$ . These solutions are known to exhibit propagation with constant speed at long times. Back in the 1930s, Kolmogorov, Petrovskii, and Piskunov showed that equation (1.1) admits a particular class of solutions in the form of *traveling waves*,  $u(x, t) = \bar{u}(x - ct)$ , moving with speed  $c$ , where the profile of the wave satisfies the ordinary differential equation

$$(1.3) \quad \bar{u}_{xx} + c\bar{u}_x + f(\bar{u}) = 0.$$

Under some extra assumptions on  $f$ , they were able to prove that there exists a continuous family of traveling wave solutions with arbitrary speeds  $c \geq c_0$ , which decay exponentially at a rate depending on  $c$  [18]. This fact already indicates that the speed of propagation for the solutions of the considered initial value problem may depend on the way the initial data go to 0 at positive infinity. However, as was already shown by Kolmogorov, Petrovskii, and Piskunov, for the initial data that decay sufficiently rapidly, in particular, if  $u_0(x) \equiv 0$  for all  $x > x_0$ , for such nonlinearities the propagation speed turns out to be equal to  $c_0$ , which, furthermore, is easily calculated from the linearization of equation (1.1) around 0.

The problem of speed selection was discussed extensively in the physics literature (see, for example, [3, 4, 5, 24, 27, 28, 29]). It was observed that depending on the nonlinearity, the propagation speed  $c^*$  for sufficiently rapidly decaying initial conditions is either equal to or greater than the speed  $c_0$  obtained by Kolmogorov, Petrovskii, and Piskunov. Depending on whether the first or the second situation is realized, the selection mechanism is termed a *linear* or *nonlinear* selection, respectively.

Van Saarloos argued that whether the first or second mechanism is realized depends on the existence of certain types of traveling wave solutions [27, 28, 29]. Rigorous studies of this problem in the general context of equations (1.1) were initiated by Aronson and Weinberger [1, 2]. They essentially resolved the problem of the speed selection with the help of the comparison arguments for a certain class of

initial data. Central to their analysis is the construction of the monotone traveling wave solutions connecting the stable equilibrium at negative infinity with the unstable equilibrium at positive infinity. Aronson and Weinberger proved that under certain assumptions, these solutions in fact exist for all  $c \geq c^*$ , where  $c^*$  is some constant that may be equal to or strictly greater than  $c_0$ . Furthermore, they proved that any nontrivial solution of the initial value problem in equation (1.1) with the initial data  $u_0(x)$  between these equilibria and decaying exponentially, faster than the traveling wave solution with speed  $c^*$ , will propagate asymptotically with speed  $c^*$  in the sense that the position  $R(t)$  of the *leading edge* of the solution of equation (1.1) behaves asymptotically as (for precise definitions and assumptions, see Section 2)

$$(1.4) \quad R(t) \sim c^*t, \quad t \rightarrow +\infty.$$

Rothe, and more recently Roquejoffre, proved that when  $c^* > c_0$  and under some mild assumptions any solution of the initial value problem in equation (1.1), with the initial data decaying sufficiently fast at positive infinity and bounded away from zero at negative infinity, converges exponentially to the traveling wave solution with speed  $c^*$  [25, 26]. Thus, they proved not only that under these assumptions the position of the leading edge behaves asymptotically according to equation (1.4), but also that the wave profile approaches uniformly a traveling wave profile with speed  $c^*$  (this result remains valid in cylinders with Neumann boundary conditions, and even in the presence of advection terms [26]). Convergence results for this type of equation were also recently discussed from a variational perspective [22].

From the discussion above it is clear that under rather general assumptions on the initial data the problem of speed selection reduces to finding the minimal speed of the monotone traveling wave solutions connecting the two equilibria of  $f$ . Although Aronson and Weinberger give a definitive answer about the existence of the propagation speed  $c^*$ , that is, the existence of the limit in equation (1.4), their techniques do not say when the value of  $c^* = c_0$  and when  $c^* > c_0$ . In other words, they do not provide a way to establish whether the linear or nonlinear selection mechanism is realized. Note that this is also a necessary ingredient for applying the results of [25, 26]. Therefore, the problem of characterizing the value of  $c^*$  is of fundamental importance for understanding the long-time behavior of the solutions of equation (1.1).

In this paper, we develop a variational characterization of the traveling wave solutions with speed  $c^* > c_0$ . We give a necessary and sufficient condition for existence of this solution and thus give a verifiable answer to the question of linear versus nonlinear selection. Our method also allows us to obtain efficient upper and lower bounds for  $c^*$  that can be easily implemented in practice.

Our paper is organized as follows: In Section 2, we introduce some notation and summarize known results about propagation for solutions of equation (1.1). We also generalize the main results of Aronson and Weinberger for the limiting

behavior of the propagation speed to initial data with exponential decay and state our main result. In Section 3, we introduce our variational problem and demonstrate its relationship to the existence of some special traveling wave solutions that determine the limiting propagation speed. In Section 4 we prove our main theorem, in Section 5 we consider a few applications of our results, and in Section 6 we make concluding remarks.

### 2 Preliminaries and the Main Result

Here we give a few basic definitions and state our main result. Let us start by giving some known results that will be used throughout the paper. We look for solutions of the initial value problem in equation (1.1)

$$(2.1) \quad u(x, t) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \quad \text{with } u_0(x) \in C(\mathbb{R}).$$

We assume that the nonlinearity  $f \in C^1([0, 1])$  and satisfies

$$(2.2) \quad f(0) = f(1) = 0, \quad f'(0) > 0 > f'(1), \quad f(u) > 0 \quad \text{for } 0 < u < 1.$$

Thus,  $u = 0$  is the unstable and  $u = 1$  is the stable equilibrium. We will further consider the initial data bounded to the strip

$$(2.3) \quad 0 \leq u_0(x) \leq 1.$$

By the maximum principle, the solutions of the initial value problem in equation (1.1) with these initial conditions will remain bounded between 0 and 1 and therefore exist for all  $t > 0$  (see, e.g., [13]).

Turn now to the traveling wave solutions. First, any bounded solution of equation (1.3) must connect the equilibria at infinity. In particular, with our assumptions on  $f$  we have the following:

**PROPOSITION 2.1** (Volpert, Volpert, and Volpert [30]) *Let  $\bar{u}$  be a solution of equation (1.3) with  $c > 0$  and  $\bar{u}(x) \in [0, 1]$  for all  $x \in \mathbb{R}$ . Then*

$$(2.4) \quad \lim_{x \rightarrow +\infty} \bar{u}(x) = 0,$$

$$(2.5) \quad \lim_{x \rightarrow -\infty} \bar{u}(x) = 1,$$

*and, furthermore,  $\bar{u}(x)$  is monotonically decreasing.*

Naturally, because of reflection symmetry for any solution of equation (1.3) there exists a mirror-symmetric solution of this equation with speed  $-c$  (hence, with the reversed order of the limits in equations (2.4) and (2.5)). Therefore, in the following we will consider only the right-moving waves and propagation toward  $x = +\infty$ .

