Magnetic skyrmions under confinement

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Abstract

We present a variational treatment of confined magnetic skyrmions in a minimal micromagnetic model of ultrathin ferromagnetic films with interfacial Dzylashinksii-Moriya interaction (DMI) in competition with the exchange energy, with a possible addition of perpendicular magnetic anisotropy. Under Dirichlet boundary conditions that are motivated by the asymptotic treatment of the stray field energy in the thin film limit we prove existence of topologically non-trivial energy minimizers that concentrate on points in the domain as the DMI strength parameter tends to zero. Furthermore, we derive the leading order non-trivial term in the Γ-expansion of the energy in the limit of vanishing DMI strength that allows us to completely characterize the limiting magnetization profiles and interpret them as particle-like states whose radius and position are determined by minimizing a renormalized energy functional. In particular, we show that in our setting the skyrmions are strongly repelled from the domain boundaries, which imparts them with stability that is highly desirable for applications. We provide explicit calculations of the renormalized energy for a number of basic domain geometries.

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1 Introduction

Magnetic skyrmions are particle-like non-collinear spin textures that were predicted to exist in non-centrosymmetric ferromagnets some 30 years ago [10, 9, 8] and have been recently observed in a number of magnetic systems [34, 46, 21, 39]. These coherent spin states are enabled by the non-trivial topological characteristics of their magnetizations [35, 19] which endow them with a considerable degree of thermal stability down to nanoscale and permit observation of magnetic skyrmions at room temperature [33, 11, 45]. The latter property makes magnetic skyrmions attractive candidates as information carriers in a new generation of spintronic devices for information technology [19, 47].

In ultrathin ferromagnetic films exhibiting skyrmions, the magnetization of the material may be described as a map from a two-dimensional plane to a three-dimensional sphere at the level of the continuum [26, 38]. As skyrmions are particle-like localized perturbations of the uniform ferromagnetic state, they must belong to a homotopy class of the equivalent (after a stereographic projection) continuous maps from $\mathbb{S}^2$ to itself. These classes are characterized by an integer topological degree, and the observed magnetic skyrmion configurations display the degree $+1$ of the identity map from $\mathbb{S}^2$ to $\mathbb{S}^2$ [35].[1] Mathematically, these configurations may be viewed as local minimizers of a suitable micromagnetic energy functional among configurations within the above homotopy class, and their existence was established for several models [32, 5, 3, 4].

[1]Note a sign error in the computation of the skyrmion number in this reference.

In a minimal model relevant to ultrathin ferromagnetic films capped with a layer of a heavy metal [38], the energy consists of a sum of the exchange energy forcing the magnetization to be constant in space, the interfacial Dzyaloshinskii-Moriya interaction (DMI) that promotes rotation of the magnetization vector, as well as the perpendicular magnetic anisotropy that forces the magnetization to align normally to the film plane and/or the Zeeman energy associated with the perpendicular applied magnetic field that has the same effect (for technical details, see section 2). Note that the problem of existence above is closely related to the one studied by Lin and Yang in a two-dimensional Skyrme model [29, 30]. Bernand-Mantel et al. established the asymptotic behavior of skyrmion solutions in the case of vanishingly small DMI strength and demonstrated that in this limit the magnetization profiles are close to the shrinking Belavin-Polyakov profiles, i.e., the degree +1 harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ [2], which in the minimal model described above are of Néel type [5, 3].

In the physics literature, the inability to continuously deform a topologically non-trivial skyrmion configuration into the topologically trivial uniform ferromagnetic state is often referred to as *topological protection* of magnetic skyrmions [35]. We note that this is somewhat of a misnomer, as a topologically non-trivial skyrmion configuration may in fact be deformed discontinuously into the uniform ferromagnetic state via core collapse by crossing a finite energy barrier [6]. In contrast, in finite samples such as nanodots or nanostrips that are of particular interest to applications, there is strictly speaking no topological obstruction that prohibits a homotopy between a skyrmion solution and the trivial solution for example by moving the skyrmion "through" the boundary. Nevertheless, these two solutions may still be separated by an energy barrier, and the question of existence of skyrmion solutions becomes more subtle.

In the minimal micromagnetic model that includes the stray field effect only via an effective anisotropy term [44], Rohart and Thiaville numerically constructed the Néel type radially-symmetric skyrmion solutions in a circular nano-dot [38]. It is unclear, however, whether these solutions always represent local energy minimizers, as the exchange energy in such a solution may be continuously lowered by moving the skyrmion towards the domain boundary, breaking the radial symmetry of the solution (see also section 2.5). Numerical studies of the minimal model in confined geometries do indicate the presence of a finite energy barrier towards skyrmion disappearance through the boundary under certain conditions [14, 13, 37]. The solutions in nanodisks were further analyzed numerically within the full micromagnetic model that includes the non-local stray field effects [41, 42, 1]. In particular, the obtained numerical profiles exhibit a strong perpendicular alignment of the magnetization at the domain edges, which can be explained by an additional contribution of the stray field enhancing the perpendicular magnetic anisotropy there (see also the experimental observations in [23]). As was shown in [16], in suitable thin film limits for the considered class of materials the effect of the stray field may be asymptotically accounted for via an effective penalty term forcing the magnetization to align with the normal to the film plane at the domain boundary, similarly to what happens in other ferromagnetic thin film problems [24]. In our problem, this should lead to the skyrmion being repelled from the sample edges.

In view of the above arguments, it is physically reasonable to consider the situation in which the magnetization at the film edge is rigidly aligned with a normal to the film plane. This may either be achieved via sending the penalization of the deviations at the boundary to infinity (corresponding to an appropriate choice of the material and geometric parameters [16]), or it could be the result of patterning the substrate of an extended ferromagnetic film with a strongly magnetically anchoring material (for a related approach, see [36]). Using these Dirichlet boundary conditions restores the possibility of topological protection, as continuous maps from a bounded two-dimensional domain to $\mathbb{S}^2$ with the boundary values pinned to a single direction can once again be classified by their topological degree. However, it is still not a priori clear whether minimizers would be attained in such a setting, as the possibility of a skyrmion shrinking to a point and collapsing is not excluded.

In this paper, we present a variational treatment of the minimal micromagnetic model of confined magnetic skyrmions in ultrathin ferromagnetic films with interfacial DMI, in which the confinement is provided by the Dirichlet boundary condition that forces the magnetization to take one direction normal to the film plane at the two-dimensional domain edge. We first prove existence of degree $+1$ minimizers of the energy consisting of the sum of the exchange and the interfacial DMI terms (with a possible addition of the perpendicular magnetic anisotropy term). We then focus on the regime in which the DMI is a perturbation to the exchange energy and develop a Γ-expansion of the energy in the limit of vanishing DMI strength. This leads to the appearance of a *renormalized energy* which determines asymptotically both the location and the radius of the skyrmion, whose shape is shown to be close to a Néel type degree $+1$ harmonic map from $\mathbb{R}^2$ to $\mathbb{S}^2$. Lastly, we explicitly construct the minimizers of the renormalized energy in the case of disk and strip domains. In particular, we show that the energy minimizing skyrmions are located in the disk center and on the strip midline, respectively, due to the effective repulsive interaction provided by the excess exchange energy from the tail of the magnetization profile. This confirms the physical expectation based on the numerical simulations that skyrmions can be robust particle-like objects even in finite samples of varying geometry.

1.1 Informal discussion of the results

From a mathematical standpoint, the confinement provided by the boundary data in fact simplifies the proof of existence of skyrmions compared to the case of the whole plane, since the translational symmetry of the problem is broken. In order to obtain the parameters describing the asymptotic behavior, we apply the rigidity of degree ± 1 harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ obtained by Bernand-Mantel, Muratov and Simon [3] after extending the magnetizations by a constant outside the domain using the Dirichlet boundary condition. This allows us to define the location, radius and rotation angle of the skyrmion. As is common in Γ-convergence arguments, we first obtain qualitative information such as linear scaling of the radius in the DMI constant or the fact that skyrmions are repelled from the boundary via non-optimal estimates, in order to obtain compactness properties of the energies.

A finer analysis requires us to also keep track of the tail correction to the skyrmion profile necessary to enforce the boundary condition. Here, the skyrmion position interacts with the boundary through the solution of the linearization of the harmonic map problem at the constant state given by the boundary condition, i.e., Laplace's equation for the in-plane components. A correction to the skyrmion core is not necessary to first order as the Belavin-Polyakov profiles are the exact degree one minimizers of the Dirichlet energy on the whole plane. The location of the optimal skyrmion is set by minimizing the interaction with the boundary, and the Néel character of the profile arises via minimizing the DMI term among all rotation angles. The radius then optimizes the balance of the two contributions. In simple domains, such as balls and strips, the Laplace's equation determining the tail correction can be explicitly solved by means of complex analysis, thus giving the full solution of the limiting problem in these cases.

Finally, we additionally include the perpendicular anisotropy at an appropriate scaling in the DMI constant. As the radius scales linearly in the DMI constant and the anisotropy energy of an exact Belavin-Polyakov profile is well-known to have a logarithmic divergence in its tail [17, 3], we consider effective anisotropies scaled down with the logarithm of the DMI strength. In this regime the anisotropy is essentially a continuous perturbation of our original problem with respect to the topology we determine the Γ-limit in. Furthermore, due to the fact that it is only the tail that contributes to the anisotropy at leading order and that it is of logarithmic character, we obtain that its contribution in the limit is in fact independent of the shape of the domain.

We note that the variational problem considered by us bears several similarities with the one for the classical Ginzburg-Landau model (without the magnetic field), in which the boundary data with a non-trivial topological degree force minimizers to form point-like vortices in the domain interior as the small parameter of the model goes to zero [7]. Our results for the magnetic skyrmion behavior in the limit of vanishing DMI strength thus provide a micromagnetic counterpart of the answer to the celebrated questions of Matano for Ginzburg-Landau vortices. In particular, we show that the skyrmion in a disk concentrates at the disk center in the limit and explicitly compute its asymptotic magnetization profile. We point out, however, that the analysis of the limit micromagnetic problem is considerably more delicate, as in contrast to the Ginzburg-Landau problem, the energy of a single skyrmion remains finite in the limit, and, therefore the tail contribution of the Dirichlet energy does not decouple from the problem for the skyrmion core. In particular, contrary to the Ginzburg-Landau vortex problem, the radius of the skyrmion turns out to be affected by the shape of the domain through the solution of the limit problem in the tail.

As in the problem of Ginzburg-Landau vortices, it is also natural to ask whether multiple skyrmion configurations may be similarly described in the vanishing DMI strength limit. In fact, the micromagnetic energy is known to exhibit a multitude of local energy minimizers other than a single magnetic skyrmion [40, 25]. However, our present analysis does not easily extend to the case of magnetization configurations of degree other than ± 1. Even at the level of existence we cannot rule out the collapse of minimizing sequences, failing to yield minimizers with a prescribed degree in this case. For the limit behavior of vanishing DMI strength, we also no longer have the quantitative rigidity estimate for the harmonic maps of arbitrary degree, which is the key tool in our analysis of a single skyrmion [3]. In fact, such an estimate has been recently shown to be false for degree 2 harmonic maps [15]. Similarly, we cannot give a positive answer to the existence of anti-skyrmions, i.e., minimizers among configurations with degree -1, as we do not know whether the basic energy bound in Lemma 3.2 holds in this class.

1.2 Outline of the paper

This paper is organized as follows. In section 2, we give the precise definitions of the micromagnetic energy, admissible classes, and the limit processes under consideration and then formulate our main results. In section 3, we prove existence of minimizers in the considered non-trivial topological class of maps with degree $+1$. In section 4, we derive the first-order term in the Γ-expansion of the energy in the DMI strength beyond the classical topological lower bound at zeroth order. Then, in section 5 we explicitly compute the renormalized energy for a number of geometries. Finally, in section 6 we show how to include the perpendicular magnetic anisotropy as a continuous perturbation to the limit energy.

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2 Main results

2.1 Definition of the energy

On a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, we consider the set of admissible functions

$$\mathcal{A} = \left\{ m \in H^1(\Omega; \mathbb{S}^2),\ m = -e_3 \text{ on } \partial\Omega,\ \mathcal{N}(m) = 1 \right\}, \tag{2.1}$$

where the degree of a function $m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ is defined as

$$\mathcal{N}(m) = \frac{1}{4\pi} \int_\Omega m \cdot (\partial_1 m \times \partial_2 m)\, \mathrm{d}x, \tag{2.2}$$

and we extend $m \in \mathcal{A}$ to the whole of $\mathbb{R}^2$ by setting $m = -e_3$ outside Ω. Here, as usual, we define

$$\mathring{H}^1(\mathbb{R}^2, \mathbb{S}^2) := \left\{ m \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^3) : \int_{\mathbb{R}^2} |\nabla m|^2\, \mathrm{d}x < \infty,\ |m| = 1 \text{ a.e. in } \mathbb{R}^2 \right\}. \tag{2.3}$$

It is well known that $\mathcal{N}(m) \in \mathbb{Z}$ for any $m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$, see Brezis and Coron [12]. For $m \in \mathcal{A}$ we wish to minimize the energy

$$\mathcal{E}_\kappa(m) = \int_\Omega \left(|\nabla m|^2 - 2\kappa m' \cdot \nabla m_3 \right) \mathrm{d}x, \tag{2.4}$$

where $\kappa \in \mathbb{R}$ is the DMI constant and we use the convention $m = (m', m_3)$, with m' taking values in $\mathbb{R}^2$. Passing from m to $\tilde{m} := (-m', m_3)$ when minimizing $\mathcal{E}_\kappa$ in the case of $\kappa < 0$, throughout the rest of the paper we may assume that $\kappa \geq 0$.

2.2 Statement of the results

We first make sure that the energy indeed admits minimizers. Due to the Dirichlet boundary conditions, minimizers exist for all $\kappa > 0$ sufficiently small even in the absence of the anisotropy penalizing the out-of-plane component of the magnetization. Note that at the same time the infimum of the energy is not attained for $\kappa = 0$ (see below).

Theorem 2.1. *There exists $\kappa_0 > 0$ depending only on Ω such that for all $0 < \kappa < \kappa_0$ there exists a minimizer of $\mathcal{E}_\kappa$ over $\mathcal{A}$.*

More importantly, we are also able to give a precise description of the minimizers for κ being small, i.e., the parameter regime in which one does have skyrmions. In particular, we can express their location and radius in terms of an optimization over the tail of the skyrmion.

