

THRESHOLD PHENOMENA FOR SYMMETRIC-DECREASING RADIAL SOLUTIONS OF REACTION-DIFFUSION EQUATIONS

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ABSTRACT. We study the long time behavior of positive solutions of the Cauchy problem for nonlinear reaction-diffusion equations in \mathbb{R}^N with bistable, ignition or monostable nonlinearities that exhibit threshold behavior. For L^2 initial data that are radial and non-increasing as a function of the distance to the origin, we characterize the ignition behavior in terms of the long time behavior of the energy associated with the solution. We then use this characterization to establish existence of a sharp threshold for monotone families of initial data in the considered class under various assumptions on the nonlinearities and spatial dimension. We also prove that for more general initial data that are sufficiently localized the solutions that exhibit ignition behavior propagate in all directions with the asymptotic speed equal to that of the unique one-dimensional variational traveling wave.

1. Introduction. This paper is a continuation of our earlier work in [38], in which we considered a one-dimensional version of the Cauchy problem for the reaction-diffusion equation

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^N, \quad t > 0, \quad (1.1)$$

with initial condition

$$u(x, 0) = \phi(x) \geq 0, \quad \phi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \quad (1.2)$$

Here $u = u(x, t) \in [0, \infty)$, and the nonlinearity f is of monostable, ignition or bistable type (for a review, see, e.g., [52]). For all three nonlinearity types, f satisfies

$$f \in C^1[0, \infty), \quad f(0) = f(\theta_0) = f(1) = 0, \quad f(u) \begin{cases} \leq 0, & \text{in } [0, \theta_0] \cup (1, \infty), \\ > 0, & \text{in } (\theta_0, 1), \end{cases} \quad (1.3)$$

for some $\theta_0 \in [0, 1)$. This type of problems appears in various applications in physics, chemistry and biology [28, 33, 34, 39]. As a prototypical nonlinearity, one may consider

$$f(u) = u(1 - u)(u - \theta_0), \quad (1.4)$$

which gives rise to what is sometimes called Nagumo's equation [32, 40] and is also a particular version of the Allen-Cahn equation [1]. Moreover, in the case when

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$\theta_0 > 0$ we assume that the $u = 1$ equilibrium is more energetically favorable than the $u = 0$ equilibrium, i.e., that

$$\int_0^1 f(s) ds > 0. \quad (1.5)$$

For the nonlinearity f from (1.4), the condition in (1.5) corresponds to $\theta_0 < \frac{1}{2}$. Note that such nonlinearities are often known to exhibit *ground states*, i.e., positive variational solutions of (for a precise definition used in our paper, see Definition 2.1)

$$\Delta v + f(v) = 0, \quad x \in \mathbb{R}^N. \quad (1.6)$$

For the problem with $N = 1$ and unbalanced bistable nonlinearities, i.e., the nonlinearities satisfying (1.3) and (1.5) for which $\theta_0 > 0$ and $f(u) < 0$ for all $u \in (0, \theta_0)$, we proved, under some mild non-degeneracy assumption for the nonlinearity f near zero, that there are exactly three alternatives for the long-time behavior of solutions of (1.1) with symmetric-decreasing initial data satisfying (1.2) [38]:

- ignition, when the solution converges locally uniformly to the equilibrium $u = 1$;
- extinction, when the solution converges uniformly to the equilibrium $u = 0$;
- convergence to the unique ground state v centered at the origin.

The solution corresponding to the third alternative serves as a kind of separatrix between the extinction and the ignition behaviors for monotone families of initial data and may be referred to as the *threshold* solution. Moreover, this solution exhibits a *sharp* threshold behavior, in the sense that for any strictly increasing family of initial data exhibiting extinction for sufficiently small values of the parameter and ignition for sufficiently large values of the parameter there is exactly one member of the family that gives rise to a threshold solution. Similar results were also obtained for the case of monostable and ignition nonlinearities [38].

We note that studies of the long time behavior of solutions of (1.1) go back to the classical work of Fife [19], in which all possible long-time behaviors of solutions of (1.1) in one space dimension were classified for a general class of initial data for nonlinearities like the one in (1.4) (for related studies, see also [16–18]). Studies of the threshold behavior go back to Kanel' [27], and more recently to those by Zlatoš [53], Du and Matano [14] and Poláčik [44], who established sharpness of the threshold in a number of general settings. In particular, for $N = 1$ and bistable nonlinearities Du and Matano proved that one of the three alternatives stated earlier holds for arbitrary bounded, compactly supported initial data, provided that the ground state v is suitably translated. Among other things, for $N > 1$ and bistable nonlinearities with $f'(0) < 0$ Poláčik showed, still for compactly supported initial data, the existence of a sharp threshold and that the threshold solution becomes asymptotically radial and symmetric-decreasing relative to some point $x^* \in \mathbb{R}^N$ as $t \rightarrow \infty$. Combining this result with those of [8] (see also [22] for a related work), one can further conclude that the threshold solution converges to a ground state. We note that in the considered situation the case of non-symmetric initial data that do not have a sufficiently fast (exponential) decay at infinity remains open, even in one space dimension.