Let us define a positive constant

$$(2.6) \quad c_0 = 2\sqrt{f'(0)}.$$

It is easy to see that this constant plays the role of the minimal possible propagation speed for monotone traveling wave solutions. Indeed, the behavior ahead of the

wave is determined by the linearization of equation (1.3) around 0. Assuming that  $\bar{u}(x) \sim e^{-\lambda x}$ , we obtain an equation

$$(2.7) \quad \lambda^2 - c\lambda + f'(0) = 0$$

relating  $c$  and  $\lambda$ , whose solutions are  $\lambda = \lambda_{\pm}(c)$ , where

$$(2.8) \quad \lambda_{\pm}(c) = \frac{1}{2}(c \pm \sqrt{c^2 - 4f'(0)}).$$

From this equation it follows that the decay of  $\bar{u}(x)$  for  $x \rightarrow +\infty$  is nonoscillatory only if  $c \geq c_0$ .

The existence of the traveling wave solutions that satisfy the conditions in equations (2.4) and (2.5) for the nonlinearities satisfying equation (2.2) were proven by Aronson and Weinberger:

**PROPOSITION 2.2** (Aronson and Weinberger [1, 2]) *Under the assumptions of equation (2.2), for each  $c \geq c^*$ , with some  $c^* \geq c_0$ , there exists a unique (up to translation) solution  $\bar{u}(x)$  of equation (1.3) satisfying equations (2.4) and (2.5). Furthermore, for  $c > c^*$  we have*

$$(2.9) \quad \ln \bar{u}(x) \sim -\lambda_-(c)x, \quad x \rightarrow +\infty,$$

whereas for  $c = c^*$  we have

$$(2.10) \quad \ln \bar{u}(x) \sim -\lambda_+(c^*)x, \quad x \rightarrow +\infty.$$

When  $c < c^*$ , there are no traveling wave solutions satisfying the assumptions of Proposition 2.1.

Let us now define the position of the leading edge  $R(t)$  of the solution of the initial value problem in equation (1.1). Let  $\alpha \in (0, 1)$  be some fixed constant so small that  $\sup_{x \in \mathbb{R}} u(x, t) \geq \alpha$  for all  $t \geq 0$ . Clearly, such a constant necessarily exists at least for large time intervals if the initial data are not 0 identically. Denote

$$(2.11) \quad R(t) = \sup\{x \in \mathbb{R} : u(x, t) \geq \alpha\}.$$

In terms of  $R(t)$ , we have the following theorem, which is a generalization of the classical results of Aronson and Weinberger.

**THEOREM 2.3** *Let  $u(x, t)$  be a solution of equation (1.1) with the initial condition not equal to 0 identically, satisfying equation (2.3) and*

$$(2.12) \quad \limsup_{x \rightarrow +\infty} u_0(x)e^{\lambda x} < +\infty,$$

where  $\lambda = \lambda_-(c^*)$ . Then there exists a constant  $\alpha$  such that equation (1.4) holds, with  $c^*$  the same as in Proposition 2.2.

**PROOF:** We first note that  $R(t)$  is well-defined for all  $t \geq 0$  if  $\alpha$  is small enough. Indeed, by the results of Aronson and Weinberger, for any nonzero initial condition we have  $\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} u(x, t) = 1$  (see [1, 2]), so it is always possible to choose  $\alpha$  such that  $\sup_{x \in \mathbb{R}} u(x, t) \geq \alpha$  for all  $t \geq 0$ . Of course,

$\sup_{x \in \mathbb{R}} u(x, t) > 0$  for all  $t \geq 0$ ; otherwise, by uniqueness of solutions of equation (1.1), we would have  $u(x, t) \equiv 0$ .

Let us first show that  $R(t) \geq ct$  for any  $c < c^*$  and large enough  $t$ . This follows immediately from the fact that by the results of Aronson and Weinberger  $u \rightarrow 1$  pointwise in any reference frame moving with speed  $0 < c < c^*$  [1, 2].

To prove that  $R(t) \leq ct + R_0$  for all  $t$ , with arbitrary  $c > c^*$  and some  $R_0$ , we construct an appropriate supersolution in the reference frame moving with speed  $c$ . To do that, let us note that for  $c > c^*$  there is a unique trajectory starting at the origin and having a slope  $-\lambda_+(c)$  in the phase plane [1, 2]. Since  $c > c^*$ , this trajectory intersects the line  $u = 1$  at some  $u_x = v_0 < 0$ . By uniqueness of the above trajectory, any other phase plane trajectory going into the origin must have slope  $-\lambda_-(c)$ . On the other hand, by Proposition 2.2, for  $c > c^*$  there exists a traveling wave solution connecting  $u = 0$  and  $u = 1$  with slope  $-\lambda_-(c)$ . Therefore, any phase plane trajectory starting at  $u = 1$  and  $u_x = v$  with  $v_0 < v < 0$  will terminate at the origin and have the slope  $-\lambda_-(c)$  as well. We now construct a supersolution  $\bar{u}(x)$  in the reference frame moving with speed  $c$  by assuming that  $\bar{u}(x)$  satisfies equation (1.3) for  $x \geq 0$  with the initial conditions  $\bar{u}(0) = 1$  and  $\bar{u}_x(0) = v$ , and take  $\bar{u} = 1$  for  $x < 0$ .

Now, observe that  $\lambda_-(c) < \lambda_-(c^*)$  for  $c > c^*$ . Then if equation (2.12) holds, we can always bound  $u_0$  with a translate of  $\bar{u}$  from above. Since  $\bar{u}$  is also monotonically decreasing and  $\bar{u}(x) \rightarrow 0$  as  $x \rightarrow +\infty$ , this implies that  $R(t) \leq ct + R_0$  with some  $R_0$ . Finally, since  $c$  can be arbitrarily close to  $c^*$  in both the upper and lower bound for  $R(t)$ , we obtain equation (1.4). □

Thus, the asymptotic propagation speed for the solutions of equation (1.1) with the initial data satisfying the assumptions of Theorem 2.3 is  $c^*$  in the sense of the asymptotic average speed of the leading edge. Note that the speed  $c^*$  is also an asymptotic lower bound for the speed of the leading edge for *any* nontrivial initial condition (see the proof above). Furthermore, note that the assumption on the decay in equation (2.12) is almost necessary, since if  $\lambda < \lambda_-(c^*)$ , then solutions whose leading edge moves faster than  $c^*$  (like the traveling waves with speed  $c > c^*$  in Proposition 2.2) are possible.