To make this statement precise, we define the standard Belavin-Polyakov profile

$$\Phi(x) := \begin{pmatrix} -\frac{2x}{1+|x|^2} & \frac{1-|x|^2}{1+|x|^2} \end{pmatrix} \tag{2.5}$$

for $x \in \mathbb{R}^2$, which is the negative of the inverse stereographic projection, and we denote the set of all Belavin-Polyakov profiles by

$$\mathcal{B} := \left\{ R\Phi\left(\rho^{-1}(\cdot - a)\right) : R \in SO(3),\ \rho > 0,\ a \in \mathbb{R}^2 \right\}. \tag{2.6}$$

They arise as configurations achieving equality in the sharp topological bound

$$\int_{\mathbb{R}^2} |\nabla m|^2 \,\mathrm{d}x \geq 8\pi |\mathcal{N}(m)| \tag{2.7}$$

with degree $\mathcal{N} = 1$. In particular, they are precisely the minimizing harmonic maps of degree one, see Belavin and Polyakov [2] and [12, Lemma A.1]. It is therefore not surprising and indeed well known [17, 3], that minimizers of micromagnetic-type energies augmented with DMI should approach the set $\mathcal{B}$ when the Dirichlet energy dominates, i.e., when $\kappa \ll 1$. We can thus attempt to express the location and the radius of the skyrmions as $a \in \mathbb{R}^2$ and $\rho > 0$ of an approached Belavin-Polyakov profile in this regime. Notice that for $\kappa = 0$ an equality in (2.7) is achieved by a sequence of truncated Belavin-Polyakov profiles with vanishing radius, which fails to converge to an element in $\mathcal{A}$. This statement remains true also in the presence of an additional out-of-plane anisotropy term (see section 2.4).

However, as we expect the radius of the minimizers to shrink compared to the size of the domain as $\kappa \to 0$, we can only expect the close-by Belavin-Polyakov profiles to converge after a rescaling. Consequently, we have to find a Belavin-Polyakov profile for each minimizer at positive κ in a controlled way. An appropriate set of tools for such a purpose has been identified by Bernand-Mantel, Muratov, and Simon in the form of a quantitative rigidity result for degree one harmonic maps:

Theorem 2.2 ([3, Theorem 2.4]). *For $m \in \mathcal{A}$, let the Dirichlet excess be*

$$Z(m) := \int_\Omega |\nabla m|^2 \,\mathrm{d}x - 8\pi \tag{2.8}$$

and the Dirichlet distance to the set of the Belavin-Polyakov profiles be

$$D(m;\mathcal{B}) := \inf_{\phi\in\mathcal{B}} \left(\int_{\mathbb{R}^2} |\nabla(m-\phi)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.9}$$

Then the infimum in the definition of $D(m;\mathcal{B})$ is achieved, i.e., there exists a Belavin-Polyakov profile closest to each $m \in \mathcal{A}$. Moreover, there exists a universal constant $\eta > 0$ such that for all $m \in \mathcal{A}$ we have

$$\eta D^2(m;\mathcal{B}) \le Z(m). \tag{2.10}$$

Shorter, alternative proofs of this statement have later been provided by Hirsch and Zemas [22] and Topping [43].

In order to identify the Belavin-Polyakov profiles corresponding to minimizers of $\mathcal{E}_\kappa$ in the limit $\kappa \to 0$, we turn to computing the Γ-limit in a suitable topology retaining the location, the radius, the global rotation and the skyrmion tail. To this end, we have to identify the correct higher order Γ-expansion of the energy. By roughly minimizing over the above quantities, we will find in Lemma 3.2 below that there exists a constant $C > 0$ depending only on Ω such that

$$\inf_{\mathcal{A}} \mathcal{E}_\kappa \le 8\pi - C\kappa^2. \tag{2.11}$$

This suggests to seek a Γ-limit of the functional $\frac{\mathcal{E}_\kappa - 8\pi}{\kappa^2}$.

However, in order to rule out some behaviors of finite energy sequences that minimizers will not exhibit, such as skyrmions shrinking too fast or their centers approaching the boundary of Ω, we will restrict our attention to magnetizations whose energy is sufficiently low, i.e. we restrict the admissible set to

$$\mathcal{A}_\kappa = \left\{ m \in \mathcal{A} : \mathcal{E}_\kappa(m) - 8\pi < 0 \right\}. \tag{2.12}$$

Furthermore, in the Γ-limit we will only consider magnetizations $m_\kappa \in \mathcal{A}_\kappa$ which satisfy

$$\liminf_{\kappa\to 0} \frac{\mathcal{E}_\kappa(m_\kappa) - 8\pi}{\kappa^2} < 0 \tag{2.13}$$

Note that this corresponds to a finite energy sequence for the functional $\frac{\kappa^2}{|\mathcal{E}_\kappa - 8\pi|}$ defined on $\mathcal{A}_\kappa$.

To specify the topology for the Γ-limit, given $m \in \mathcal{A}_\kappa$ we choose $\phi_m(x) := R\Phi(\rho^{-1}(x-a))$ for $R \in SO(3)$, $\rho > 0$ and $a \in \mathbb{R}^2$ to minimize the Dirichlet distance to m after extension to $\mathbb{R}^2$ by $-e_3$. In addition, we will also consider the tail of the skyrmion $w_m := m + e_3 - \phi_m - Re_3$. Guessing from the construction of Lemma 3.2, we expect $\rho \sim \kappa$ and $\|\nabla w_m\|_{L^2(\mathbb{R}^2)} \sim \rho$. It turns out that the information $m = -e_3$ in $\mathbb{R}^2 \setminus \Omega$ will translate into an asymptotic expression for $\rho^{-1} w_m$ outside Ω, see Lemma 4.2. This motivates the following:

Definition 2.3. *Let*

$$\widetilde{\mathcal{A}_0} := \left\{ R_0 \in SO(3) : R_0 e_3 = e_3 \right\} \times (0, \infty) \times \Omega. \tag{2.14}$$

We then say that a sequence $m_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ *BP-converges to* $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0}$ *as* $\kappa_n \to 0$ *if and only if the following holds: There exist* $R_n \in SO(3)$*,* $\rho_n > 0$*,* $a_n \in \Omega$ *such that for* $\phi_n := R_n \Phi(\rho_n^{-1}(\bullet - a_n)) \in \mathcal{B}$ *we have*

$$\limsup_{n\to\infty} \kappa_n^{-2} \int_{\mathbb{R}^2} |\nabla(m_{\kappa_n} - \phi_n)|^2 \, \mathrm{d}x < \infty, \tag{2.15}$$

$$R_0 = \lim_{n\to\infty} R_n, \tag{2.16}$$

$$r_0 = \lim_{n\to\infty} \frac{\rho_n}{\kappa_n}, \tag{2.17}$$

$$a_0 = \lim_{n\to\infty} a_n. \tag{2.18}$$

Remark 2.4. *By the first condition and the triangle inequality in* $\mathring{H}^1(\mathbb{R}^2)$*, one can see that BP-limits are unique.*

We are now in a position to give the Γ-limit of $\frac{\mathcal{E}_\kappa - 8\pi}{\kappa^2}$ with respect to the above convergence.

Definition 2.5. *For* $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0}$ *let*

$$\mathcal{E}_0(R_0, r_0, a_0) := r_0^2 T(a_0) - 2r_0 \int_{\mathbb{R}^2} (R_0 \Phi)' \cdot \nabla \Phi_3 \, \mathrm{d}x, \tag{2.19}$$

where the Dirichlet contribution of the tail correction is

$$T(a_0) := \inf \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x : u(x) = 2\frac{x - a_0}{|x - a_0|^2} \text{ in } \mathbb{R}^2 \setminus \Omega \right\}. \tag{2.20}$$

We furthermore define a restricted admissible set

$$\mathcal{A}_0 := \left\{ (R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0} : \mathcal{E}_0(R_0, r_0, a_0) < 0 \right\}. \tag{2.21}$$

We can then state the Γ-convergence.

Theorem 2.6. *The* Γ*-limit as* $\kappa \to 0$ *of the functionals* $\frac{\mathcal{E}_\kappa - 8\pi}{\kappa^2}$ *restricted to* $\mathcal{A}_\kappa$ *with respect to the BP-convergence is given by* $\mathcal{E}_0$ *restricted to* $\mathcal{A}_0$ *in the sense that we have the following:*

(i) For every sequence of $\kappa_n \to 0$ *and* $m_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ *with* $\liminf_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} < 0$ *there exists a subsequence (not relabeled) and* $(R_0, r_0, a_0) \in \mathcal{A}_0$ *such that* m_{κ_n} *BP-converges to* (R_0, r_0, a_0)*.*

(ii) Let $\kappa_n \to 0$, let $m_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ BP-converge to $(R_0, r_0, a_0) \in \mathcal{A}_0$ and let

$$\liminf_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} < 0. \tag{2.22}$$

Then we have

$$\liminf_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} \geq \mathcal{E}_0(R_0, r_0, a_0). \tag{2.23}$$

(iii) For every $(R_0, r_0, a_0) \in \mathcal{A}_0$ and every sequence of $\kappa_n \to 0$ there exist $m_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ BP-converging to (R_0, r_0, a_0) such that

$$\limsup_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} \leq \mathcal{E}_0(R_0, r_0, a_0). \tag{2.24}$$

Remark 2.7. *The above version of Γ-convergence is equivalent to the usual notion for the functionals $\frac{\kappa^2}{|\mathcal{E}_\kappa - 8\pi|}$ and $|\mathcal{E}_0|^{-1}$ restricted to $\mathcal{A}_\kappa$ and $\mathcal{A}_0$, respectively.*

Notice that the last term in the definition of $\mathcal{E}_0$ is clearly minimized by $R_0 = \mathrm{id}$ among all R_0 satisfying $R_0 e_3 = e_3$, since this achieves an absolute maximum of the integrand by pointwise Cauchy-Schwarz inequality in view of the fact that Φ' is collinear to $\nabla\Phi_3$. Thus from the fact that $\int_{\mathbb{R}^2} \Phi' \cdot \nabla\Phi_3 \,\mathrm{d}x = 4\pi$, see [3, Lemma A.5], we have

$$\mathcal{E}_0(R_0, r_0, a_0) \geq \mathcal{E}_0(\mathrm{id}, r_0, a_0) = T(a_0)\left(r_0 - \frac{4\pi}{T(a_0)}\right)^2 - \frac{16\pi^2}{T(a_0)}. \tag{2.25}$$

Upon minimizing $\mathcal{E}_0$ over $\widetilde{\mathcal{A}_0}$, we can saturate the lower bound in (2.25) and obtain the following characterization of the minimizers of $\mathcal{E}_\kappa$.

Theorem 2.8. *Let $\kappa_n \to 0$ as $n \to \infty$ and let m_{κ_n} be minimizers of $\mathcal{E}_{\kappa_n}$ over $\mathcal{A}$. Then there exists a subsequence (not relabeled) and $a_0 \in \operatorname{argmin}_{a\in\Omega} T(a)$ such that with*

$$r_0 := \frac{4\pi}{T(a_0)}, \tag{2.26}$$

$$R_0 := \mathrm{id} \tag{2.27}$$

we get for $\phi_n := \Phi\left(\frac{\bullet - a_0}{r_0\kappa_n}\right) \in \mathcal{B}$ and all $n \in \mathbb{N}$ that

$$\int_{\mathbb{R}^2} |\nabla(m_{\kappa_n} - \phi_n)|^2 \,\mathrm{d}x \leq C\kappa_n^2 \tag{2.28}$$

and

$$\lim_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} = -\frac{16\pi^2}{T(a_0)}. \tag{2.29}$$

In particular, this theorem says that as $\kappa_n \to 0$ the appropriately translated and dilated minimizer $m_{\kappa_n}(r_0\kappa_n(\bullet) + a_0)$ converges to the canonical Belavin-Polyakov profile Φ in Dirichlet distance, up to a subsequence. In the original variables, the energy minimizing profile is, therefore, close to the Belavin-Polyakov profile of Néel type centered at a_0 and with the small radius $\rho_n = r_0\kappa_n$.

2.3 Explicit minimizers for specific domains

We next give several examples of geometries, in which an explicit minimizer of the limit problem may be obtained. We use the standard identification of the complex plane with $\mathbb{R}^2$ and write $z \in \mathbb{C}$ to denote a vector in the plane. The symbol $\bar{z}$ denotes the complex conjugate of z. We also introduce the Wirtinger derivatives $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, acting on $u : \mathbb{C} \to \mathbb{C}$.

Clearly, the infimum in (2.20) is attained by the unique harmonic extension of u from $\partial\Omega$ into Ω. With $u_{z_0} : \mathbb{C} \to \mathbb{C}$ solving

$$\Delta u_{z_0} = 0 \text{ in } \Omega, \qquad u_{z_0}(z) = \frac{2}{\bar{z} - \bar{z}_0} \text{ in } \mathbb{C} \setminus \Omega, \tag{2.30}$$

one can then write the limit energy associated with a skyrmion centered at $z_0 \in \mathbb{C}$ as

$$T(z_0) = \int_{\mathbb{R}^2} \nabla \bar{u}_{z_0} \cdot \nabla u_{z_0} \, \mathrm{d}x. \tag{2.31}$$

The following proposition allows us to reduce the computation of $T(z_0)$ to evaluating a derivative of $u_{z_0}(z)$ at $z = z_0$ for the considered geometries.

Proposition 2.9. *Let $\Omega \subset \mathbb{C}$ be a simply connected bounded domain with a boundary of class $C^{1,\alpha}$, for some $\alpha \in (0,1)$. Then we have*

$$T(z_0) = 8\pi \partial_z u_{z_0}(z_0). \tag{2.32}$$

We note that due to the continuous dependence of the boundary values of u on z_0, the function $T(z_0)$ is continuous for all $z_0 \in \Omega$. Moreover, it is not difficult to see that $T(z_0) \to +\infty$ as z_0 approaches $\partial\Omega$, so $T(z_0)$ always attains its minimum for some $z_0 \in \Omega$.

2.3.1 Disks

For the special choice $\Omega = B_\ell(0)$ with $\ell > 0$ we can fully solve the above minimization problem, obtaining that the skyrmion will be located in the disk's center.

Proposition 2.10. *For $\Omega = B_\ell(0)$ and $z_0 \in \Omega$, the map achieving $T(z_0)$ is given by*

$$u_{z_0}(z) = \begin{cases} \frac{2z}{\ell^2 - \bar{z}_0 z} & \text{if } z \in B_\ell(0), \\ \frac{2}{\bar{z} - \bar{z}_0} & \text{if } z \in \mathbb{C} \setminus B_\ell(0). \end{cases} \tag{2.33}$$

Its energy is given by

$$T(z_0) = \frac{16\pi\ell^2}{(\ell^2 - |z_0|^2)^2}, \tag{2.34}$$

which is minimized by $z_0 = 0$ *with* $T(0) = \frac{16\pi}{\ell^2}$. *The rescaled skyrmion radius is* $r_0 = \frac{\ell^2}{4}$ *and the corresponding limiting energy is* $\mathcal{E}_0\left(\mathrm{id}, \frac{\ell^2}{4}, 0\right) = -\pi\ell^2$.