For $N \geq 2$, the problem of classifying the long time behaviors for solutions of (1.1) with nonlinearities as in (1.4) was treated by Jones [26]. For radial non-increasing initial data with values in the unit interval and crossing the threshold value of $u = \theta_0$, Jones used dynamical systems arguments to prove that the ω -limit set of each solution consists only of the stable homogeneous equilibria $u = 0$ and

$u = 1$, and of ground states. Under an extra assumption that the set of all ground states is discrete, Jones' analysis shows that any solution of the initial value problem considered in [26] converges either uniformly to $u = 0$, or locally uniformly to $u = 1$, or uniformly to one of the ground states as $t \rightarrow \infty$ (however, for existence of non-convergent solutions in a related context, see [42]). Alternatively, convergence to a ground state as the third alternative follows from the results of [8] for exponentially decaying initial data (the latter assumption is dropped in a recent work [20]). We note that in contrast to the $N = 1$ case, in higher dimensions it is generally not known whether or not (1.6) may exhibit continuous families of ground states, even for non-degenerate bistable nonlinearities (for examples of nonlinearities exhibiting arbitrarily large numbers of distinct ground states, see [3]). Some general sufficient conditions establishing the absence of multiplicity of the ground states were provided by Serrin and Tang [47] (existence of such solutions under very general assumptions on f goes back to the classical works of Berestycki and Lions [5] and of Berestycki, Lions and Peletier [6]; the possibility of multiple ground states for $N \geq 2$ and nonlinearities having zero as a locally stable equilibrium was pointed out in [41]). In particular, the results of Serrin and Tang apply to the nonlinearity in (1.4), thus establishing the expected multiplicity of the long time behaviors for Nagumo's equation for radial symmetric-decreasing data in all dimensions, with the *unique* ground state as the limit of the threshold solution. Another example of a bistable nonlinearity to which the uniqueness result in [47] applies is

$$f(u) = -u^r + (1 + \gamma)u^p - \gamma u^q, \quad 1 < r < p < q, \quad \gamma > \frac{(p-r)(q+1)}{(q-p)(r+1)}, \quad (1.7)$$

which satisfies (1.3) and (1.5).

At the same time, for monostable nonlinearities such a conclusion about the ground state multiplicity is easily seen to be false. For example, if $N \geq 3$ and $f(u) = u^p$ for $u \leq \frac{1}{2}$, with $p = p_S$, where $p_S := (N + 2)/(N - 2)$ is the Sobolev critical exponent (here and in the rest of the paper, we use the notations of [45] for the critical exponents; for the reader's convenience, the values of the relevant critical exponents are collected in Table 1), one gets a continuous family of ground states

$$v_\lambda(x) := \left(\lambda + \frac{|x|^2}{\lambda N(N-2)} \right)^{-(N-2)/2}, \quad (1.8)$$

for all $\lambda \in [2^{2/(N-2)}, \infty)$. Here v_λ are the unique, up to translations, ground states such that $\|v_\lambda\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{2}$ [11]. Very recently, Poláčik and Yanagida showed that the ω -limit sets for such problems may not consist only of stationary solutions, even in the radial case [43]. Therefore, the long time behavior of solutions is expected to be more delicate in the case of monostable nonlinearities.

In this paper, we revisit the problem of threshold behavior for radial symmetric-decreasing solutions of (1.1) in dimensions $N > 1$ whose studies were initiated by Jones for bistable nonlinearities. Our main contribution in the latter case is to remove the strong non-degeneracy assumptions of [26, 44], which read $f'(0) < 0$ and $f'(1) < 0$ in the context of the nonlinearities considered in this paper, and to establish the picture of sharp threshold behavior for radial symmetric-decreasing L^2 initial data, under a generic assumption on the structure of the set of all ground states. Note that our results are new even in the case $N = 1$, since, in contrast with [38], we do not impose any non-degeneracy assumptions on f any more, at the expense of not being able to determine precisely the limit energy of the threshold

TABLE 1. List of critical exponents.