REMARK 2.4 We point out that in view of [1] the function  $R(t)$  is defined for any  $\alpha \in (0, 1)$  when  $t$  is large enough (see also the proof above), so the statement of Theorem 2.3 remains valid for all such  $\alpha$ . Also note that equation (1.4) holds independently of  $\alpha$ .

We have two possibilities that we need to discern, so, following van Saarloos [27, 28, 29], we introduce the following:

DEFINITION 2.5 Under the assumptions of Theorem 2.3, we call the case  $c^* = c_0$  the *linear selection* and the case  $c^* > c_0$  the *nonlinear selection* mechanism.

The purpose of what follows is to characterize linear versus nonlinear selection for a given nonlinearity  $f(u)$  obeying equation (2.2) within the context of Theorem 2.3. We start by defining the exponentially weighted Sobolev spaces we will be working in.

DEFINITION 2.6 For  $c > 0$ , denote by  $H_c^1(\mathbb{R})$  the completion of  $C_0^\infty(\mathbb{R})$  with respect to the norm

$$\|u\|_{1,c} = \|u\|_c + \|u_x\|_c, \quad \|u\|_c^2 = \int_{\mathbb{R}} e^{cx} u^2 dx.$$

These are in fact natural spaces for working with the solutions of equation (1.3), since they provide control of the exponential decay of the solution at positive infinity for different choices of the constant  $c$  (see below, and also [22]). Also, a simple observation about the decay of the traveling wave solutions in Proposition 2.2, together with the fact that  $|\bar{u}_x| \leq C\bar{u}$  with some constant  $C$  (see [1, 2]), gives the following:

COROLLARY 2.7 Let  $c > c_0$ , and let  $\bar{u}$  be the traveling wave solution from Proposition 2.2. Then  $\bar{u} \in H_c^1(\mathbb{R})$  if and only if  $c = c^*$ .

Therefore, the question of linear versus nonlinear selection is determined by whether there exists a traveling wave solution with speed  $c > c_0$  and satisfying the assumptions of Proposition 2.1, which lies in  $H_c^1(\mathbb{R})$ . It is precisely the existence of this solution that we are going to characterize.

For  $u \in H_c^1(\mathbb{R})$ , define the functional

$$(2.13) \quad \Phi_c[u] = \int_{\mathbb{R}} e^{cx} \left( \frac{1}{2} u_x^2 + V(u) \right) dx,$$

where the function  $V(u)$  is given by

$$(2.14) \quad V(u) = \begin{cases} \frac{1}{2} |f'(0)| u^2, & u < 0, \\ -\int_0^u f(s) ds, & 0 \leq u \leq 1, \\ -\int_0^1 f(s) ds + \frac{1}{2} |f'(1)| (u - 1)^2, & u > 1. \end{cases}$$

From the assumptions on  $f(u)$  in equation (2.2), it follows that  $V(u) \in C^1(\mathbb{R})$  and that  $|V(u)| \leq Cu^2$  for some  $C$ , so  $\Phi_c[u]$  is well-defined for all  $u$  in the considered class.

We now state our main result.

THEOREM 2.8 Under the assumptions of Theorem 2.3, the nonlinear selection mechanism is realized if and only if there exists  $u \in H_c^1(\mathbb{R})$ ,  $u \not\equiv 0$ , such that

$$(2.15) \quad \Phi_c[u] \leq 0$$

for some  $c > c_0$ .

Thus, the functional  $\Phi_c$  provides a complete characterization of the speed selection mechanism within the framework of Aronson and Weinberger.

### 3 Constrained Variational Problem

We now formulate the problem of existence of the traveling wave solutions with speed  $c^*$  as a constrained variational problem. We point out that our method provides a very general way of constructing the traveling wave solutions and is not limited to the case of the nonlinearities specified in equation (2.2) or to one-dimensional scalar reaction diffusion equations. The general treatment of this problem from the variational perspective will be presented elsewhere [19].

For  $u \in H_c^1(\mathbb{R})$ , introduce an auxiliary functional

$$(3.1) \quad \Gamma_c[u] = \frac{1}{2} \int_{\mathbb{R}} e^{cx} u_x^2 dx.$$

Note that both  $\Phi_c$  and  $\Gamma_c$  transform similarly under translation.

LEMMA 3.1 *Let  $u \in H_c^1(\mathbb{R})$  and  $u_a(x) = u(x - a)$ . Then,*

$$(3.2) \quad \Phi_c[u_a] = e^{ca} \Phi_c[u] \quad \text{and} \quad \Gamma_c[u_a] = e^{ca} \Gamma_c[u].$$

Now, by setting

$$(3.3) \quad \mathcal{B}_c = \{u \in H_c^1(\mathbb{R}) : \Gamma_c[u] = 1\},$$

we obtain the following constrained minimization problem:

$$(P) \quad \text{Find } u_c \in \mathcal{B}_c \text{ satisfying } \Phi_c[u_c] = \inf_{\mathcal{B}_c} \Phi_c[u].$$

The connection between the solutions of problem (P) and the solutions of equation (1.3) is established by the following:

PROPOSITION 3.2 *Let  $u_c(x)$  be a solution of problem (P) with  $\Phi_c[u_c] \leq 0$ . Then*

$$(3.4) \quad \bar{u}(x) = u_c(x\sqrt{1 - \Phi_c[u_c]})$$

*is the traveling wave solution with speed  $c^\dagger = c\sqrt{1 - \Phi_c[u_c]} \geq c$  that satisfies the assumptions of Proposition 2.1. Furthermore,  $\bar{u} \in H_{c^\dagger}^1(\mathbb{R})$ .*

We prove this proposition via a sequence of lemmas.

LEMMA 3.3 *Let  $u_c(x)$  be a solution of problem (P) with  $\Phi_c[u_c] \leq 0$ . Then  $0 \leq u_c(x) \leq 1$ .*

PROOF: We argue by contradiction. First note that  $u \in H_c^1(\mathbb{R})$  implies  $u \in C(\mathbb{R})$ . Observe that for any  $u \in H_c^1(\mathbb{R})$  we can define

$$(3.5) \quad \tilde{u}(x) = \begin{cases} 0, & u(x) < 0, \\ u(x), & 0 \leq u(x) \leq 1, \\ 1, & u(x) > 1. \end{cases}$$

Since  $V(u)$  is strictly increasing outside the interval  $u \in [0, 1]$ , we have

$$(3.6) \quad \Phi_c[\tilde{u}] < \Phi_c[u] \leq 0.$$

We also have  $\Gamma_c[\tilde{u}] \leq 1$ . Furthermore,  $\Gamma_c[\tilde{u}] > 0$ , since otherwise  $\tilde{u} = 0$  and hence  $\Phi_c[\tilde{u}] = 0$ , contradicting equation (3.6). Therefore, by Lemma 3.1 it is always possible to find a value of  $a \geq 0$  that  $\tilde{u}_a(x) = \tilde{u}(x - a) \in \mathcal{B}_c$ , too. So, if  $u$  is the solution of problem (P), then so is  $\tilde{u}_a$ , which by Lemma 3.1 and equation (3.6) gives a lower value of  $\Phi_c$ .  $\square$