Note that the minimizer achieving $T(a_0)$ has the special property of being a holomorphic function in Ω. In the next example of strips we will see that this does not necessarily have to be the case.

2.3.2 Strips

We can also consider the energy (2.4) on strips $\Omega_\ell = \mathbb{R} \times (-\ell/2, \ell/2)$ for $\ell > 0$. Technically, the previous statements do not apply as Ω_ℓ is not bounded. However, the arguments can be adjusted straightforwardly as strips support Poincaré inequalities. We will give the modifications in section 5.2 below.

The only change in the resulting statement is that in the BP-convergence we will, due to the translational invariance of Ω_ℓ in the first component, only track the second component of the skyrmion center, so that the limiting set is

$$\widetilde{\mathcal{A}_0} := \left\{R_0 \in SO(3) \,:\, R_0 e_3 = e_3\right\} \times (0, \infty) \times \left(-\frac{\ell}{2}, \frac{\ell}{2}\right). \tag{2.35}$$

Furthermore, the limiting energy is given by

$$\mathcal{E}_0(R_0, r_0, y_0) := r_0^2 T(iy_0) - 2r_0 \int_{\mathbb{R}^2} (R_0\Phi)' \cdot \nabla\Phi_3 \,\mathrm{d}x, \tag{2.36}$$

where

$$T(iy_0) := \inf\left\{ \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x \,:\, u(z) = \frac{2}{\bar{z} + iy_0} \text{ in } \mathbb{C} \setminus \Omega_\ell \right\}. \tag{2.37}$$

Also this problem can be solved explicitly.

Proposition 2.11. *For* $\ell > 0$, $\Omega_\ell = \mathbb{R} \times (-\ell/2, \ell/2)$, *and* $y_0 \in (-\ell/2, \ell/2)$, *the map achieving* $T(iy_0)$ *is given by*

$$u_{y_0}(z) = \begin{cases} \frac{\pi}{\ell} \tanh\left(\frac{\pi}{2\ell}(z + iy_0)\right) - \frac{\pi}{\ell}\coth\left(\frac{\pi}{2\ell}(\bar{z} + iy_0)\right) + \frac{2}{\bar{z}+iy_0} & \text{if } z \in \Omega_\ell, \\ \frac{2}{\bar{z}+iy_0} & \text{if } z \in \mathbb{C} \setminus \Omega_\ell. \end{cases} \tag{2.38}$$

Its energy is given by

$$T(iy_0) = \frac{4\pi^3}{\ell^2 \cos^2\left(\frac{\pi y_0}{\ell}\right)}, \tag{2.39}$$

which is minimized by $y_0 = 0$ *with* $T(0) = \frac{4\pi^3}{\ell^2}$. *The rescaled skyrmion radius is* $r_0 = \frac{\ell^2}{\pi^2}$ *and the corresponding limiting energy is* $\mathcal{E}_0\left(\mathrm{id}, \frac{\ell^2}{\pi^2}, 0\right) = -\frac{4\ell^2}{\pi}$.

The formula for u_{y_0} above was obtained by computing the harmonic extension of the boundary data in Fourier space, but to verify its validity we only need to check that it satisfies the conditions defining u_{y_0}.

2.3.3 Half-plane

For the half-space $\Omega = \mathbb{R} \times (-\infty, 0)$ our rigorous arguments cannot be salvaged, and indeed in this case the energy can be easily seen to be unbounded from below. However, we may still consider the problem as arising from a limiting procedure where the distance of the skyrmion center to the boundary of growing, smooth domains is fixed. Then we obtain the problem

$$\mathcal{E}_0(R_0, r_0, y_0) := r_0^2 T(iy_0) - 2r_0 \int_{\mathbb{R}^2} (R_0\Phi)' \cdot \nabla\Phi_3 \,\mathrm{d}x, \tag{2.40}$$

defined on

$$\widetilde{\mathcal{A}_0} := \left\{ R_0 \in SO(3) : R_0 e_3 = e_3 \right\} \times (0, \infty) \times (-\infty, 0) \tag{2.41}$$

and where

$$T(iy_0) := \inf\left\{ \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x : u(z) = \frac{2}{\bar{z} + iy_0} \text{ in } \mathbb{C} \setminus \Omega \right\}, \tag{2.42}$$

where the skyrmion is located at $z_0 = iy_0$ with $y_0 < 0$ in the limit.

The straightforward solution then gives information about how the energy of the skyrmion behaves as it approaches the boundary. Of course, the repelling effect of the boundary can also be seen from our rigorous analysis. However, in this situation, the estimate is especially transparent.

Proposition 2.12. *For* $\Omega = \mathbb{R} \times (-\infty, 0)$, *and* $y_0 \in (-\infty, 0)$, *the map achieving* $T(iy_0)$ *is given by*

$$u_{y_0}(z) = \begin{cases} \frac{2}{z + iy_0} & \text{if } z \in \Omega, \\ \frac{2}{\bar{z} + iy_0} & \text{if } z \in \mathbb{C} \setminus \Omega. \end{cases} \tag{2.43}$$

Its energy is given by $T(iy_0) = \frac{4\pi}{y_0^2}$, *the corresponding rescaled skyrmion radius is* $r_0 = y_0^2$ *and the limiting energy is* $\mathcal{E}_0\left(\mathrm{id}, y_0^2, y_0\right) = -4\pi y_0^2$.

2.4 Adding anisotropy

We may also consider the case where we augment our energy by an anisotropy term. In order not to significantly change the behavior of the Γ-limit, we choose the quality factor Q in dependence of κ such that, after the renormalization in the thin film limit by the local contribution of the stray field term, the resulting term is essentially a compact perturbation of our above results. As it is well known that the anisotropy contribution of a skyrmion with radius $\rho > 0$ behaves like $\rho^2|\log\rho|$, see for example [17, 3], and as in our case $\rho \sim \kappa$, the appropriate scaling is $Q - 1 = \lambda|\log\kappa|^{-1}$ for some $\lambda > 0$. Consequently, we obtain the modified energy

$$\mathcal{E}_{\kappa,\lambda}(m) := \int_\Omega \left(|\nabla m|^2 - 2\kappa m' \cdot \nabla m_3 + \frac{\lambda}{|\log\kappa|}|m'|^2 \right) \mathrm{d}x. \tag{2.44}$$

Notice that the statement of Theorem 2.1 remains valid for minimizers of $\mathcal{E}_{\kappa,\lambda}$.

The following proposition then implies that the Γ-limit with respect to the BP-convergence at order κ^2 is given by

$$\mathcal{E}_0(R_0, r_0, a_0) := r_0^2(T(a_0) + 8\pi\lambda) - 2r_0 \int_{\mathbb{R}^2} (R_0\Phi)' \cdot \nabla\Phi_3 \,\mathrm{d}x \tag{2.45}$$

for $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0}$.

Proposition 2.13. *For $\kappa_n \to 0$ as $n \to \infty$, let $m_{\kappa_n} \in \mathcal{A}$ BP-converge to $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0}$. Then we have*

$$\lim_{n\to\infty} \frac{1}{\kappa_n^2|\log\kappa_n|} \int_\Omega |m'_{\kappa_n}|^2 \,\mathrm{d}x = 8\pi r_0^2. \tag{2.46}$$

In particular, the above result shows that the addition of anisotropy does not affect the center of the skyrmion in the limit $\kappa \to 0$ in the considered regime. As before, the limit of $\mathcal{E}_0$ is achieved by the Néel profile, $R_0 = \mathrm{id}$, the center $a_0 = \mathrm{argmin}_{a_0\in\Omega} T(a_0)$ and

$$r_0 = \frac{4\pi}{\min_{a_0\in\Omega} T(a_0) + 8\pi\lambda}. \tag{2.47}$$

The corresponding minimal energy is

$$\min_{\mathcal{A}_0} \mathcal{E}_0 = -\frac{16\pi^2}{\min_{a_0\in\Omega} T(a_0) + 8\pi\lambda}. \tag{2.48}$$

2.5 A note on free boundary conditions

We wish to also mention the difference between the behavior of the energy for BP-converging sequences for the Dirichlet problem associated with the admissible class $\mathcal{A}$ and that of the analogous free problem in which the Dirichlet boundary condition at $\partial\Omega$ is absent. We point out that in this case the BP-limit does not give rise to a well-behaved energy whose minimization would yield the position of the skyrmion in Ω as $\kappa \to 0$. Indeed, in the latter case the restriction to Ω of the Néel-type Belavin-Polyakov profile $\phi_n = R_n \Phi(\rho_n^{-1}(\bullet - a_n))$ with $R_n = \text{id}$, $\rho_n/\kappa_n \to r_0$ and $a_n \to a_0 \in \Omega$ in Definition 2.3 is an example of a BP-convergent sequence, and, therefore, we have an upper bound on the Dirichlet energy excess for a sequence of m_{κ_n} BP-converging to (id, r_0, a_0) by

$$Z(\phi_n) = -\int_{\mathbb{R}^2 \setminus \Omega} |\nabla \phi_n|^2 \, \mathrm{d}x. \tag{2.49}$$

A straightforward computation shows that as $\kappa_n \to 0$ we have

$$\begin{aligned} \lim_{n\to\infty} \kappa_n^{-2} Z(\phi_n) &= -\lim_{n\to\infty} \kappa_n^{-2} \int_{\mathbb{R}^2 \setminus \Omega} |\nabla \phi_n'|^2 \, \mathrm{d}x \\ &= -8 r_0^2 \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{|x - a_0|^4} \, \mathrm{d}x = -4 r_0^2 \int_{\partial\Omega} \frac{(x - a_0) \cdot \nu(x)}{|x - a_0|^4} \, \mathrm{d}\mathcal{H}^1(x), \end{aligned} \tag{2.50}$$

where ν is the outward unit normal to $\partial\Omega$, and in the last line we carried out an integration by parts. This is a negative contribution that goes to negative infinity as a_0 approaches $\partial\Omega$.

For example, if, as in section 2.3.3, we take $\Omega = \mathbb{R} \times (-\infty, 0)$ and $a_0 = (0, y_0)$ with some $y_0 \in (-\infty, 0)$, then by (2.50) we have explicitly for the renormalized energy:

$$\mathcal{E}_0(\text{id}, r_0, y_0) := \inf_{m_{\kappa_n}} \liminf_{n\to\infty} \frac{E_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} \le \lim_{n\to\infty} \frac{E_{\kappa_n}(\phi_n) - 8\pi}{\kappa_n^2} = -\frac{2\pi r_0^2}{y_0^2} - 8\pi r_0, \tag{2.51}$$

where the infimum is over sequences of m_{κ_n} that BP-converge to (id, r_0, a_0). This energy clearly does not have a minimum in r_0, suggesting that the skyrmion is not able to stabilize its radius at a fixed distance towards the boundary. Similarly, at fixed radius the skyrmion is attracted towards the boundary. Dynamically this would give rise to the disappearance of a skyrmion from Ω via escape towards the boundary, with zero energy barrier. This is in contrast with the case of the Dirichlet boundary conditions considered in section 2.3.3, in which the exchange contribution has the opposite sign.

Finally, notice that an addition of a sufficiently strong anisotropy as in section 2.4 may restore existence of local minimizers. To get some sense for this, consider again a skyrmion in the half-plane as in the previous paragraph. With the addition of anisotropy we would then get

$$\mathcal{E}_0(\text{id}, r_0, y_0) \le r_0^2 \left(8\pi\lambda - \frac{2\pi}{y_0^2} \right) - 8\pi r_0, \tag{2.52}$$

and it is clear that the skyrmion should experience a repulsive interaction and have a well-defined optimal radius far enough from $\partial\Omega$, while it would still be attracted towards the boundary close enough to $\partial\Omega$.

Notation and presentation

Throughout the rest of the paper, we extend $m \in \mathcal{A}$ to $\mathbb{R}^2$ by $-e_3$. Furthermore, unless explicitly stated otherwise, the letters C, C' denote generic, positive constants only depending on Ω and which may change from line to line. Each subsection first lists its statements and provides a description for their proof and use throughout the rest of the paper. The actual proofs are collected at the end of the respective subsections.

3 Existence of minimizers

We first provide a simple lower bound for the energy that controls the L^2-norm of ∇m for sufficiently small κ.

Lemma 3.1. *For all $\kappa > 0$ and $m \in H^1(\Omega; \mathbb{S}^2)$ satisfying $m = -e_3$ on $\partial\Omega$ we have*

$$\mathcal{E}_\kappa(m) \geq (1 - C\kappa) \int_\Omega |\nabla m|^2 \,\mathrm{d}x, \tag{3.1}$$

for some $C > 0$ depending only on Ω. Furthermore, if also $m \in \mathcal{A}_\kappa$ and $\kappa < 1/(2C)$ we have

$$Z(m) \leq 16\pi C\kappa. \tag{3.2}$$

Next, we show by a construction that the infimum energy is strictly below the topological lower bound for the case of the pure Dirichlet energy.

Lemma 3.2. *For all $\kappa > 0$ we have*

$$\inf_{\mathcal{A}} \mathcal{E}_\kappa < 8\pi. \tag{3.3}$$

In particular, the restricted admissible sets $\mathcal{A}_\kappa$, see definition (2.12)*, are non-empty. Furthermore, there exist constants $C > 0$ and $\kappa_0 > 0$ depending only on Ω such that for all $\kappa \in (0, \kappa_0)$, we have*

$$\inf_{\mathcal{A}} \mathcal{E}_\kappa \leq 8\pi - C\kappa^2. \tag{3.4}$$

Proof of Lemma 3.1. For $m \in H^1(\Omega; \mathbb{S}^2)$ satisfying $m = -e_3$ on $\partial\Omega$ we have by Cauchy-Schwarz and Poincaré inequalities

$$\begin{aligned} \mathcal{E}_\kappa(m) &\geq \int_\Omega |\nabla m|^2 \,\mathrm{d}x - 2\kappa \left(\int_\Omega |m'|^2 \,\mathrm{d}x \int_\Omega |\nabla m_3|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\geq \int_\Omega |\nabla m|^2 \,\mathrm{d}x - C\kappa \left(\int_\Omega |\nabla m'|^2 \,\mathrm{d}x \int_\Omega |\nabla m_3|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}, \end{aligned} \tag{3.5}$$

from which (3.1) follows. Furthermore, under the assumption $m \in \mathcal{A}_\kappa$ we have $\mathcal{E}_\kappa(m) < 8\pi$, so combining this with (3.1) and a bound on κ we obtain $\int_\Omega |\nabla m|^2 \,\mathrm{d}x < 16\pi$. Using this fact together with (3.1), we obtain (3.2). $\square$

Proof of Lemma 3.2. Step 1: Truncation of the Belavin-Polyakov profile

We truncate the standard Belavin-Polyakov profile by choosing $L > 1$ and setting

$$f_L(r) := \begin{cases} \frac{2r}{1+r^2} & \text{if } r < L, \\ \frac{2}{1+L^2}(2L - r) & \text{if } L \le r < 2L, \\ 0 & \text{if } 2L \le r, \end{cases} \tag{3.6}$$

for $r > 0$ and

$$\Phi_L(x) := \left(-f_L(|x|)\frac{x}{|x|}, \operatorname{sign}(1 - |x|)(1 - f_L^2(|x|))^{\frac{1}{2}} \right) \tag{3.7}$$

for $x \in \mathbb{R}^2$.