Name	Exponent	Validity	$N = 3$
Fujita	$p_F = (N + 2)/N$	$N \geq 1$	5/3
Serrin	$p_{sg} = N/(N - 2)$	$N \geq 3$	3
Sobolev	$p_S = (N + 2)/(N - 2)$	$N \geq 3$	5
Joseph-Lundgren	$p_{JL} = 1 + 4/(N - 4 - 2\sqrt{N - 1})$	$N \geq 11$	–

solution. In addition, to the best of our knowledge this is a first general study of threshold phenomena for ignition and monostable nonlinearities for $N > 1$. In particular, we show that the character of the threshold behavior depends rather delicately on the dimension of space and may become quite intricate for $N \geq 3$.

Our paper is organized as follows. In Sec. 2, we discuss the motivations for our results and present the precise statements in Theorems 1–9. In Sec. 3, we present a number of auxiliary results. In Sec. 4, we prove Theorems 1–3 that are concerned with ignition and propagation phenomena. In Sec. 5, we prove Theorem 4 that treats bistable nonlinearities. In Sec. 6, we prove Theorems 5 and 6 dealing with ignition nonlinearities. Finally, in Sec. 7 we prove Theorems 7–9 treating monostable nonlinearities.

2. Statement of results. Our approach to the problem takes advantage of two variational structures possessed by (1.1). The first one is well known and expresses the fact that (1.1) is an L^2 gradient flow generated by the energy (for justification of this and the following statements, see the next section)

$$E[u] := \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla u|^2 + V(u) \right) dx, \quad V(u) := - \int_0^u f(s) ds, \quad (2.1)$$

which is well-defined for all $u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. In particular, this implies the energy dissipation identity for solutions $u(x, t)$ of (1.1) that belong to $H^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for each $t > 0$:

$$\frac{dE[u(\cdot, t)]}{dt} = - \int_{\mathbb{R}^N} u_t^2(x, t) dx, \quad (2.2)$$

and, therefore, the energy evaluated on solutions of (1.1) is non-increasing in time. Yet, in contrast to problems on bounded domains, E does not serve as a Lyapunov functional for (1.1), since it is not bounded from below a priori.

From (2.2), one easily deduces that whenever $\lim_{t \rightarrow \infty} E[u(\cdot, t)] \neq -\infty$, the ω -limit set of $u(x, t)$ may consist only of stationary solutions of (1.1). Indeed, in this case there exists a sequence of $t_n \in [n, n + 1)$ such that $u_t(\cdot, t_n) \rightarrow 0$ in $L^2(\mathbb{R}^N)$ as $n \rightarrow \infty$. Therefore, multiplying (1.1) by a test function $\varphi \in C_c^\infty(\mathbb{R}^N)$ and integrating, we can see from the obtained equation:

$$\int_{\mathbb{R}^N} \varphi u_t dx = - \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \varphi - f(u)\varphi) dx, \quad (2.3)$$

that if $u(\cdot, t_n)$ converges to some limit in $H_{loc}^1(\mathbb{R}^N)$, that limit satisfies (1.6) distributionally (hence also classically [23]). In view of the standard parabolic regularity, the latter is true, at least on a subsequence of $t_{n_k} \rightarrow \infty$. Furthermore, if this limit is independent of the subsequence, then by the uniform in space Hölder regularity of $u(x, \cdot)$ (see Proposition 3.3 below) the obtained limit is a full limit as $t \rightarrow \infty$

locally uniformly. Nevertheless, despite the energy $E[u(\cdot, t_n)]$ being bounded from below in this situation for all n , we cannot yet conclude that the obtained limit is a critical point of E , in the sense that the limit has *finite* energy. In this paper, we refer to those solutions of (1.6) that do as ground states. More precisely, we have the following definition.

Definition 2.1. We call $v \in C^2(\mathbb{R}^N)$ solving (1.6) a *ground state*, if $v > 0$, $v(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $|\nabla v| \in L^2(\mathbb{R}^N)$ and $V(v) \in L^1(\mathbb{R}^N)$.

One naturally expects that for a variety of initial data the solutions of (1.1) go locally uniformly to the equilibrium $u = 1$, whose energy under (1.5) is formally equal to negative infinity. The latter is intimately related to the phenomenon of *propagation*, whereby the solution at long times may look asymptotically like a radially divergent front invading the $u = 0$ equilibrium by the $u = 1$ equilibrium with finite propagation speed, even for non-radial initial data [2, 25, 26].