LEMMA 3.4 *Let  $u = u_c(x)$  be a solution of problem (P). Then  $u \in C^2(\mathbb{R})$  and satisfies*

$$(3.7) \quad (1 - \mu)(u_{xx} + cu_x) + f(u) = 0.$$

Moreover,

$$(3.8) \quad \mu = \Phi_c[u_c].$$

PROOF: We have  $\Phi_c$  and  $\Gamma_c$  of class  $C^1$ . Let  $D\Gamma_c[u]v$  be the Frechet derivative of  $\Gamma_c$  at  $u$  acting on  $v$ . Since

$$D\Gamma_c[u]u = \int_{\mathbb{R}} e^{cx} u_x^2 dx = 2 \quad \forall u \in \mathcal{B}_c,$$

we get  $D\Gamma_c[u] \neq 0$  on  $\mathcal{B}_c$ . Thus, applying the Lagrange multiplier theorem (see, e.g., [9, section 3.5]), we obtain

$$(3.9) \quad \int_{\mathbb{R}} e^{cx} \{(1 - \mu)u_x \varphi_x + V'(u)\varphi\} dx = 0 \quad \forall \varphi \in H_c^1(\mathbb{R}),$$

where  $\mu$  is the Lagrange multiplier.

To proceed, we first show that  $\mu \neq 1$ . Indeed, if  $\mu = 1$ , from equation (3.9) we get  $V'(u) \equiv 0$ . Since  $u$  is also continuous, according to equation (2.2) this means that  $u \equiv 0 \notin \mathcal{B}_c$ , leading to a contradiction.

So,  $\mu \neq 1$ , and from elliptic regularity theory (see, e.g., [15]) and Lemma 3.3 we deduce that  $u \in C^2(\mathbb{R})$  and satisfies equation (3.7). From this equation and the fact that  $|f(u)| \leq C|u|$ , it follows that  $u_x \in H_c^1(\mathbb{R})$ , and therefore we can use  $\varphi = u_x$  in equation (3.9). Integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}} e^{cx} \{(1 - \mu)u_x u_{xx} + V'(u)u_x\} dx \\ &= -c \int_{\mathbb{R}} e^{cx} \left( \frac{1}{2}(1 - \mu)u_x^2 + V(u) \right) \\ &= -c(\Phi_c[u] - \mu\Gamma_c[u]). \end{aligned}$$

So, taking into account that  $u \in \mathcal{B}_c$ , we obtain equation (3.8).  $\square$

PROOF OF PROPOSITION 3.2: We are now ready to prove Proposition 3.2. By Lemma 3.4, the solution  $u = u_c$  of problem (P) satisfies equation (3.7) with  $\mu \leq 0$  given by equation (3.8). Introducing  $\bar{u}(x) = u_c(x\sqrt{1-\mu})$  and using equation (3.7), we obtain that  $\bar{u}$  is a traveling wave with speed  $c^\dagger = c\sqrt{1-\mu} \geq c$ . Furthermore, by Lemma 3.3 we have  $\bar{u} \in [0, 1]$  and  $\bar{u} \in H^1_{c^\dagger}(\mathbb{R})$ . Indeed, with  $u = u_c$  we have

$$\begin{aligned} \int_{\mathbb{R}} e^{c^\dagger x} \bar{u}^2 dx &= \int_{\mathbb{R}} e^{cx\sqrt{1-\mu}} u^2(x\sqrt{1-\mu}) dx \\ &= \frac{1}{\sqrt{1-\mu}} \int_{\mathbb{R}} e^{cx} u^2 dx < \infty, \\ \int_{\mathbb{R}} e^{c^\dagger x} \bar{u}_x^2 dx &= \int_{\mathbb{R}} e^{cx\sqrt{1-\mu}} u_x^2(x\sqrt{1-\mu}) dx \\ &= \sqrt{1-\mu} \int_{\mathbb{R}} e^{cx} u_x^2 dx < \infty. \end{aligned}$$

□

REMARK 3.5 We point out that the speed  $c^\dagger$  of the obtained traveling wave is independent of the value of  $c$  in Proposition 3.2. This is in fact a general property of the considered variational problem (for more details, see [19, 22]).

To show that the assumption of Theorem 2.8 is necessary, we note simply that if  $c^* > c_0$  in Proposition 2.2, then by Corollary 2.7 we have  $\bar{u} \in H^1_{c^*}(\mathbb{R})$ , where  $\bar{u}$  is a traveling wave solution with speed  $c^*$ . Furthermore,  $\bar{u}$  satisfies equation (3.7) with  $c = c^*$  and  $\mu = 0$ . This implies that  $\Phi_{c^*}[\bar{u}] = 0$ , which gives the desired result.

### 4 Existence of Minimizers

We now show that under the assumption of Theorem 2.8 problem (P) always has a solution. To prove that, we need to establish a few auxiliary results regarding the properties of the functional  $\Phi_c$ . First, the following lemma is of fundamental importance to the study of problem (P) and gives the analogue of the Poincaré inequality for spaces  $H^1_c(\mathbb{R})$ .

LEMMA 4.1 For all  $u \in H^1_c(\mathbb{R})$ , we have

$$(4.1) \quad \frac{c^2}{4} \int_{\mathbb{R}} e^{cx} u^2 dx \leq \int_{\mathbb{R}} e^{cx} u_x^2 dx \quad \forall R \in \mathbb{R},$$

$$(4.2) \quad u^2(R) \leq \frac{e^{-cR}}{c} \int_{\mathbb{R}} e^{cx} u_x^2 dx \quad \forall R \in \mathbb{R}.$$

PROOF: Let us first prove equation (4.1). We have

$$\begin{aligned} \frac{c}{2} \int_R^\infty e^{cx} u^2 dx &= -\frac{1}{2} e^{cR} u^2(R) - \int_R^\infty e^{cx} u u_x dx \\ &\leq \left( \int_R^\infty e^{cx} u^2 dx \right)^{1/2} \left( \int_R^\infty e^{cx} u_x^2 dx \right)^{1/2}, \end{aligned}$$

which implies (4.1).