One may then compute, see [3, equation (A.66)], that

$$|\nabla \Phi_L|^2(x) = \frac{|f_L'|^2(|x|)}{1 - f_L^2(|x|)} + \frac{f_L^2(|x|)}{|x|^2}, \tag{3.8}$$

so that

$$\int_{B_L(0)} |\nabla \Phi_L|^2(x)\,\mathrm{d}x = \frac{8\pi L^2}{1 + L^2} \tag{3.9}$$

and

$$|\nabla \Phi_L|^2(x) \le C L^{-4} \tag{3.10}$$

for all $x \in B_{2L}(0) \setminus B_L(0)$. Consequently, we have

$$\int_{B_{2L}(0)} |\nabla \Phi|^2\,\mathrm{d}x \le 8\pi + C L^{-2}. \tag{3.11}$$

Furthermore, as can be seen by a direct computation we have

$$-2 \int_{B_{2L}(0)} \Phi_L' \cdot \nabla \Phi_{L,3}\,\mathrm{d}x \le -8\pi + C L^{-2}. \tag{3.12}$$

Step 2: Construction of competitors

Without loss of generality, we may assume that $B_r(0) \subset \Omega$ realizes the maximum in the definition of the in-radius. Let now $\rho > 0$ and $L > 1$ be such that $2L\rho \le r$. Then the function $\phi_{\rho,L}(x) := \Phi_L(\rho^{-1}x)$ satisfies $\phi_{\rho,L} \in \mathcal{A}$. For $\kappa < 1$ we compute

$$\mathcal{E}_\kappa(\phi_{\rho,L}) \le 8\pi - 8\pi\kappa\rho + C L^{-2}. \tag{3.13}$$

To minimize the preceding expression, we need to choose ρ as big as possible, i.e., $\rho = \frac{r}{2L}$. This yields

$$\mathcal{E}_\kappa(\phi_{\rho,L}) \leq 8\pi - 4\pi\kappa r L^{-1} + C L^{-2}. \tag{3.14}$$

In particular, choosing L big enough, we obtain that

$$\mathcal{E}_\kappa(\phi_{\rho,L}) < 8\pi, \tag{3.15}$$

which yields (3.3). Furthermore, optimizing in L gives $L = c/(r\kappa)$ for some suitably chosen $c > 0$ depending only on Ω, so that $L > 1$ for $\kappa < c/r$ and

$$\mathcal{E}_\kappa(\phi_{\rho,L}) \leq 8\pi - C r^2 \kappa^2, \tag{3.16}$$

which completes the proof. □

Proof of Theorem 2.1. Let $(m_n) \in \mathcal{A}$ be a minimizing sequence. By Lemma 3.1, assuming that κ is small enough, we get that (m_n) is uniformly bounded in $H^1(\Omega; \mathbb{S}^2)$. Consequently, there exists a subsequence (not relabeled) and $m_\infty \in H^1(\Omega; \mathbb{S}^2)$ such that $m_n \to m_\infty$ in L^2 and $\nabla m_n \rightharpoonup \nabla m_\infty$ in L^2 as $n \to \infty$. Furthermore, by a weak-times-strong argument, we get $\int_{\mathbb{R}^2} m_n' \cdot \nabla m_{n,3} \, \mathrm{d}x \to \int_{\mathbb{R}^2} m_\infty' \cdot \nabla m_{\infty,3} \, \mathrm{d}x$ and, therefore, we have

$$\mathcal{E}_\kappa(m_\infty) \leq \liminf_{n\to\infty} \mathcal{E}_\kappa(m_n) = \inf_{\mathcal{A}} \mathcal{E}_\kappa. \tag{3.17}$$

Thus, it remains to prove that $m_\infty \in \mathcal{A}$, i.e., that $\mathcal{N}(m_\infty) = 1$.

Arguing as in [12, 32], we complete the squares to get for all $m \in H^1(\Omega; \mathbb{S}^2)$ that

$$\int_\Omega |\nabla m|^2 \, \mathrm{d}x \pm 8\pi \mathcal{N}(m) = \int_\Omega |\partial_1 m \mp m \times \partial_2 m|^2 \, \mathrm{d}x. \tag{3.18}$$

As a result, by the lower semicontinuity of the right-hand side in (3.18) and the continuity of the DMI term we have

$$\mathcal{E}_\kappa(m_\infty) \pm 8\pi\mathcal{N}(m_\infty) \leq \liminf_{n\to\infty} \left(\mathcal{E}_\kappa(m_n) \pm 8\pi\mathcal{N}(m_n)\right) = \liminf_{n\to\infty} \mathcal{E}_\kappa(m_n) \pm 8\pi. \tag{3.19}$$

Therefore, for small enough κ we get with the help of Lemmas 3.1 and 3.2:

$$\pm 8\pi\mathcal{N}(m_\infty) \leq \mathcal{E}_\kappa(m_\infty) \pm 8\pi\mathcal{N}(m_\infty) \leq \liminf_{n\to\infty} \mathcal{E}_\kappa(m_n) \pm 8\pi < 8\pi \pm 8\pi, \tag{3.20}$$

from which $\mathcal{N}(m_\infty) = 1$ immediately follows. □

4 The next-order Γ-limit

4.1 Preliminaries

Before we turn to the actual proof of Theorem 2.6, we establish a number of preliminary statements designed to provide compactness in BP-convergence. First, we prove that in fact the skyrmion center a_κ for $0 < \kappa \ll 1$ satisfies $a_\kappa \in \Omega$, as well as a lower bound for the Dirichlet excess of a minimizer in terms of its radius ρ and $\operatorname{dist}(a_\kappa, \mathbb{R}^2 \setminus \Omega)$. The idea is that for ϕ achieving the Dirichlet distance from m to $\mathcal{B}$ we have $\nabla(m - \phi)(x) = -\nabla\phi(x)$ for all $x \in \mathbb{R}^2 \setminus \Omega$, so that the control of Theorem 2.2 can be translated into a control over the radius and the center.

Lemma 4.1. *There exist $\kappa_0 > 0$ and $C, C' > 0$ depending only on Ω such that for all $0 < \kappa < \kappa_0$ the following statement holds:*

Let $m \in \mathcal{A}_\kappa$ and let $\phi(x) = R\Phi(\rho^{-1}(x - a))$ with $R \in SO(3)$, $\rho > 0$, $a \in \mathbb{R}^2$ achieve the Dirichlet distance from $\mathcal{B}$ to m. Then we have $a \in \Omega$ and

$$\frac{\rho^2}{\operatorname{dist}^2(a, \mathbb{R}^2 \setminus \Omega)} \le C Z(m) \le C' \kappa. \tag{4.1}$$

We now record some basic estimates for the skyrmion tail. As we wish to apply this result also in the construction of a recovery sequence, we take care to only assume $m \in \mathcal{A}$, not $m \in \mathcal{A}_\kappa$.

Lemma 4.2. *There exists a constant $C > 0$ only depending on Ω such that we have the following statement:*

Let $m \in \mathcal{A}$ and $\phi = R\Phi(\rho^{-1}(\bullet - a))$ with $R \in SO(3)$, $\rho \in \left(0, \frac{1}{2}\right)$, $a \in \Omega$. We define $w := m + e_3 - \phi - Re_3$. Then, for all $x \in \mathbb{R}^2 \setminus \Omega$ we have

$$\left|\frac{1}{\rho} w(x) - 2R\left(\frac{x-a}{|x-a|^2}, 0\right)\right| \le C \frac{\rho}{|x-a|^2}, \tag{4.2}$$

$$\left|\frac{1}{\rho} \nabla w(x) - 2\nabla\left(R\left(\frac{x-a}{|x-a|^2}, 0\right)\right)\right| \le \frac{\rho}{|x-a|^3} \tag{4.3}$$

as well as

$$\int_\Omega |w|^2 \,\mathrm{d}x \le C\left(\rho^2 + \int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 \,\mathrm{d}x\right). \tag{4.4}$$

We note that for $x \in \mathbb{R}^2 \setminus \Omega$ we have $m(x) + e_3 = 0$ and therefore $w = -\phi - Re_3$ in the lemma above.

With these bounds we can give an estimate for the DMI term. Again, we only assume $m \in \mathcal{A}$ to keep the statement applicable for the construction of the recovery sequence.

Lemma 4.3. *Let $\kappa \in (0,1)$, $m \in \mathcal{A}$ and $\phi = R\Phi(\rho^{-1}(\bullet - a))$ with $R \in SO(3)$, $\rho \in \left(0, \frac{1}{2}\right)$, $a \in \Omega$. Then there exist $C_1 > 0$ universal and $C_2 = C_2(\Omega, a) > 0$ such that the following holds:*

$$\left| \int_{\mathbb{R}^2} (R(\Phi + e_3))' \cdot \nabla (R\Phi)_3 \,\mathrm{d}x \right| \leq C_1. \tag{4.5}$$

and

$$\begin{aligned} \left| -2\kappa \int_\Omega m' \cdot \nabla m_3 \,\mathrm{d}x + 2\kappa\rho \int_{\mathbb{R}^2} (R(\Phi + e_3))' \cdot \nabla (R\Phi)_3 \,\mathrm{d}x \right| & \\ \leq C_2 \left(\int_{\mathbb{R}^2} |\nabla (m - \phi)|^2 \,\mathrm{d}x + \rho^{\frac{6}{5}} \right) \kappa & \end{aligned} \tag{4.6}$$

If additionally $m \in \mathcal{A}_\kappa$ and ϕ achieves the Dirichlet distance of m to $\mathcal{B}$, then there exists $\kappa_0 > 0$ and $C_3 > 0$ depending only on Ω such that for all $\kappa \in (0, \kappa_0)$ the estimate takes the form

$$\left| -2\kappa \int_\Omega m' \cdot \nabla m_3 \,\mathrm{d}x + 2\kappa\rho \int_{\mathbb{R}^2} (R(\Phi + e_3))' \cdot \nabla (R\Phi)_3 \,\mathrm{d}x \right| \leq C_3 \left(Z(m) + \rho^{\frac{6}{5}} \right) \kappa. \tag{4.7}$$

Proof of Lemma 4.1. Step 1 : $a \in \mathrm{conv}(\Omega)$, the convex envelope of Ω.

Towards a contradiction, we assume that $a \notin \mathrm{conv}(\Omega)$. Then there exists $n \in \mathbb{S}^1$ such that $a \cdot n \geq x \cdot n$ for all $x \in \Omega \subset \mathrm{conv}(\Omega)$. Consequently, $\{x \in \mathbb{R}^2 : (x - a) \cdot n > 0\} \subset \mathbb{R}^2 \setminus \Omega$ and by the estimates (2.10) and (3.2), recalling that $m(x) = -e_3$ for all $x \in \mathbb{R}^2 \setminus \Omega$, we have

$$4\pi = \int_{\{(x-a)\cdot n > 0\}} |\nabla \phi|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^2 \setminus \Omega} |\nabla \phi|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^2} |\nabla (\phi - m)|^2 \,\mathrm{d}x \leq C\kappa. \tag{4.8}$$

For small enough κ we have a contradiction.

Step 2: There exist $C, C' > 0$ such that

$$\rho^2 \leq C Z(m) \leq C'\kappa. \tag{4.9}$$

We have $\mathrm{diam}(\Omega) = \mathrm{diam}(\mathrm{conv}(\Omega))$. Since from Step 1 we know that $a \in \mathrm{conv}(\Omega)$, it follows that $\Omega \subset B_{\mathrm{diam}(\Omega)}(a)$. By a direct computation (as in [3, Equation (A.67)]) and (2.10), we have

$$\begin{aligned} \frac{8\pi}{1 + \rho^{-2}\,\mathrm{diam}^2(\Omega)} &= \int_{\mathbb{R}^2 \setminus B_{\rho^{-1}\,\mathrm{diam}(\Omega)}(0)} |\nabla \Phi|^2 \,\mathrm{d}x = \int_{\mathbb{R}^2 \setminus B_{\mathrm{diam}(\Omega)}(a)} |\nabla \phi|^2 \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^2 \setminus \Omega} |\nabla \phi|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^2} |\nabla (\phi - m)|^2 \,\mathrm{d}x \leq C Z(m). \end{aligned} \tag{4.10}$$

Therefore, together with (3.2) and taking κ small enough, we have $\rho^2 \leq C Z(m) \leq C'\kappa$.

Step 3: $a \in \Omega$, provided κ is small enough.

Towards a contradiction, let us assume that $a \notin \Omega$. We claim that since Ω is a Lipschitz domain, there exist $\alpha > 0$ and $\tilde{r} > 0$ depending only on Ω such that $\mathcal{C}_\alpha(a) \cap B_{\tilde{r}}(a) \subset \mathbb{R}^2 \setminus \Omega$, where $\mathcal{C}_\alpha(a)$

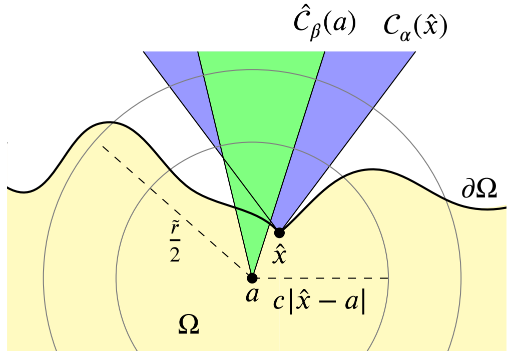

Figure 1: Sketch of Ω and the cones $C_\alpha(\hat{x})$ and $\hat{C}_\beta(a)$.

is a cone with vertex at a and the opening angle α. Indeed, we may assume that a is sufficiently close to Ω, and near a the set Ω is locally a subgraph of a Lipschitz function. Translating the point a vertically down towards $\partial\Omega$, we obtain a cone $C_\alpha(\tilde{a})$ pointing up with $\tilde{a} \in \partial\Omega$ that lies above Ω in $B_r(\tilde{a})$ for some $\alpha > 0$ and $r > 0$ depending only on Ω. Hence the claim follows by translating the cone $C_\alpha(\tilde{a})$ vertically upward until its vertex coincides with a.