Let us now prove equation (4.2). Since,  $\int_R^\infty e^{cx} (\sqrt{c}u + \frac{1}{\sqrt{c}}u_x)^2 dx \geq 0$ , we get

$$\begin{aligned} \frac{1}{c} \int_R^\infty e^{cx} u_x^2 dx + c \int_R^\infty e^{cx} u^2 dx &\geq -2 \int_R^\infty e^{cx} u u_x dx \\ &= u^2(R) e^{cR} + c \int_R^\infty e^{cx} u^2 dx, \end{aligned}$$

which gives (4.2). □

An immediate consequence of equation (4.1) is the following:

COROLLARY 4.2 *If  $u \in H_c^1(\mathbb{R})$ , then*

$$(4.3) \quad \frac{c^2}{4} \int_{\mathbb{R}} e^{cx} u^2 dx \leq \int_{\mathbb{R}} e^{cx} u_x^2 dx.$$

Observe that since under our assumptions  $V(u) \geq -Cu^2$  with some  $C \geq 0$ , the functional  $\Phi_c$  will be positive for all nonzero  $u \in H_c^1(\mathbb{R})$  for sufficiently large  $c$ . More precisely, we have the following (see also [22]):

LEMMA 4.3 *Let  $c_{\max}$  be defined as*

$$(4.4) \quad c_{\max} = \min \left\{ c \geq 0 : \frac{1}{8} c^2 s^2 + V(s) \geq 0 \quad \forall s \in \mathbb{R} \right\}.$$

*Then, for any  $c > c_{\max}$  we have  $\Phi_c[u] > 0$  for all  $u \in H_c^1(\mathbb{R})$  such that  $u \not\equiv 0$ .*

Combined with Theorem 2.8, this result gives an upper bound for the value of  $c^*$ .

PROPOSITION 4.4 *The value of  $c_{\max}$  in equation (4.4) gives an upper bound for the propagation speed  $c^*$  in Theorem 2.3.*

PROOF: First of all, we note that  $c_{\max} < \infty$ . Second, we must have  $c_{\max} \geq c_0$ . Indeed, we have

$$V(s) \sim -\frac{1}{2} f'(0) s^2, \quad s \rightarrow 0^+,$$

with  $f'(0) > 0$ . Recalling the definition of  $c_0$  in equation (2.6), we see that  $c_{\max} \geq c_0$  in order for the inequality in equation (4.4) to hold for small  $s$ . Now, if  $c^* > c_{\max}$ , then  $c^* > c_0$ , so by the argument at the end of Section 3 we have  $\Phi_{c^*}[\bar{u}] = 0$ , where  $\bar{u}$  is the traveling wave solution with speed  $c^*$ , contradicting Lemma 4.3. □

REMARK 4.5 The statement of Proposition 4.4 remains valid in a much more general context (see [22]).

Before proceeding further to the proof of existence of solutions of problem (P), let us introduce the following notation. For given  $-\infty \leq a < b \leq +\infty$ , define

$$(4.5) \quad \Phi_c[u, (a, b)] = \int_a^b e^{cx} \left( \frac{1}{2} u_x^2 + V(u) \right) dx .$$

LEMMA 4.6 *Assume  $c > c_0$ ; then there exists  $R > 0$  such that*

- (1)  $\Phi_c[u, (R, +\infty)] \geq 0$  for all  $u \in \mathcal{B}_c$ ;
- (2) letting  $u_n \rightharpoonup u$  in  $H_c^1(\mathbb{R})$ , then

$$\liminf_{n \rightarrow \infty} \Phi_c[u_n, (-\infty, R)] \geq \Phi_c[u, (-\infty, R)] .$$

PROOF: (1) Since  $c > c_0$ , we can choose  $0 < \epsilon < c^2/4 - f'(0)$ . By the definition of  $V$ , there exists some  $s_0 > 0$  such that

$$V(s) \geq -\frac{1}{2}(f'(0) + \epsilon)s^2, \quad |s| \leq s_0 .$$

From Lemma 4.1, there exists  $R_0$  such that

$$|u(x)| < s_0 \quad \forall x > R_0, \quad \forall u \in \mathcal{B}_c .$$

Therefore, given  $R > R_0$ , we get

$$(4.6) \quad \int_R^\infty e^{cx} V(u) dx \geq -\frac{1}{2}(f'(0) + \epsilon) \int_R^\infty e^{cx} u^2 dx .$$

Thus, from equations (4.1) and (4.6) and the choice of  $\epsilon$ , we obtain

$$\Phi_c[u, (R, +\infty)] \geq \frac{1}{2} \left( \frac{c^2}{4} - f'(0) - \epsilon \right) \int_R^\infty e^{cx} u^2 dx \geq 0 ,$$

which concludes the proof.

- (2) Since  $V(u)$  is bounded from below and  $\int_{-\infty}^R e^{cx} dx < \infty$ , this follows by standard semicontinuity results; see, for example, [11, propositions 2.1 and 2.2].

□

We are now ready to prove our existence result. We note that our method has a number of features in common with the technique developed by Berestycki and Lions for scalar field equations [6]. Similar techniques were also used by Heinze in [17].

PROPOSITION 4.7 *Suppose there exists  $u \in \mathcal{B}_c$  such that  $\Phi_c[u] \leq 0$  for some  $c > c_0$ . Then problem (P) has a solution.*

PROOF: Let  $(u_n)$  be a minimizing sequence of problem (P), i.e.,  $u_n \in \mathcal{B}_c$  with  $\Phi_c[u_n] \rightarrow \inf_{\mathcal{B}_c} \Phi_c$ . By assumption,  $\inf_{\mathcal{B}_c} \Phi_c \leq 0$ . Since  $\Gamma_c[u_n] = 1$ , from inequality (4.3) we get that  $\int_{\mathbb{R}} e^{cx} u_n^2 dx \leq 8/c^2$ . Thus,  $(u_n)$  is bounded in  $H_c^1(\mathbb{R})$  and therefore converges weakly to some  $u \in H_c^1(\mathbb{R})$ . Furthermore,

$$\begin{aligned}
 \inf_{\mathcal{B}_c} \Phi_c &= \liminf_{n \rightarrow \infty} \Phi_c[u_n] \\
 &\geq \liminf_{n \rightarrow \infty} \{ \Phi_c[u_n, (-\infty, R)] \} + \liminf_{n \rightarrow \infty} \{ \Phi_c[u_n, (R, +\infty)] \} \\
 (4.7) \quad &\geq \Phi_c[u, (-\infty, R)] \\
 &= \Phi_c[u] - \Phi_c[u, (R, +\infty)]
 \end{aligned}$$

for large enough  $R$ . Now, by letting  $R \rightarrow +\infty$  and noting that

$$\lim_{R \rightarrow +\infty} \Phi_c[u, (R, +\infty)] = 0,$$

equation (4.7) leads to

$$0 \geq \inf_{\mathcal{B}_c} \Phi_c \geq \Phi_c[u].$$

If  $\inf_{\mathcal{B}_c} \Phi_c = 0$ , we deduce that  $u$  in the assumption of this proposition is a minimizer. Therefore, let us assume that  $\inf_{\mathcal{B}_c} \Phi_c < 0$ . Then  $u \neq 0$ , and by standard semicontinuity results [11]

$$1 = \liminf_{n \rightarrow \infty} \Gamma_c[u_n] \geq \Gamma_c[u] > 0.$$

Then we can, by using Lemma 3.1, choose  $a \geq 0$  such that

$$\Gamma_c[u_a] = 1 \quad \text{with} \quad u_a(x) = u(x - a).$$

Since  $\inf_{\mathcal{B}_c} \Phi_c \leq 0$  and  $a \geq 0$ , we derive

$$\Phi_c[u_a] = e^{ca} \Phi_c[u] \leq \Phi_c[u] \leq \inf_{\mathcal{B}_c} \Phi_c.$$

Therefore,  $\Phi_c[u_a] = \inf_{\mathcal{B}_c} \Phi_c$ , and  $u_a$  solves problem (P). □

Theorem 2.8 then follows by noting that if  $\Phi_c[u] \leq 0$  for some  $u \neq 0$ , then, according to Lemma 3.1, we can make  $u \in \mathcal{B}_c$  by an appropriate shift. Therefore, by Propositions 3.2 and 4.7 there exists a traveling wave solution with speed  $c^\dagger$  that lies in  $H_{c^\dagger}^1(\mathbb{R})$  and satisfies the assumptions of Proposition 2.1 and hence is one of the solutions from Proposition 2.2. Then, by Corollary 2.7 we have  $c^* = c^\dagger \geq c > c_0$ , which completes the proof.

REMARK 4.8 It is not difficult to see that Propositions 3.2 and 4.7 remain valid when  $f(u)$  is not necessarily positive for all  $u \in (0, 1)$ .

REMARK 4.9 Observe that by Proposition 3.2 we have  $c^* \geq c$ , so the value of  $c$  in Theorem 2.8 also provides the lower bound for the propagation speed.

Thus, our method also provides a general method of constructing the traveling wave solutions, as well as obtaining upper and lower bounds for their speed. Moreover, in the case of  $f'(0) \leq 0$  there is no need for any assumptions on  $c$  (since  $c^2 - 4f'(0) > 0$  in Lemma 4.6 for all  $c > 0$ ), and the assumption of Proposition 4.7 is always satisfied as long as  $V(u) < 0$  for some  $0 < u \leq 1$  (see Section 5). We also note that by the result of Rothe the existence of the traveling wave solution with  $c^* > c_0$  established above implies that under essentially the same assumptions as those in Theorem 2.3 the solutions of the initial value problem for equation (1.1) will converge uniformly to a translate of this traveling wave as  $t \rightarrow +\infty$  [25].

### 5 Some Applications

In this section we first give two results concerning sufficient conditions for linear and nonlinear selection mechanisms, respectively. We then perform a variational study of a particular example, namely, equation (1.2).

Let us first formulate the general sufficient condition for the linear selection mechanism.

**THEOREM 5.1** *If for all  $0 < u \leq 1$*

$$(5.1) \quad \frac{2}{u^2} \int_0^u f(s)ds \leq f'(0) ,$$

*then the linear selection is realized.*

**PROOF:** From equation (5.1) we obtain

$$V(u) \geq -\frac{1}{2}f'(0)u^2 = -\frac{1}{8}c_0^2u^2 ,$$

so  $c_{\max} = c_0$  in equation (4.4). Therefore, by Lemma 4.3 we have  $\Phi_c[u] > 0$  for all  $u \neq 0$  and  $c > c_0$ . The result then follows from the “only if” statement of Theorem 2.8. □

Naturally, this implies that  $c^* = c_0$ . Observe that this result is the generalization of that of Kolmogorov, Petrovskii, and Piskunov, who required that  $f'(u) \leq f'(0)$  [18], as well as the result of Aronson and Weinberger, who obtained an upper bound for  $c^*$ , which is equal to  $c_0$  if  $f(u)/u \leq f'(0)$  [2].

We now give a sufficient criterion of the nonlinear selection mechanism. We note that better criteria can be obtained for a given nonlinearity by using suitable trial functions (see below). This criterion, nevertheless, gives a more precise meaning to the “ZFK” case considered in [26] and is relevant to combustion. We also point out that this criterion was obtained earlier by Berestycki and Nirenberg [7] as a sufficient condition for  $c^* > c_0$  in Proposition 2.2.

**THEOREM 5.2** *If*

$$(5.2) \quad f'(0) \leq \frac{1}{2} \int_0^1 f(u)du ,$$

then the nonlinear selection mechanism is realized.

PROOF: Let us take

$$u_\lambda(x) = \begin{cases} 1, & x \leq 0, \\ e^{-\lambda x}, & x > 0. \end{cases}$$

Then for  $u = u_\lambda(x)$  we have

$$\begin{aligned} \Phi_c[u] &< \frac{1}{2} \int_0^\infty e^{cx} u_x^2 dx + \int_{-\infty}^0 e^{cx} V(u) dx \\ &= \frac{\lambda^2}{2(2\lambda - c)} + \frac{V(1)}{c} \end{aligned}$$

as long as  $\lambda > c/2$ . Minimizing this expression with respect to  $\lambda$ , we obtain that the minimum is achieved at  $\lambda = \lambda_{\min} = c$ . This means that

$$\Phi_c[u_{\lambda_{\min}}] < \frac{c}{2} + \frac{V(1)}{c}.$$

Recalling that  $V(1) < 0$ , we see that the expression above is negative whenever

$$(5.3) \quad c = c_{\min} = \sqrt{-2V(1)} = \sqrt{2 \int_0^1 f(u) du}.$$

By continuity, there exists  $c > c_{\min}$  such that  $\Phi_c[u_{\lambda_{\min}}] \leq 0$ , so  $c > c_0$  (see equation (2.6)) and  $u_{\lambda_{\min}}$  satisfies the assumption of Theorem 2.8.  $\square$

Note that the expression in the right-hand side of equation (5.3) tends to  $c^*$  in the limit of the narrow reaction zone (that is, when  $f(u)$  is concentrated around  $u = 1$ ). Of course, it also provides a lower bound for  $c^*$ . Also note that for general  $f(u)$  (not necessarily positive) this proof can be modified to obtain an analogous estimate as long as  $V(1) < 0$ , which is a necessary condition for the existence of traveling waves with  $c > 0$  (see, for example, [30]).