We now compute (again, as in [3, Equation (A.67)], and using (2.10) and (3.2))

$$\begin{aligned}\frac{\rho^{-2}\tilde{r}^2}{1+\rho^{-2}\tilde{r}^2} \le C \int_{(C_\alpha(a)-a)\cap B_{\rho_\kappa^{-1}\tilde{r}}(0)} |\nabla\Phi|^2 \,\mathrm{d}x = C \int_{C_\alpha(a)\cap B_{\tilde{r}}(a)} |\nabla\phi|^2 \,\mathrm{d}x \\ \le C \int_{\mathbb{R}^2\setminus\Omega} |\nabla\phi|^2 \,\mathrm{d}x \le C\kappa.\end{aligned} \tag{4.11}$$

By step 2, the left-hand side is uniformly bounded from below, giving a contradiction for κ small enough.

Step 4: We have estimate (4.1).

Once again, since Ω is a Lipschitz domain there exist $\alpha > 0$ and $\tilde{r} > 0$ depending only on Ω such that for any $\hat{x} \in \partial\Omega$ we have $C_\alpha(\hat{x}) \cap B_{\tilde{r}}(\hat{x}) \subset \mathbb{R}^2 \setminus \Omega$. As $a \in \Omega$, there is $\hat{x} \in \partial\Omega$ such that $|\hat{x} - a| = \mathrm{dist}(a, \mathbb{R}^2 \setminus \Omega)$. We next fix $\delta > 0$ small enough (depending only on α). If $\mathrm{dist}(a, \mathbb{R}^2 \setminus \Omega) \ge \delta\tilde{r}$ then by the estimate (4.9) shown in Step 2 we have $\frac{\rho^2}{\mathrm{dist}^2(a,\mathbb{R}^2\setminus\Omega)} \le \frac{C\kappa}{\delta^2\tilde{r}^2} \le C'\kappa$ with C' depending only on Ω. If, on the other hand, $\mathrm{dist}(a, \mathbb{R}^2 \setminus \Omega) < \delta\tilde{r}$ (meaning $|\hat{x} - a|$ is very small comparing to $\tilde{r}$) then using basic geometry arguments we deduce that there exist a cone $\hat{C}_\beta(a)$ with the opening angle $\beta > 0$, and a constant $c > 0$ (both depending only on α and δ) such that $(\hat{C}_\beta(a) \cap B_{\frac{\tilde{r}}{2}}(a)) \setminus B_{c|\hat{x}-a|}(a) \subset C_\alpha(\hat{x}) \cap B_{\tilde{r}}(\hat{x}) \subset \mathbb{R}^2 \setminus \Omega$, see Figure 1.

Similarly to the previous calculations (see (4.11)), we obtain

$$\frac{\rho^{-2}\frac{\tilde{r}^2}{4}}{1+\rho^{-2}\frac{\tilde{r}^2}{4}} - \frac{\rho^{-2}c^2|\hat{x}-a|^2}{1+\rho^{-2}c^2|\hat{x}-a|^2} \le C \int_{(\hat{C}_\beta(a)\cap B_{\frac{\tilde{r}}{2}}(a))\setminus B_{c|\hat{x}-a|}(a)} |\nabla\phi|^2 \,\mathrm{d}x \le C Z(m). \tag{4.12}$$

We further calculate

$$
\begin{aligned}
\frac{\rho^{-2}\frac{\tilde r^2}{4}}{1+\rho^{-2}\frac{\tilde r^2}{4}} - \frac{\rho^{-2}c^2|\hat x - a|^2}{1+\rho^{-2}c^2|\hat x - a|^2} &= 1 - \frac{1}{1+\rho^{-2}\frac{\tilde r^2}{4}} - \frac{\rho^{-2}c^2|\hat x - a|^2}{1+\rho^{-2}c^2|\hat x - a|^2} \\
&= \frac{1}{1+\rho^{-2}c^2|\hat x - a|^2} - \frac{1}{1+\rho^{-2}\frac{\tilde r^2}{4}} \\
&\geq \frac{1}{1+\rho^{-2}c^2|\hat x - a|^2} - \frac{4\rho^2}{\tilde r^2}.
\end{aligned}
\tag{4.13}
$$

By step 2, we know that $\rho^2 \leq CZ(m)$ and, therefore, we have

$$
\frac{1}{1+\rho^{-2}c^2|\hat x - a|^2} \leq CZ(m). \tag{4.14}
$$

Taking κ small enough and recalling $Z(m) \leq C\kappa$, we deduce that

$$
\frac{\rho^2}{\operatorname{dist}^2(a, \mathbb{R}^2 \setminus \Omega)} \leq CZ(m) \leq C'\kappa, \tag{4.15}
$$

as claimed. □

Proof of Lemma 4.2. If $x \in \mathbb{R}^2 \setminus \Omega$, then $w(x) = -R\left(\Phi(\rho^{-1}(x-a)) + e_3\right)$. It is straightforward to compute

$$
\frac{1}{\rho}\left(\Phi(\rho^{-1}(x-a)) + e_3\right) = \left(-\frac{2(x-a)}{\rho^2 + |x-a|^2}, \frac{2\rho}{\rho^2+|x-a|^2}\right). \tag{4.16}
$$

Therefore, we have

$$
\frac{1}{\rho}w(x) - 2R\left(\frac{x-a}{|x-a|^2}, 0\right) = -R\left(\frac{2\rho^2(x-a)}{(\rho^2+|x-a|^2)|x-a|^2}, \frac{2\rho}{\rho^2+|x-a|^2}\right). \tag{4.17}
$$

Using the fact that $\rho^2 + |x-a|^2 \geq 2\rho|x-a|$ and the fact that $R \in SO(3)$, we obtain (4.2). Taking the gradient of both parts of (4.17) we arrive at (4.3) in a similar way.

For $x \in \mathbb{R}^2 \setminus B_{\operatorname{diam}(\Omega)}(a)$, from estimate (4.2) we get

$$
|w(x)| \leq C\rho, \tag{4.18}
$$

Therefore, with the help of Friedrichs' inequality [31, Corollary 6.11.2] we obtain

$$
\begin{aligned}
\int_\Omega |w|^2\,\mathrm{d}x \leq \int_{B_{\operatorname{diam}(\Omega)}(a)} |w|^2\,\mathrm{d}x &\leq C\left(\int_{B_{\operatorname{diam}(\Omega)}(a)} |\nabla w|^2\,\mathrm{d}x + \int_{\partial B_{\operatorname{diam}(\Omega)}(a)} |w|^2\,\mathrm{d}\mathcal{H}^1(x)\right) \\
&\leq C\left(\rho^2 + \int_{\mathbb{R}^2} |\nabla(m-\phi)|^2\,\mathrm{d}x\right).
\end{aligned}
\tag{4.19}
$$

which is the estimate (4.4). □

Proof of Lemma 4.3. Letting $w := m + e_3 - \phi - Re_3$, we compute

$$\begin{aligned}\int_\Omega m' \cdot \nabla m_3 \,\mathrm{d}x &= \int_\Omega (m+e_3)' \cdot \nabla (m+e_3)_3 \,\mathrm{d}x \\ &= \int_\Omega (\phi + Re_3)' \cdot \nabla(\phi_3 + Re_3)\,\mathrm{d}x + \int_\Omega (\phi + Re_3)' \cdot \nabla w_3 \,\mathrm{d}x \\ &\quad + \int_\Omega w' \cdot \nabla(\phi + Re_3)\,\mathrm{d}x + \int_\Omega w' \cdot \nabla w_3 \,\mathrm{d}x.\end{aligned} \tag{4.20}$$

The first term gives

$$\int_\Omega (\phi + Re_3)' \cdot \nabla(\phi_3 + Re_3)\,\mathrm{d}x = \rho \int_{\rho^{-1}(\Omega - a)} (R(\Phi + e_3))' \cdot \nabla (R\Phi)_3 \,\mathrm{d}x. \tag{4.21}$$

As $|(R(\Phi + e_3))' \cdot \nabla(R\Phi)_3| \le C/(1+|x|^3)$ uniformly in R, we get

$$\begin{aligned}\left|\rho \int_{\mathbb{R}^2 \setminus B_{\rho^{-1}\operatorname{dist}(a,\mathbb{R}^2\setminus\Omega)}(0)} (R(\Phi + e_3))' \cdot \nabla (R\Phi)_3 \,\mathrm{d}x\right| &\le C\rho \int_{\rho^{-1}\operatorname{dist}(a,\mathbb{R}^2\setminus\Omega)}^\infty \frac{r}{1+r^3}\,\mathrm{d}r \\ &\le C \frac{\rho^2}{\operatorname{dist}(a, \mathbb{R}^2 \setminus \Omega)}.\end{aligned} \tag{4.22}$$

Similarly, we get the estimate (4.5). In total, we obtain

$$\left|\int_\Omega (\phi + Re_3)' \cdot \nabla(\phi + Re_3)\,\mathrm{d}x - \rho \int_{\mathbb{R}^2} (R(\Phi + e_3))' \cdot \nabla(R\Phi)_3 \,\mathrm{d}x\right| \le C \frac{\rho^2}{\operatorname{dist}(a, \mathbb{R}^2 \setminus \Omega)}. \tag{4.23}$$

We treat the second term by using Young's inequality to get

$$\left|\int_\Omega (\phi + Re_3)' \cdot \nabla w_3 \,\mathrm{d}x\right| \le \frac{1}{2}\left(\rho^2 \int_{\rho^{-1}(\Omega - a)} |\Phi + e_3|^2 \,\mathrm{d}x + \int_\Omega |\nabla w|^2 \,\mathrm{d}x\right). \tag{4.24}$$

As $|\Phi + e_3| \le C/(1+|x|)$, we have

$$\int_{\rho^{-1}(\Omega - a)} |\Phi + e_3|^2 \,\mathrm{d}x \le C \int_0^{\rho^{-1}\operatorname{diam}(\Omega)} \frac{r}{(1+r)^2}\,\mathrm{d}r \le C' |\log \rho|. \tag{4.25}$$

Therefore, we get

$$\left|\int_\Omega (\phi + Re_3)' \cdot \nabla w_3 \,\mathrm{d}x\right| \le C\left(\rho^2 |\log\rho| + \int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 \,\mathrm{d}x\right). \tag{4.26}$$

For the third term, we find by Hölder's inequality that for $p > 2$ and $p' = \frac{p}{p-1} \in (1,2)$ we have

$$\left|\int_\Omega w' \cdot \nabla(\phi + Re_3)\,\mathrm{d}x\right| \le \|w'\|_{L^p(\Omega)} \|\nabla\phi\|_{L^{p'}(\Omega)}. \tag{4.27}$$

Noticing that $|\nabla\Phi|^{p'}$ decays sufficiently fast to be integrable, we furthermore compute

$$\int_\Omega |\nabla\phi|^{p'}\,\mathrm{d}x \le \rho^{2-p'} \int_{\mathbb{R}^2} |\nabla\Phi|^{p'}\,\mathrm{d}x \le C\rho^{2-p'}. \tag{4.28}$$

By the Sobolev embedding, for some $C_p > 0$ depending only on Ω and p we have

$$\|w'\|_{L^p(\Omega)} \le C_p \left(\|\nabla w'\|_{L^2(\Omega)} + \|w'\|_{L^2(\Omega)} \right). \tag{4.29}$$

Therefore, together with the estimates (4.4) and (4.27) we see that

$$\left| \int_\Omega w' \cdot \nabla(\phi + Re_3)\,\mathrm{d}x \right| \le C_p \rho^{\frac{2}{p'}-1} \left(\rho + \left(\int_{\mathbb{R}^2} |\nabla(m-\phi)|^2\,\mathrm{d}x \right)^{\frac{1}{2}} \right). \tag{4.30}$$

Applying Young's inequality to $\rho^{\frac{2}{p'}-1} \left(\int_{\mathbb{R}^2} |\nabla(m-\phi)|^2\,\mathrm{d}x \right)^{\frac{1}{2}}$ and choosing $p = 5$, we obtain

$$\begin{aligned} \left| \int_\Omega w' \cdot \nabla(\phi + Re_3)\,\mathrm{d}x \right| &\le C_p \left(\rho^{\frac{2}{p'}} + \rho^{\frac{4}{p'}-2} + \int_{\mathbb{R}^2} |\nabla(m-\phi)|^2\,\mathrm{d}x \right) \\ &\le C \left(\rho^{\frac{6}{5}} + \int_{\mathbb{R}^2} |\nabla(m-\phi)|^2\,\mathrm{d}x \right). \end{aligned} \tag{4.31}$$

For the last term, we have by Young's inequality and estimate (4.4) that

$$\left| \int_\Omega w' \cdot \nabla w_3\,\mathrm{d}x \right| \le C \left(\rho^2 + \int_{\mathbb{R}^2} |\nabla(m-\phi)|^2\,\mathrm{d}x \right). \tag{4.32}$$

Combining the estimates (4.23), (4.26), (4.31), and (4.32) in (4.20), we get the desired estimate (4.6). Furthermore, the only dependence of the constant C on a in (4.6) is through estimate (4.23), and for ϕ achieving the Dirichlet distance we can uniformly absorb this term into $Z(m)$ using Lemma 4.1. This gives us estimate (4.7). □

4.2 Γ-convergence

With the preliminary statements above, we can now argue for all the relevant compactness properties. Essentially, the centers a_κ cannot approach the boundary since otherwise the Dirichlet excess will be too large by Lemma 4.1. Estimates for the radii ρ_κ and the Dirichlet excess $Z(m_\kappa)$ easily follow. To control pinning of the rotation, we refer to the Moser-Trudinger-type inequality [3, Lemma 2.5] to side-step the fact that in two dimensions H^1 does not embed into L^∞. Recall that $\mathcal{A}_0$ and $\widetilde{\mathcal{A}_0}$ are defined in (2.21) and (2.14), respectively.

Lemma 4.4. *For every sequence of $\kappa_n \to 0$ and $m_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ with*

$$\limsup_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} < 0 \tag{4.33}$$

there exists a subsequence (not relabeled) and $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0}$ such that m_{κ_n} BP-converges to (R_0, r_0, a_0).

Note that the above lemma does not yet yield a limit in $\mathcal{A}_0$.

We now proceed to formulate the Γ-convergence result by first giving the lower bound statement. The convergence of the DMI term will follow from Lemma 4.3, and therefore we only need to deal with the Dirichlet excess. In particular, we have to prove that deviations from the Belavin-Polyakov profile are only energetically favorable in the tail of the skyrmion. To this end, we split the Dirichlet excess into a part localized in the core and a tail contribution. The localized part turns out to be given by the Hessian of the Dirichlet energy after using a new parametrization by mapping $\mathbb{R}^2$ to the sphere, with the Belavin-Polyakov profile closest to m. Since Belavin-Polyakov profiles are minimizers of the Dirichlet energy, the Hessian is non-negative and thus the contribution of any possible core correction is non-negative.