Let us now perform a variational study of equation (1.2). This equation with  $\mu > 0$  satisfies the assumptions in equation (2.2) after a trivial rescaling of  $u$ . The stable positive equilibrium  $u = u_{\max}$  is given by

$$(5.4) \quad u_{\max} = \frac{\sqrt{1 + \sqrt{1 + 4\mu}}}{\sqrt{2}}.$$

The nice thing about equation (1.2) is that it admits a traveling wave solution with speed  $c > 0$  for  $\mu > -\frac{3}{16}$ , which can be found exactly [24, 28]. Furthermore, for  $0 < \mu < \frac{3}{4}$  this is precisely the solution with speed  $c^*$ , which is explicitly given by

$$(5.5) \quad c^* = \frac{2\sqrt{1 + 4\mu} - 1}{\sqrt{3}}.$$

Numerical evidence also suggests that for  $\mu \geq \frac{3}{4}$  we have  $c^* = c_0 = 2\sqrt{\mu}$ . Note that the expression in equation (5.5) for the speed of the wave also remains valid for  $\mu < 0$ ; it corresponds to the unique (up to translation) traveling wave solution in this case [1]. Also, when  $\mu = 0$ , it corresponds to the unique traveling wave solution that decays exponentially at positive infinity (for an existence proof in this degenerate case, see [8, 20]). Since our proof of existence of the traveling wave solution in fact extends to the case of  $f'(0) \leq 0$ , we will treat equation (1.2) for all  $\mu > -\frac{3}{16}$ . In short, the availability of an analytical expression for the speed  $c^*$  makes it possible to assess the effectiveness of our variational method, as well as to study the parametric dependence of  $c^*$ .

We start by looking for the value of  $c_{\max}$  as a function of  $\mu$ . Observe that for  $0 \leq u \leq 1$  we have

$$\frac{1}{8}c^2u^2 + V(u) = \frac{1}{2}\left(\frac{c^2}{4} - \mu - \frac{1}{2}u^2 + \frac{1}{3}u^4\right)u^2.$$

So, minimizing the expression in the brackets and demanding that it remains positive for all  $u$ , we obtain

$$(5.6) \quad c_{\max} = \frac{1}{2}\sqrt{3 + 16\mu}.$$

Note that the value of  $c_{\max}$  in equation (5.6) provides an upper bound for  $c^*$  and is typically within 20% of the exact value in equation (5.5). This equation also shows that  $c_{\max} > 0$  only for  $\mu > -\frac{3}{16}$ , as expected from the exact solution.

We are now going to analyze the existence of the traveling wave solutions with speed  $c^* > c_0$  and obtain the lower bounds for the speed, using our variational approach. To proceed, let us choose a very simple trial function that looks like a front:

$$(5.7) \quad u_\lambda(x) = \begin{cases} \frac{1}{2}u_{\max}e^{-\lambda x}, & x \geq 0, \\ \frac{1}{2}u_{\max}(2 - e^{\lambda x}), & x < 0, \end{cases}$$

where  $u_{\max}$  is given by equation (5.4). So,  $u_\lambda$  is a  $C^1$  function connecting the unstable equilibrium  $u = 0$  at positive infinity with the stable equilibrium  $u = u_{\max}$  at negative infinity, characterized by just a single parameter  $\lambda$  that gives the rate of exponential decay at positive infinity. What we are going to show below is that this choice of the trial function already allows us to determine the value of  $c^*$  with an accuracy of just a few percent in the entire range of  $\mu$ .

Let us substitute  $u_\lambda$  into the functional. It is straightforward, although tedious, to perform the integration; the resulting expression is a rational function of  $\lambda > \frac{c}{2}$  (the algebraic calculations were performed using *Mathematica* software):

$$\begin{aligned}
 \Phi_c[u_\lambda] = & - \left( \lambda^2 \left( 16c^7 \lambda (1 + \sqrt{1 + 4\mu}) - 92160 \lambda^6 (1 + \sqrt{1 + 4\mu} + \mu(6 + 4\sqrt{1 + 4\mu})) \right) \right. \\
 & - 8c^4 \lambda^2 (9(1 + \sqrt{1 + 4\mu}) + 832 \lambda^2 (1 + \sqrt{1 + 4\mu}) - 3\mu(13 + 19\sqrt{1 + 4\mu})) \\
 & + 192c \lambda^5 (53(1 + \sqrt{1 + 4\mu}) + 720 \lambda^2 (1 + \sqrt{1 + 4\mu}) - 2\mu(222 + 275\sqrt{1 + 4\mu})) \\
 & + 48c^2 \lambda^4 (295(1 + \sqrt{1 + 4\mu}) + 1536 \lambda^2 (1 + \sqrt{1 + 4\mu}) + \mu(973 + 383\sqrt{1 + 4\mu})) \\
 & - 12c^3 \lambda^3 (-187(1 + \sqrt{1 + 4\mu}) + 272 \lambda^2 (1 + \sqrt{1 + 4\mu}) - \mu(1153 + 779\sqrt{1 + 4\mu})) \\
 & - c^5 \lambda (592 \lambda^2 (1 + \sqrt{1 + 4\mu}) + 9(5 + 27\mu + 5\sqrt{1 + 4\mu}) + 17\mu\sqrt{1 + 4\mu}) \\
 & \left. - c^6 (-128 \lambda^2 (1 + \sqrt{1 + 4\mu}) + 3(1 + \sqrt{1 + 4\mu} + \mu(7 + 5\sqrt{1 + 4\mu}))) \right) / \\
 & (64c(c - 6\lambda)(c - 4\lambda)(c - 2\lambda)(c + 2\lambda)(c + 3\lambda)(c + 4\lambda)(c + 5\lambda)(c + 6\lambda)).
 \end{aligned}$$

The situation is complicated somewhat by the  $\mu$ -dependence, so before we apply our techniques, we need to make sure that  $\Phi_c[u_\lambda]$  can in fact be negative in a certain range of  $\mu$  for  $c > c_0 = 2\sqrt{\mu}$ . To do this, we will use the following trick. Take a sufficiently small  $\epsilon > 0$  and consider  $\lambda_\epsilon = \sqrt{\mu} + \epsilon$  and  $c = 2\sqrt{\mu} + \epsilon^2$ . Then clearly  $u_{\lambda_\epsilon} \in H_{c_\epsilon}^1(\mathbb{R})$ . A direct calculation then shows that

$$(5.8) \quad \lim_{\epsilon \rightarrow 0^+} \Phi_{c_\epsilon}[u_{\lambda_\epsilon}] = \frac{\mu(331 + 5785\sqrt{1 + 4\mu}) - 2727(1 + \sqrt{1 + 4\mu})}{215040\sqrt{\mu}}.$$

Analyzing this expression as a function of  $\mu$ , we see that it is negative as long as  $0 < \mu < \mu_{\max}$ , where

$$(5.9) \quad \mu_{\max} = \frac{23\,212\,224}{33\,466\,225} \simeq 0.6936.$$

Now, since the limit in equation (5.8) exists, for all  $0 < \mu < \mu_{\max}$  it is possible to choose  $\epsilon$  such that  $\Phi_{c_\epsilon}[u_{\lambda_\epsilon}] \leq 0$ , so  $u_{\lambda_\epsilon}$  satisfies the assumption of Theorem 2.8. Thus, we have just proven, using the trial function in equation (5.7), that the nonlinear selection mechanism is realized at least for  $0 < \mu < \mu_{\max}$ , with  $\mu_{\max}$  given explicitly by equation (5.9). Note that numerical analysis of the expression for  $\Phi_{c_\epsilon}[u_{\lambda_\epsilon}]$  shows that the value of  $\mu_{\max}$  obtained above is in fact optimal. Comparing this with the known exact result, we see that  $\mu_{\max}$  slightly underestimates the maximal value of  $\mu = \frac{3}{4}$  below which the nonlinear selection mechanism is realized [24, 28].

Let us now look at how closely our variational procedure allows us to estimate the propagation speed  $c^*$  from below. First of all, note that  $c_0 = 2\sqrt{\mu}$  is a natural lower bound for the speed  $c^*$  for  $\mu > 0$ . In fact, for  $\mu \geq \mu_{\max}$  the difference between  $c^*$  and  $c_0$  is less than 0.03%. Now, for  $\mu < \mu_{\max}$  and fixed  $c$ ,  $\Phi_c[u_\lambda]$

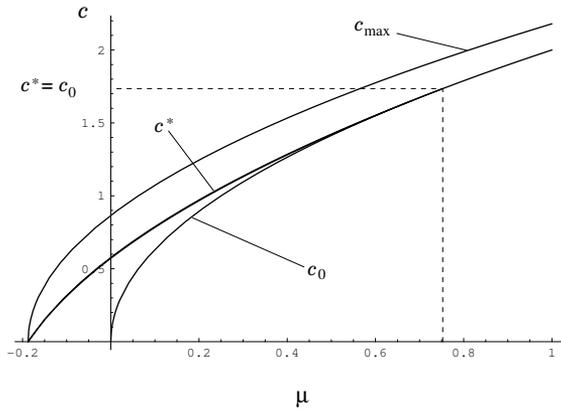


FIGURE 5.1. Speed of the traveling wave as a function of  $\mu$ .

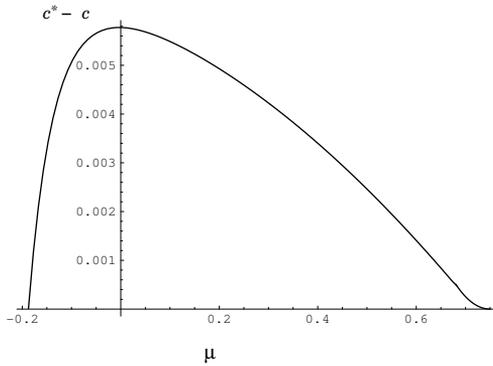


FIGURE 5.2. The absolute error  $c^* - c$  as a function of  $\mu$ .

as a function of  $\lambda$  has a minimum for  $\lambda > c/2$ , since  $\Phi_c[u_\lambda] \rightarrow +\infty$  as  $\lambda \rightarrow c/2 + 0$  (since in this case the integral is dominated by the quadratic terms at positive infinity, which for  $c > c_0$  are positive) or  $\lambda \rightarrow +\infty$ . Therefore, we look for the values of  $c$  and  $\lambda$  that solve

$$\Phi_c[u_\lambda] = 0, \quad \frac{\partial \Phi_c[u_\lambda]}{\partial \lambda} = 0.$$

The solution is obtained numerically for all  $-\frac{3}{16} < \mu < \mu_{\max}$ . The results are shown in Figure 5.1. This figure shows the speed  $c$  obtained above, together with the exact speed  $c^*$  given by equation (5.5), as well as  $c_{\max}$  given by equation (5.6) and  $c_0$ , all as functions of  $\mu$ . Note that the dependencies  $c(\mu)$  and  $c^*(\mu)$  are virtually indistinguishable. To see the agreement between  $c$  and  $c^*$ , we plot the difference between the two as a function of  $\mu$  in Figure 5.2. For  $\mu \geq \mu_{\max}$ , we use  $c = c_0$  as the lower bound for  $c^*$ . One can see that the error is within 1% of the true value for all  $\mu > 0$  and rapidly decreases as  $\mu$  increases. We also verified that the speed  $c$  remains within 2.5% of the exact value in the *entire* range of  $\mu$  (including

$-\frac{3}{16} < \mu \leq 0$ ). This accuracy is quite remarkable, especially considering the fact that it is achieved by tuning only one parameter,  $\lambda$ , in the trial function. Note that we experimented with a number of different choices of the trial functions; they all give comparable results. Thus, our method provides a very efficient way of estimating the propagation speed, as well as establishing the ranges of parameters for the nonlinear selection mechanism.

## 6 Concluding Remarks

To conclude, we have obtained a variational characterization of certain traveling wave solutions for scalar reaction-diffusion equations. These traveling waves are special in the sense that they have a nongeneric exponential decay at positive infinity (see equation (2.10)). In the context of the theory of Aronson and Weinberger, these solutions determine the propagation speed for the sufficiently localized initial conditions, so our method gives an easily verifiable answer to the problem of a linear versus nonlinear selection mechanism. We showed that our method allows us to obtain very accurate lower bounds for the propagation speed, as well as simple upper bounds.

We remark that various other variational characterizations of the speed  $c^*$  exist. One method is based on the minimax characterization of the traveling wave solutions (see, for example, [16, 30]). Another technique, introduced by Benguria and Depassier, uses integral variational principles in the phase plane [4, 5]. We point out that these methods are formulated within phase plane and may therefore be rather sensitive to the choices of the trial functions. We, on the other hand, demonstrated that in our variational approach there is very little sensitivity to the choice of the trial functions, which allows very accurate estimates of the propagation speed, at least for the example considered above. We also point out that our method does not just characterize the speed  $c^*$ , but also gives an alternative proof of existence of the traveling wave solutions of certain types, and is in fact much more general than the original setup (see also [19]).

Concerning the problem of propagation, the framework of Aronson and Weinberger allows us to associate the speed  $c^*$  with the average asymptotic speed of the leading edge for the solutions of equation (1.1) with suitable initial conditions. An interesting question arising here is whether this remains true for more general classes of initial conditions, for example, in the case of equation (1.2), for initial conditions that are not necessarily positive (see, e.g., [24]). Such problems have recently been addressed in [22], where the generalization of the functional from equation (2.13) was used to study questions of propagation and convergence to traveling wave solutions for reaction-diffusion systems of gradient type in cylinders in a very general setting. It is interesting to see whether the selection problem discussed in this paper is meaningful in this more general context and in fact reduces to the problem of existence of the special traveling waves whose existence can be captured by our variational procedure.

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