Proposition 4.5. *Let $\kappa_n \to 0$ and let $m_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ BP-converge to $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0}$ with*

$$\limsup_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} < 0. \tag{4.34}$$

Then we have

$$\liminf_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} \geq \mathcal{E}_0(R_0, r_0, a_0) \tag{4.35}$$

and, in particular, $(R_0, r_0, a_0) \in \mathcal{A}_0$.

We next turn to the construction of a recovery sequence. Essentially, we take the Belavin-Polyakov profile determined by the limit problem and modify the tail according to the harmonic function u arising as the minimizer of $T(a_0)$, see (2.20). The DMI term has again been treated in Lemma 4.3, so that after an appropriate construction only the Dirichlet term remains to be analyzed. To ensure that the tail correction does not affect the skyrmion core, we modify u to satisfy $u = 0$ in a small neighborhood of a_0, which is possible since points have zero capacity in $H^1(\mathbb{R}^2)$.

Proposition 4.6. *For every $(R_0, r_0, a_0) \in \mathcal{A}_0$ and all sequences of $\kappa_n \to 0$ there exists a sequence $m_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ BP-converging to (R_0, r_0, a_0) such that*

$$\limsup_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(m_{\kappa_n}) - 8\pi}{\kappa_n^2} \leq \mathcal{E}_0(R_0, r_0, a_0). \tag{4.36}$$

Throughout the proofs of these statements, we will omit the index n from the notation by abuse of notation.

Proof of Lemma 4.4. Let $m_\kappa \in \mathcal{A}_\kappa$ be a sequence satisfying condition (4.33). We take $\phi_\kappa(x) = R_\kappa \Phi(\rho_\kappa^{-1}(x - a_\kappa))$ with $R_\kappa \in SO(3)$, $\rho_\kappa > 0$ and $a_\kappa \in \mathbb{R}^2$ to be a Belyavin-Polyakov profile achieving the Dirichlet distance of m_κ to $\mathcal{B}$. We would like to show existence of a subsequence m_κ (not relabelled) BP-converging to $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0}$ (see Definition 2.3). Due to compactness of

$SO(3)$ it is clear that there exists $R_0 \in SO(3)$ such that $R_\kappa \to R_0$, however, we need to show that $R_0 e_3 = e_3$. Using Lemma 4.1, we also know that $a_\kappa \in \Omega$ and hence $a_\kappa \to a_0 \in \overline{\Omega}$, so we only need to show that $a_0 \in \Omega$. We begin by estimating ρ_κ and the Dirichlet distance through the Dirichlet excess.

Step 1: Estimate $\operatorname{dist}(a_\kappa, \mathbb{R}^2 \setminus \Omega)$ *and* $Z(m_\kappa)$*, and prove*

$$0 < \liminf_{\kappa \to 0} \frac{\rho_\kappa}{\kappa} \le \limsup_{\kappa \to 0} \frac{\rho_\kappa}{\kappa} < \infty. \tag{4.37}$$

By our assumption (4.33), for $\kappa > 0$ small enough there exists a subsequence (not relabeled) and $C_1 > 0$, such that we have

$$Z(m_\kappa) - 2\kappa \int_\Omega m_\kappa' \cdot \nabla m_{\kappa,3} \,\mathrm{d}x = \mathcal{E}_\kappa(m_\kappa) - 8\pi \le -C_1 \kappa^2. \tag{4.38}$$

Due to estimate (4.7) from Lemma 4.3, there exists $C_2 > 0$ with

$$Z(m_\kappa) - C_2 \kappa \left(\rho_\kappa + Z(m_\kappa) + \rho_\kappa^{\frac{6}{5}} \right) \le -C_1 \kappa^2. \tag{4.39}$$

Noting that $\rho_\kappa \le C\kappa^{\frac{1}{2}}$ by Lemma 4.1, if κ is small enough we can absorb $Z(m_\kappa)$ and $\rho_\kappa^{\frac{6}{5}}$ in the second term on the left-hand side into the other terms, giving

$$\frac{1}{2} Z(m_\kappa) - 2C_2 \kappa \rho_\kappa \le -C_1 \kappa^2. \tag{4.40}$$

We may thus use Lemma 4.1 again to obtain

$$\frac{\rho_\kappa^2}{\operatorname{dist}^2(a_\kappa, \mathbb{R}^2 \setminus \Omega)} - 2C_2 C \kappa \rho_\kappa \le -C_1 C \kappa^2. \tag{4.41}$$

Completing the square on the left-hand side, we get

$$\left(\frac{\rho_\kappa}{\operatorname{dist}(a_\kappa, \mathbb{R}^2 \setminus \Omega)} - C_2 C \kappa \operatorname{dist}(a_\kappa, \mathbb{R}^2 \setminus \Omega) \right)^2 \le \left(C_2^2 C^2 \operatorname{dist}^2(a_\kappa, \mathbb{R}^2 \setminus \Omega) - C_1 C \right) \kappa^2 \tag{4.42}$$

Since the left-hand must be non-negative and constants C, C_1, C_2 are positive, we obtain the estimate

$$\liminf_{\kappa \to 0} \operatorname{dist}(a_\kappa, \mathbb{R}^2 \setminus \Omega) > 0, \tag{4.43}$$

so that after passing to a further subsequence we have $a_\kappa \to a_0$ with $a_0 \in \Omega$.

Continuing from (4.42), we also have

$$\left| \frac{\rho_\kappa}{\kappa \operatorname{dist}(a_\kappa, \mathbb{R}^2 \setminus \Omega)} - C_2 C \operatorname{dist}(a_\kappa, \mathbb{R}^2 \setminus \Omega) \right| \le \left(C_2^2 C^2 \operatorname{dist}^2(a_\kappa, \mathbb{R}^2 \setminus \Omega) - C_1 C \right)^{\frac{1}{2}}. \tag{4.44}$$

Consequently, we obtain $\limsup_{\kappa\to 0} \frac{\rho_\kappa}{\kappa} < \infty$. Since

$$C_2 C \operatorname{dist}(a_\kappa, \mathbb{R}^2 \setminus \Omega) > \left(C_2^2 C^2 \operatorname{dist}^2(a_\kappa, \mathbb{R}^2 \setminus \Omega) - C_1 C\right)^{\frac{1}{2}}, \tag{4.45}$$

we also obtain $\liminf_{\kappa\to 0} \frac{\rho_\kappa}{\kappa} > 0$. Extracting a further subsequence, if necessary, we find $r \in (0, \infty)$ such that $\frac{\rho_\kappa}{\kappa} \to r$.

By the estimate (4.40), Theorem 2.2 and taking into account $\limsup_{\kappa\to 0} \frac{\rho_\kappa}{\kappa} < \infty$, we also get

$$\limsup_{\kappa\to\infty} \kappa^{-2} \int_{\mathbb{R}^2} |\nabla(m_\kappa - \phi_\kappa)|^2 \,\mathrm{d}x < \infty. \tag{4.46}$$

Step 2: Prove $R_0 e_3 = e_3$.

As was already mentioned, the existence of $R_0 \in SO(3)$ such that $R_\kappa \to R_0$ along a subsequence simply follows from compactness of $SO(3)$. Therefore, we are left with showing $R_0 e_3 = e_3$. By [3, Lemma 2.5 and Lemma A.2], there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} \exp\left(\frac{2\pi}{3} \frac{|m_\kappa - \phi_\kappa|^2}{\|\nabla(m_\kappa - \phi_\kappa)\|^2_{L^2(\mathbb{R}^2)}}\right) |\nabla \phi_\kappa|^2 \,\mathrm{d}x \le C. \tag{4.47}$$

Additionally, for κ small enough, on $\mathbb{R}^2 \setminus B_{\operatorname{diam}(\Omega)}(a_\kappa) \subset \mathbb{R}^2 \setminus \Omega$ we have $m_\kappa = -e_3$, as well as $|\phi_\kappa + R_\kappa e_3| \le C\kappa$ by Lemma 4.2 and estimate (4.37). As a result, also using estimate (4.46), for some constant $c > 0$ we have

$$\exp\left(c \frac{|R_\kappa e_3 - e_3|^2}{\kappa^2}\right) \int_{\mathbb{R}^2 \setminus B_{\operatorname{diam}(\Omega)}(a_\kappa)} |\nabla \phi_\kappa|^2 \,\mathrm{d}x \le C. \tag{4.48}$$

The usual integration in polar coordinates therefore gives

$$\exp\left(c \frac{|R_\kappa e_3 - e_3|^2}{\kappa^2}\right) \rho_\kappa^2 \le C, \tag{4.49}$$

which together with estimate (4.37) implies $\lim_{\kappa\to 0} R_\kappa e_3 = e_3$. □

Proof of Proposition 4.5. Let $m_\kappa \in \mathcal{A}_\kappa$ BP-converge to $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}_0}$. We first choose a subsequence in κ (not relabeled) such that

$$\liminf_{\kappa\to 0} \frac{\mathcal{E}_\kappa(m_\kappa) - 8\pi}{\kappa^2} = \lim_{\kappa\to 0} \frac{\mathcal{E}_\kappa(m_\kappa) - 8\pi}{\kappa^2}, \tag{4.50}$$

so that we may pass to further subsequences if necessary. Then by Lemma 4.4, and using the fact that BP-limits are unique, see Remark 2.4, we may further suppose that $R_\kappa \in SO(3)$, ρ_κ and a_κ determine a Belavin-Polyakov profile $\phi_\kappa := R_\kappa \Phi(\rho_\kappa^{-1}(\bullet - a_\kappa))$ achieving the Dirichlet distance of m_κ to $\mathcal{B}$. In particular, we can apply Lemmas 4.1–4.3.

The fact that the DMI term converges to the corresponding expression in the Γ-limit follows immediately from Lemma 4.3 and the assumptions (2.15) and (2.17) of BP-convergence, with an error of order $\kappa^{-1}Z(m_\kappa)+\kappa^{\frac{1}{5}}$ as $\kappa\to 0$. We therefore only have to deal with the limit behavior of the Dirichlet energy excess, which satisfies $Z(m_\kappa)\le C\kappa^2$ by (4.34) for κ small enough.

To this end, we note that $\phi_\kappa:\mathbb{R}^2\to\mathbb{S}^2$ is a bijective, conformal mapping. Therefore we may introduce the function $v_\kappa:\mathbb{S}^2\to\mathbb{R}^3$ defined as $v_\kappa:=\rho_\kappa^{-1}(m_\kappa\circ\phi_\kappa^{-1}-\mathrm{id}_{\mathbb{S}^2})$. From assumptions (2.15) and (2.17), and the change of variables formula [3, Lemma A.2] for conformal mappings we get

$$\limsup_{\kappa\to 0}\int_{\mathbb{S}^2}|\nabla v_\kappa|^2\,\mathrm{d}\mathcal{H}^2(z)<\infty. \tag{4.51}$$

Thus, applying the Poincaré-type estimate [3, Lemma 2.5] to $\rho_\kappa v_\kappa$ we get

$$\limsup_{\kappa\to 0}\int_{\mathbb{S}^2}\left(|v_\kappa|^2+|\nabla v_\kappa|^2\right)\,\mathrm{d}\mathcal{H}^2(z)<\infty. \tag{4.52}$$

Consequently, up to a subsequence there exists $v_0\in H^1(\mathbb{S}^2;\mathbb{R}^3)$ such that $v_\kappa\rightharpoonup v_0$ weakly in $H^1(\mathbb{S}^2;\mathbb{R}^3)$.

For $z\in\mathbb{S}^2$, we compute

$$v_\kappa(z)\cdot z=\rho_\kappa^{-1}(m_\kappa\circ\phi_\kappa^{-1}(z)-z)\cdot z=\rho_\kappa^{-1}(m_\kappa\circ\phi_\kappa^{-1}(z)\cdot z-1)=-\frac{1}{2\rho_\kappa}|m_\kappa\circ\phi_\kappa^{-1}(z)-z|^2. \tag{4.53}$$

Therefore, another application of [3, Lemma 2.5] and assumptions (2.15) and (2.17) gives

$$\limsup_{\kappa\to 0}\kappa^{-2}\int_{\mathbb{S}^2}|v_\kappa(z)\cdot z|^2\,\mathrm{d}\mathcal{H}^2(z)=\limsup_{\kappa\to 0}\int_{\mathbb{S}^2}\frac{1}{4\rho_\kappa^2\kappa^2}|m_\kappa\circ\phi_\kappa^{-1}(z)-z|^4\,\mathrm{d}\mathcal{H}^2(z)<\infty. \tag{4.54}$$

As a result, in the limit $\kappa\to 0$, we get that

$$v_0(z)\cdot z=0 \text{ for } \mathcal{H}^2\text{-a.e. } z\in\mathbb{S}^2. \tag{4.55}$$

In particular, v_0 is an H^1-regular, tangent vector field on the sphere.

We define a set $U_\kappa:=\phi_\kappa(B_{\sqrt{\rho_\kappa}}(a_\kappa))\subset\mathbb{S}^2$ and a function $w_\kappa:=m_\kappa+e_3-\phi_\kappa-R_\kappa e_3$. By [3, Lemma A.4], see also [27, Lemma 9], the excess can be rewritten as

$$\begin{aligned}\kappa^{-2}Z(m_\kappa)&=\kappa^{-2}\left(\int_{\mathbb{R}^2}|\nabla(m_\kappa-\phi_\kappa)|^2\,\mathrm{d}x-\int_{\mathbb{R}^2}|m_\kappa-\phi_\kappa|^2|\nabla\phi_\kappa|^2\,\mathrm{d}x\right)\\&=\frac{\rho_\kappa^2}{\kappa^2}\int_{\mathbb{R}^2\setminus B_{\sqrt{\rho_\kappa}}(a_\kappa)}\left|\nabla\left(\rho_\kappa^{-1}w_\kappa\right)\right|^2\,\mathrm{d}x+\frac{\rho_\kappa^2}{\kappa^2}\left(\int_{U_\kappa}|\nabla v_\kappa|^2\,\mathrm{d}\mathcal{H}^2(z)-2\int_{\mathbb{S}^2}|v_\kappa|^2\,\mathrm{d}\mathcal{H}^2(z)\right),\end{aligned} \tag{4.56}$$

where the transformation of the expressions onto the sphere is again via [3, Lemma A.2].

Due to assumptions (2.15) and Lemma 4.2, there exists $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ such that

$$\rho_\kappa^{-1} w'_\kappa \rightharpoonup u \tag{4.57}$$

in $H^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ as $\kappa \to 0$. Let $R'_0 \in SO(2)$ be such that $R_0 v = (R'_0 v', v_3)$ for all $v \in \mathbb{R}^3$. Note that, again by Lemma 4.2 and assumption (2.17), the limit for $x \in \mathbb{R}^2 \setminus \Omega$ satisfies

$$u(x) = 2R'_0 \frac{x-a_0}{|x-a_0|^2}. \tag{4.58}$$

Let $\delta \in (0, \frac{1}{2})$. For $\kappa > 0$ small enough we have by BP-convergence that $B_{\sqrt{\rho_\kappa}}(a_\kappa) \subset B_\delta(a_0)$. Similarly, due to (2.15) and the definition of Φ (see (2.5)), for the set $V_\delta := \{z \in \mathbb{S}^2 : |z+e_3| > \delta\}$ we obtain $V_\delta \subset (R_\kappa \Phi)(B_{\rho_\kappa^{-1/2}}(0)) = U_\kappa$. Therefore, for any fixed $\tilde{r} > \delta$ and κ small enough, we obtain

$$\kappa^{-2} Z(m_\kappa) \geq \frac{\rho_\kappa^2}{\kappa^2} \int_{B_{\tilde{r}}(a_0)\setminus B_\delta(a_0)} \left|\nabla\left(\rho_\kappa^{-1} w'_\kappa\right)\right|^2 \mathrm{d}x + \frac{\rho_\kappa^2}{\kappa^2}\left(\int_{V_\delta} |\nabla v_\kappa|^2 \,\mathrm{d}\mathcal{H}^2(z) - 2\int_{\mathbb{S}^2} |v_\kappa|^2 \,\mathrm{d}\mathcal{H}^2(z)\right). \tag{4.59}$$

Together with the compact Sobolev embedding $H^1(\mathbb{S}^2) \hookrightarrow L^2(\mathbb{S}^2)$, we therefore have in the limit $\kappa \to 0$ that

$$\liminf_{\kappa\to 0} \kappa^{-2} Z(m_\kappa) \geq r^2 \int_{B_{\tilde{r}}(a_0)\setminus B_\delta(a_0)} |\nabla u|^2 \,\mathrm{d}x + r^2\left(\int_{V_\delta} |\nabla v_0|^2 \,\mathrm{d}\mathcal{H}^2(z) - 2\int_{\mathbb{S}^2} |v_0|^2 \,\mathrm{d}\mathcal{H}^2(z)\right). \tag{4.60}$$

Letting $\delta \to 0$ and $\tilde{r} \to \infty$, we consequently get

$$\liminf_{\kappa\to 0} \kappa^{-2} Z(m_\kappa) \geq r^2 \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + r^2\left(\int_{\mathbb{S}^2} |\nabla v_0|^2 \,\mathrm{d}\mathcal{H}^2(z) - 2\int_{\mathbb{S}^2} |v_0|^2 \,\mathrm{d}\mathcal{H}^2(z)\right). \tag{4.61}$$

The non-negativity of the second variation of the Dirichlet energy at minimizers for tangent vector fields on the sphere [3, (4.5)] finally implies

$$\liminf_{\kappa\to 0} \kappa^{-2} Z(m_\kappa) \geq r^2 \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x = r^2 \int_{\mathbb{R}^2} |\nabla \tilde{u}|^2 \,\mathrm{d}x \geq r^2 T(a_0), \tag{4.62}$$

where we noted that $\tilde{u} := (R'_0)^{-1} u$ satisfies the boundary data required in the definition of T, see identity (4.58). □

Proof of Proposition 4.6. In contrast to the rest of the paper, in this proof the constant C may depend on r_0 and a_0. We fix $(R_0, r_0, a_0) \in \mathcal{A}_0$ and take $u \in \mathring{H}^1(\mathbb{R}^2;\mathbb{R}^2) \cap L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ with $u(x) = 2\frac{x-a_0}{|x-a_0|^2}$ for $x \in \mathbb{R}^2 \setminus \Omega$ and achieving

$$\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x = T(a_0). \tag{4.63}$$

In particular, u is harmonic in Ω, unique, and by the maximum principle satisfies $u \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$.

We start by constructing two auxiliary functions which will be useful in the construction of the recovery sequence.

Step 1: Truncating u at a_0.

For $\delta \in \left(0, \frac{1}{2} \wedge \frac{1}{2}\operatorname{dist}^2(a_0, \mathbb{R}^2 \setminus \Omega)\right)$ and $x \in \mathbb{R}^2$, we define

$$\eta_\delta(x) := \begin{cases} 1 & \text{if } |x-a_0| \le \delta, \\ \frac{2\log|x-a_0|}{\log\delta} - 1 & \text{if } \delta < |x-a_0| \le \delta^{\frac{1}{2}}, \\ 0 & \text{else.} \end{cases} \tag{4.64}$$

This function satisfies $\eta_\delta \in H^1(\mathbb{R}^2)$, $\operatorname{supp}\eta_\delta \subset B_{\sqrt{\delta}}(a_0) \subset \Omega$ and

$$\int_{\mathbb{R}^2} |\nabla\eta_\delta|^2 \,\mathrm{d}x \le \frac{C}{|\log\delta|}. \tag{4.65}$$

We now set $u_\delta := (1-\eta_\delta)u$ in order to enforce $u_\delta = 0$ in $B_\delta(a_0)$. Then we still have $u_\delta \in \mathring{H}^1(\mathbb{R}^2;\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2;\mathbb{R}^2)$, $u_\delta(x) = 2\frac{x-a_0}{|x-a_0|^2}$ for $x \in \mathbb{R}^2 \setminus \Omega$ and

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla u_\delta|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x &= \int_{B_{\sqrt{\delta}}(a_0)} \left(|-u\otimes\nabla\eta_\delta + (1-\eta_\delta)\nabla u|^2 - |\nabla u|^2\right) \,\mathrm{d}x \\ &\le C\int_{B_{\sqrt{\delta}}(a_0)} \left(|\nabla\eta_\delta|^2 + |\nabla u|^2\right) \,\mathrm{d}x \\ &= o_\delta(1) \end{aligned} \tag{4.66}$$

by the estimate (4.65) and $|\nabla u| \in L^2(\mathbb{R}^2)$.

Step 2: Construct the boundary data corrector $v_\kappa : \mathbb{R}^2 \to \mathbb{R}^2$ with $v_\kappa(x) = 0$ for all $x \in B_{\operatorname{dist}(a_0,\partial\Omega)/2}(a_0)$ and

$$\|v_\kappa\|_{L^\infty(\mathbb{R}^2)} + \|\nabla v_\kappa\|_{L^2(\mathbb{R}^2)} \le C\kappa^2. \tag{4.67}$$

Let $\phi_\kappa := R_0\Phi((r_0\kappa)^{-1}(\bullet - a_0))$ and again let $R_0' \in SO(2)$ be such that $R_0 v = (R_0'v', v_3)$ for all $v \in \mathbb{R}^3$. In order to achieve the correct boundary data, we define $v_\kappa(x) := -\phi_\kappa'(x) - r_0\kappa R_0' u$ for $x \in \mathbb{R}^2 \setminus \Omega$ and $v_\kappa = 0$ in $B_{\operatorname{dist}(a_0,\partial\Omega)/2}(a_0)$. Exploiting the estimates (4.2) and (4.3), we can extend $v_\kappa(x)$ using [18, Theorem 3.1] to a Lipschitz function on $\mathbb{R}^2$ such that

$$\|v_\kappa\|_{W^{1,\infty}(\mathbb{R}^2)} \le C\kappa^2. \tag{4.68}$$

The L^2 estimate for the gradient on the whole space follows.

Step 3: Definition of the recovery sequence.

Having introduced the boundary corrector v_κ, we may now define the test magnetizations $m_{\kappa,\delta} \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ as follows:

$$m_{\kappa,\delta}(x) := \begin{cases} \phi_\kappa(x) & \text{if } x \in B_\delta(a_0), \\ p\left(\phi'_\kappa + r_0 \kappa R'_0 u_\delta + v_\kappa\right) & \text{if } x \in \mathbb{R}^2 \setminus B_\delta(a_0), \end{cases} \tag{4.69}$$

where the map $p : \mathbb{R}^2 \to \mathbb{S}^2$ lifts $v \in B_1(0) \subset \mathbb{R}^2$ to $\mathbb{S}^2$ via $p(v) := \left(v, -\sqrt{1-v^2}\right)$. It is clear that since u and u_δ coincide outside $B_{\sqrt{\delta}}(a_0)$, using the definition of v_κ, we have $m_{\kappa,\delta}(x) = -e_3$ for all $x \in \mathbb{R}^2 \setminus \Omega$. Furthermore, for small enough $\delta > 0$ we have $u_\delta = 0$ and $v_\kappa = 0$ in $\overline{B_\delta}(a_0)$ and therefore the test configuration $m_{\kappa,\delta}$ is well defined for all κ sufficiently small depending on δ. We also have

$$\|\phi'_\kappa + r_0 \kappa R'_0 u_\delta + v_\kappa\|_{L^\infty(\mathbb{R}^2 \setminus B_\delta(a_0))} \leq C_\delta \kappa, \tag{4.70}$$

by the definition of Φ', boundedness of u_δ, and estimate (4.67). Here and in the following, the symbol $C_\delta > 0$ denotes a generic positive constant depending only on Ω, r_0, a_0, and δ.

Let $q_{\kappa,\delta}(x) := m_{\kappa,\delta}(x) - \phi_\kappa(x) - r_0 \kappa R_0(u_\delta(x), 0)$. For $x \in B_\delta(a_0)$ we of course have $q_{\kappa,\delta} = 0$. For $x \in \mathbb{R}^2 \setminus B_\delta(a_0)$ we compute

$$q_{\kappa,\delta}(x) = \left(v_\kappa, p_3\left(\phi'_\kappa + r_0 \kappa R'_0 u_\delta + v_\kappa\right) - p_3\left(\phi'_\kappa\right)\right). \tag{4.71}$$

Using estimates (4.67) and (4.70), as well as Lipschitz continuity of the square root near 1, we get

$$\|q_{\kappa,\delta}\|_{L^\infty(\mathbb{R}^2)} \leq C\left(\kappa^2 + \|\phi'_\kappa + r_0 \kappa R'_0 u_\delta + v_\kappa\|^2_{L^\infty(\mathbb{R}^2 \setminus B_\delta(a_0))} + \|\phi'_\kappa\|^2_{L^\infty(\mathbb{R}^2 \setminus B_\delta(a_0))}\right) \leq C_\delta \kappa^2. \tag{4.72}$$

To estimate the H^1-norm of $q_{\kappa,\delta}$, note that for any $v \in H^1(\mathbb{R}^2)$ with $\|v\|_{L^\infty(\mathbb{R}^2)} < 1$ we have $\nabla p_3(v) = \frac{v}{\sqrt{1-v^2}} \nabla v$ a.e. in $\mathbb{R}^2$ by the weak chain rule. Therefore, arguing as in [3, (A.67)] and using (4.66), (4.67) and (4.70), for all κ small enough we get

$$\left\|\nabla\left(p_3(\phi'_\kappa)\right)\right\|_{L^2(\mathbb{R}^2 \setminus B_\delta(a_0))} \leq C_\delta \kappa^2, \tag{4.73}$$

$$\left\|\nabla\left(p_3\left(\phi'_\kappa + r_0 \kappa R'_0 u_\delta + v_\kappa\right)\right)\right\|_{L^2(\mathbb{R}^2 \setminus B_\delta(a_0))} \leq C_\delta \kappa^2, \tag{4.74}$$

so that again with (4.67) we have

$$\|\nabla q_{\kappa,\delta}\|_{L^2(\mathbb{R}^2)} \leq C_\delta \kappa^2. \tag{4.75}$$

In particular, by the definition of $q_{\kappa,\delta}$ and (4.66) we have the estimate

$$\int_{\mathbb{R}^2} |\nabla(m_{\kappa,\delta} - \phi_\kappa)|^2 \, \mathrm{d}x \leq C_\delta \kappa^2, \tag{4.76}$$

for all κ sufficiently small depending on δ. In particular, by the definition of $\mathcal{N}(m_{\kappa,\delta})$ in (2.2) and the fact that $\mathcal{N}(m_{\kappa,\delta}) \in \mathbb{Z}$ we have $\mathcal{N}(m_{\kappa,\delta}) = 1$ and, therefore, $m_{\kappa,\delta} \in \mathcal{A}$ for small enough κ.

Step 4: Computation of the energy.

By Lemma 4.3 and estimate (4.76), we have

$$\begin{aligned} &\left| -2\kappa \int_{\Omega} m'_{\kappa,\delta} \cdot \nabla m_{\kappa,\delta,3} \,\mathrm{d}x + 2\kappa\rho \int_{\mathbb{R}^2} (R_0(\Phi + e_3))' \cdot \nabla \Phi_3 \,\mathrm{d}x \right| \\ &\leq C_\delta \kappa^{\frac{11}{5}}. \end{aligned} \tag{4.77}$$

For the Dirichlet energy, we again use [3, Lemma A.4] to get

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla m_{\kappa,\delta}|^2 \,\mathrm{d}x - 8\pi &= \int_{\mathbb{R}^2} |\nabla (m_{\kappa,\delta} - \phi_\kappa)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} |m_{\kappa,\delta} - \phi_\kappa|^2 |\nabla \phi_\kappa|^2 \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^2} |\nabla (m_{\kappa,\delta} - \phi_\kappa)|^2 \,\mathrm{d}x. \end{aligned} \tag{4.78}$$

By the estimates (4.75) and (4.66), we get

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla m_{\kappa,\delta}|^2 \,\mathrm{d}x - 8\pi &\leq r_0^2 \kappa^2 \int_{\mathbb{R}^2} |\nabla u_\delta|^2 \,\mathrm{d}x + C_\delta \kappa^3 \\ &\leq r_0^2 \kappa^2 \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + C_\delta \kappa^3 + \kappa^2 o_\delta(1). \end{aligned} \tag{4.79}$$

Therefore, we have

$$\limsup_{\kappa \to 0} \frac{\mathcal{E}_\kappa(m_{\kappa,\delta}) - 8\pi}{\kappa^2} \leq \mathcal{E}_0(R_0, r_0, a_0) + o_\delta(1). \tag{4.80}$$

By a diagonal argument, the statement then follows. □

Proof of Theorem 2.6. After we established the compactness of sequences obeying (4.33) with respect to the BP-convergence in Lemma 4.4, the statement of Theorem 2.6 follows by combining Propositions 4.5 and 4.6, and noting that by Proposition 4.5 the limit of the sequence in Lemma 4.4 belongs to $\mathcal{A}_0$. □

5 Analyzing the limit problem

Proof of Theorem 2.8. By the properties of Γ-convergence, minimizers m_κ of $\mathcal{E}_\kappa$ BP-converge to minimizers $(R_0, r_0, a_0) \in \mathcal{A}_0$ of $\mathcal{E}_0$ as $\kappa \to 0$ with the rate

$$\int_{\mathbb{R}^2} |\nabla (m_\kappa - \phi_\kappa)|^2 \,\mathrm{d}x \leq C\kappa^2, \tag{5.1}$$

where $\phi_\kappa := R_0\Phi\left(\frac{\bullet - a_0}{\kappa r_0}\right)$. Note that Theorem 2.6 does apply to minimizers of $\mathcal{E}_\kappa$ over $\mathcal{A}$ in view of Lemma 3.2.

Recall that

$$\mathcal{E}_0(R_0, r_0, a_0) = r_0^2 T(a_0) - 2r_0 \int_{\mathbb{R}^2} (R_0\Phi)' \cdot \nabla\Phi_3 \, \mathrm{d}x. \tag{5.2}$$

Since by the Cauchy-Schwarz inequality the integrand of the DMI term is minimized at each point when $R'\Phi'$ and $\nabla\Phi_3$ are parallel and since Φ' is parallel to $\nabla\Phi_3$, the DMI term as a whole is minimized for $R_0 = \mathrm{id}$. Direct calculation or [3, Lemma A.5] gives

$$2\int_{\mathbb{R}^2} \Phi' \cdot \nabla\Phi_3 \, \mathrm{d}x = 8\pi. \tag{5.3}$$

For $a_0 \in \operatorname{argmin}_{a\in\Omega} T(a)$ minimizing in r_0 therefore gives

$$r_0 = \frac{4\pi}{T(a_0)}, \tag{5.4}$$

$$\mathcal{E}_0(R_0, r_0, a_0) = -\frac{16\pi^2}{T(a_0)}. \tag{5.5}$$

which completes the proof. $\square$

Proof of Proposition 2.9. First of all, observe that under our assumptions, (2.30) is uniquely solvable in $C^\infty(\Omega;\mathbb{C}) \cap C^{1,\alpha}(\overline{\Omega};\mathbb{C})$, see [20, Theorem 8.34]. The expression in (2.31) may be conveniently rewritten as an integral over $\partial\Omega$. Integrating by parts and using that u_{z_0} is harmonic both inside and outside Ω (it is anti-holomorphic in Ω^c), we obtain

$$\begin{aligned} T(z_0) &= \int_\Omega \nabla \cdot (\bar{u}_{z_0} \nabla u_{z_0}) \, \mathrm{d}x + \int_{\Omega^c} \nabla \cdot (\bar{u}_{z_0} \nabla u_{z_0}) \, \mathrm{d}x \\ &= \int_{\partial\Omega} \bar{u}_{z_0} \left(\partial_\nu u_{z_0}\Big|_\Omega - \partial_\nu u_{z_0}\Big|_{\Omega^c} \right) \mathrm{d}\mathcal{H}^1(z), \end{aligned} \tag{5.6}$$

where ∂_ν denotes the derivative in the direction of the outward unit normal to $\partial\Omega$.

Since Ω is simply connected, both the real and the imaginary parts of the harmonic function u_{z_0} possess the unique, up to constants, harmonic conjugates in $\overline{\Omega}$ that belong to $C^{1,\alpha}(\overline{\Omega})$. Therefore u_{z_0} admits a decomposition

$$u_{z_0}(z) = f(z) + \overline{g(z)} \qquad z \in \overline{\Omega}, \tag{5.7}$$

for two functions $f(z)$ and $g(z)$ that are holomorphic in $\overline{\Omega}$.

Recall that if $\nu : \partial\Omega \to \mathbb{C}$ represents the outward unit normal to $\partial\Omega$ and f is holomorphic in $\overline{\Omega}$, we have $\partial_\nu f = \nu f'$ on $\partial\Omega$, where the prime denotes the usual derivative of a holomorphic function. Similarly, if $\tau := i\nu$ represents the unit tangent to $\partial\Omega$ in the counter-clockwise direction, the tangential derivative $\partial_\tau f = \tau f'$ on $\partial\Omega$. Therefore, using (5.7) we can write

$$\partial_\nu u_{z_0}\Big|_\Omega = \nu f' + \overline{\nu g'}, \qquad \partial_\nu u_{z_0}\Big|_{\Omega^c} = -\frac{2\bar{\nu}}{(\bar{z} - \bar{z}_0)^2} \tag{5.8}$$

on $\partial\Omega$. At the same time, by continuity of u_{z_0} across $\partial\Omega$ we have

$$\tau f' + \overline{\tau g'} = -\frac{2\bar{\tau}}{(\bar{z} - \bar{z}_0)^2} \tag{5.9}$$

on $\partial\Omega$. Thus we have

$$\partial_\nu u_{z_0}\Big|_\Omega - \partial_\nu u_{z_0}\Big|_{\Omega^c} = -2i\tau f' \tag{5.10}$$

on $\partial\Omega$, and the integral in (5.6) can be rewritten as a Cauchy type contour integral

$$T(z_0) = -4i \oint_{\partial\Omega} \frac{f'(z)}{z - z_0}\,\mathrm{d}z. \tag{5.11}$$

Finally, applying the residue theorem, we obtain $T(z_0) = 8\pi f'(z_0)$, which is precisely (2.32). □

5.1 Disks

Proof of Proposition 2.10. Without loss of generality, we may assume $\ell = 1$. Let u_{z_0} be defined by (2.33). We recall that for $z \in \mathbb{C}\backslash B_1(0)$ we have

$$u_{z_0}(z) = \frac{2}{\bar{z} - \bar{z}_0}. \tag{5.12}$$

In particular, up to complex conjugation, u_{z_0} is invariant under the Kelvin transform, i.e., for all $z \in \overline{B_1(0)}$ we have

$$u_{z_0}\left(\bar{z}^{-1}\right) = \frac{2z}{1 - \bar{z}_0 z} = u_{z_0}(z), \tag{5.13}$$

which is holomorphic and, therefore, harmonic in $B_1(0)$. Furthermore, the function u_{z_0} is continuous across $\partial B_1(0)$. By the uniqueness of boundary value problems for harmonic functions and continuity at the boundary, u_{z_0} is indeed the function achieving $T(z_0)$. By Proposition 2.9, we obtain

$$T(z_0) = \frac{16\pi}{(1 - |z_0|^2)^2}. \tag{5.14}$$

Clearly, this expression is minimized for $z_0 = 0$. The rest of the statement is obtained by a direct substitution. □

5.2 Strips

Instead of giving a full proof for the Γ-convergence in the case of a strip $\Omega_\ell := \mathbb{R} \times (-\ell/2, \ell/2)$ for $\ell > 0$, we point out the places in which the proof for bounded sets needs to be adjusted.

The first adjustment concerns Lemma 3.1, where a Poincaré estimate still holds for all admissible m. Furthermore, the proof of existence of a minimizer follows the lines of the proof in the whole space [3] to deal with the non-compact invariance under horizontal shifts.

Step 1 of Lemma 4.1 applies verbatim. Lemma 4.2 also works similarly, one just needs to use the decay behavior of the leading order contribution of w_m when applying a Poincaré type estimate on slices $\{x_1\} \times (-\ell, \ell)$ in order to achieve finite L^2 norm of w_m on Ω_ℓ. In Lemma 4.3, additional care needs to be taken in the integration (4.25), i.e.,

$$\int_{\rho^{-1}(\Omega_\ell - a)} |\Phi + e_3|^2 \,\mathrm{d}x \leq C|\log \rho|, \tag{5.15}$$

although the result is unchanged. Furthermore, the Sobolev embedding $H^1(\Omega_\ell) \hookrightarrow L^5(\Omega_\ell)$ still works by virtue of Ω_ℓ being an extension domain and [28, Theorem 8.5(ii)].

The remaining arguments work the same, up to the adjustment that due to translation invariance the component $(a_\kappa)_1$ cannot be controlled.

Proof of Proposition 2.11. Again, without loss of generality we may assume $\ell = \pi/2$. Let $\Omega := \mathbb{R} \times (-\frac{\pi}{4}, \frac{\pi}{4})$, and let u_{y_0} be the map defined in equation (2.38) for $\ell = \pi/2$ and $y_0 \in (-\frac{\pi}{4}, \frac{\pi}{4})$. The fact that u_{y_0} satisfies the boundary conditions follows from the two elementary identities:

$$\tanh\left(z \pm \frac{i\pi}{4}\right) = \coth\left(z \mp \frac{i\pi}{4}\right) \quad \text{for all } z \in \mathbb{C}. \tag{5.16}$$

Furthermore, as u_{y_0} is a sum of a holomorphic and an anti-holomorphic functions in Ω, it is harmonic in all the points where it is finite. As is well-known, the only singularities of both $\tanh z$ and $\coth z$ are simple poles on the imaginary axis. Since $|\operatorname{Im}(z + iy_0)| < \frac{\pi}{2}$ for all $z \in \Omega$, the tanh contribution is smooth in Ω. For the same reason, the coth contribution only has a singularity at $z = iy_0$, which, however, is precisely counterbalanced by $\frac{2}{\bar{z}+iy_0}$, as can be seen from the Laurent series of coth at the origin. Therefore u_{y_0} is indeed the map achieving $T(iy_0)$. As u_{y_0} decays sufficiently quickly at infinity, the arguments leading to Proposition 2.9 may also be adapted to the setting of strips, whereby we have

$$T(iy_0) = \frac{16\pi}{\cos^2(2y_0)}. \tag{5.17}$$

This function is clearly minimized by $y_0 = 0$, giving the statement. □

5.3 Half-plane

Proof of Proposition 2.12. Checking that u_{y_0} is indeed the minimizer realizing $T(iy_0)$ is trivial, and the rest of the statement is obtained via Proposition 2.9, again, adapted to the half-plane setting. □

6 Anisotropy as a continuous perturbation

Proof of Proposition 2.13. Due to the properties of the BP-convergence, there exist sequences $R_n \in SO(3)$, $\rho_n > 0$ and $a_n \in \Omega$ such that with $\phi_n(x) := R_n\Phi(\rho_n^{-1}(x - a_n))$ for $x \in \mathbb{R}^2$ the estimates of Definition 2.3 hold. Again, by Friedrichs' inequality [31, Corollary 6.11.2] we have

$$\begin{aligned}&\int_\Omega \left| m_{\kappa_n} + e_3 - \phi_n - R_n e_3 \right|^2 \,\mathrm{d}x \\ &\leq C\left(\int_\Omega \left| \nabla(m_{\kappa_n} - \phi_n) \right|^2 \,\mathrm{d}x + \int_{\partial B_{\operatorname{diam}(\Omega)}(a_0)} \left| \phi_n + R_n e_3 \right|^2 \,\mathrm{d}\mathcal{H}^1(x) \right),\end{aligned} \tag{6.1}$$

which allows to control the L^2–distance between m_{κ_n} and the Belavin-Polyakov profile ϕ_n that approximates it. In particular, by Lemma 4.2 and the properties of the BP-convergence, the two error terms in the right-hand side of (6.1) are of order κ_n^2 for all n large enough. Therefore, additionally reparametrizing the integral by the factor ρ_n^{-1} in the second step and using the assumption $\lim_{n\to\infty} \frac{\rho_n}{\kappa_n} = r_0$, we have that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\kappa_n^2 |\log \kappa_n|} \int_\Omega |m'_{\kappa_n}|^2 \,\mathrm{d}x &= \lim_{n\to\infty} \frac{1}{\kappa_n^2 |\log \kappa_n|} \int_\Omega \left| \left((R_n(\Phi(\rho_n^{-1}(x - a_n)) + e_3) \right)' \right|^2 \,\mathrm{d}x \\ &= \lim_{n\to\infty} \frac{r_0^2}{|\log \kappa_n|} \int_{\rho_n^{-1}(\Omega - a_n)} \left| \left((R_n(\Phi(x) + e_3) \right)' \right|^2 \,\mathrm{d}x.\end{aligned} \tag{6.2}$$

Since $\Phi_3 + 1 \in L^2(\mathbb{R}^2)$, the contribution of $R_n(0, \Phi_3 + 1)$ in the last integral is negligible, so that by expanding the square we get

$$\begin{aligned}&\lim_{n\to\infty} \frac{r_0^2}{|\log \kappa_n|} \int_{\rho_n^{-1}(\Omega - a_n)} \left| \left((R_n(\Phi(x) + e_3) \right)' \right|^2 \,\mathrm{d}x \\ &= \lim_{n\to\infty} \frac{r_0^2}{|\log \kappa_n|} \int_{\rho_n^{-1}(\Omega - a_n)} \left| \left(R_n \left(\frac{2x}{1 + |x|^2}, 0 \right) \right)' \right|^2 \,\mathrm{d}x.\end{aligned} \tag{6.3}$$

Therefore, together with the fact that $|(R_n v)'| \leq |v|$ for all $v \in \mathbb{R}^3$ we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{r_0^2}{|\log \kappa_n|} \int_{\rho_n^{-1}(\Omega - a_n)} \left| \left(R_n \left(\frac{2x}{1 + |x|^2}, 0 \right) \right)' \right|^2 \,\mathrm{d}x \\ &\leq \liminf_{n\to\infty} \frac{8\pi r_0^2}{|\log \kappa_n|} \left(\int_0^1 s^3 \,\mathrm{d}s + \int_1^{\rho_n^{-1} \operatorname{diam}(\Omega)} \frac{\mathrm{d}s}{s} \right) = 8\pi r_0^2,\end{aligned} \tag{6.4}$$

where we recalled that $\rho_n/\kappa_n \to r_0$ as $n \to \infty$ by the BP-convergence.

At the same time, as $\lim_{n\to\infty} a_n = a_0 \in \Omega$, there exists an $\tilde{s} > 0$ such that $B_{\tilde{s}}(a_n) \subset \Omega$ for all $n \in \mathbb{N}$. By the estimate (6.4) the expression on the right-hand side of estimate (6.3) is bounded, so that we can pass to the limit in the rotation $\lim_{n\to\infty} R_n = R_0$ and exploit $R_0 e_3 = e_3$, to get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{r_0^2}{|\log \kappa_n|} \int_{\rho_n^{-1}(\Omega - a_n)} \left| \left(R_n \left(\frac{2x}{1+|x|^2}, 0 \right) \right)' \right|^2 \mathrm{d}x \\
&\geq \limsup_{n\to\infty} \frac{r_0^2}{|\log \kappa_n|} \int_{B_{\rho_n^{-1}\tilde{s}}(0)} \frac{4|x|^2}{(1+|x|^2)^2} \mathrm{d}x \\
&= \limsup_{n\to\infty} \frac{8\pi r_0^2}{|\log \kappa_n|} \int_1^{\rho_n^{-1}\tilde{s}} \frac{\mathrm{d}s}{s} \\
&= 8\pi r_0^2.
\end{aligned} \tag{6.5}$$

The statement then follows from combining the estimates (6.4) and (6.5). □

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