To discern between different classes of long time limit behaviors of solutions of (1.1), it is useful to take advantage of a different variational structure of (1.1) that was pointed out in [35]. In the case of radial solutions of (1.1), we may formulate this variational structure as follows. Let $x = (y, z) \in \mathbb{R}^N$, where $y \in \mathbb{R}^{N-1}$ and $z \in \mathbb{R}$ (this notation is used throughout the rest of the paper). For a fixed $c > 0$, define

$$\tilde{u}(y, z, t) := u(y, z + ct, t), \tag{2.4}$$

which corresponds to $u(x, t)$ in the reference frame moving with constant speed c in the z -direction. Then (1.1) written in terms of \tilde{u} takes the following form:

$$\tilde{u}_t = \Delta \tilde{u} + c\tilde{u}_z + f(\tilde{u}). \tag{2.5}$$

This equation is a gradient flow in the exponentially weighted space $L_c^2(\mathbb{R}^N)$, defined to be the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{L_c^2(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} e^{cz} |u|^2 dx \right)^{1/2}, \tag{2.6}$$

and is generated by the functional

$$\Phi_c[u] := \int_{\mathbb{R}^N} e^{cz} \left(\frac{1}{2} |\nabla u|^2 + V(u) \right) dx, \tag{2.7}$$

which is well-defined for all $u \in H_c^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where $H_c^1(\mathbb{R}^N)$ is the exponentially weighted Sobolev space similarly obtained from $C_c^\infty(\mathbb{R}^N)$ via completion with respect to the norm

$$\|u\|_{H_c^1(\mathbb{R}^N)} := \left(\|u\|_{L_c^2(\mathbb{R}^N)}^2 + \|\nabla u\|_{L_c^2(\mathbb{R}^N)}^2 \right)^{1/2}. \tag{2.8}$$

Note that the space obtained in this way is a Hilbert space with the naturally defined inner product.

The above formulation captures *propagation* of solutions of (1.1) [35, 36]. Notice that in the radial context we arbitrarily chose the last component of x as the axis of propagation. More generally, one can still use the above variational structure to analyse propagation in an arbitrary direction in \mathbb{R}^N by rotating the initial condition appropriately. The dissipation identity for the solutions of (2.5) that belong to

$H_c^2(\mathbb{R}^N)$ (the space of all functions in $H_c^1(\mathbb{R}^N)$ whose first derivatives also belong to $H_c^1(\mathbb{R}^N)$) takes the form:

$$\frac{d\Phi_c[\tilde{u}(\cdot, t)]}{dt} = - \int_{\mathbb{R}^N} e^{cz} \tilde{u}_t^2(\cdot, t) dx. \tag{2.9}$$

The constant $c > 0$ above is arbitrary and can be suitably chosen for the purposes of the analysis. One particular value of c is special, however.

Proposition 2.2. *Let $N = 1$ and let (1.3) hold with some $\theta_0 \in [0, 1)$. Also let $f'(0) = 0$ if $\theta_0 = 0$, or let (1.5) hold if $\theta_0 > 0$. Then there exists a unique $c^\dagger > 0$ and a unique $\bar{u} \in C^2(\mathbb{R}) \cap H_{c^\dagger}^1(\mathbb{R})$ such that $0 < \bar{u} < 1$, $\bar{u}' < 0$, $\bar{u}(+\infty) = 0$, $\bar{u}(-\infty) = 1$, $\bar{u}(0) = \frac{1}{2}$, and \bar{u} minimizes Φ_{c^\dagger} over all $u \in H_{c^\dagger}^1(\mathbb{R})$ such that $0 \leq u \leq 1$. Furthermore, $u(x, t) = \bar{u}(x - c^\dagger t)$ solves (1.1).*

This proposition is an immediate corollary to [38, Proposition 2.3]. The solution $u(x, t)$ in Proposition 2.2 is an example of a *variational traveling wave* and plays an important role for the long time behavior of solutions of (1.1) [35, 36]. Its existence allows us to make a very general conclusion about propagation of the trailing and the leading edges of the solution with localized initial data. For $\delta \in (0, 1)$, we define

$$R_\delta^+(t) := \sup_{x \in \mathbb{R}^N} \{|x| : u(x, t) > \delta\}, \tag{2.10}$$

$$R_\delta^-(t) := \inf_{x \in \mathbb{R}^N} \{|x| : u(x, t) < \delta\}. \tag{2.11}$$

The functions $R_\delta^\pm(t)$ represent, respectively, the positions of the leading and the trailing edges of radially divergent solutions at level δ . Then we have the following result, which is a consequence of the gradient flow structure generated by Φ_c .

Theorem 1 (Propagation). *Let (1.3) hold with some $\theta_0 \in [0, 1)$, and let $f'(0) = 0$ if $\theta_0 = 0$, or let (1.5) hold if $\theta_0 > 0$. Assume that $u(x, t)$ is a solution of (1.1) satisfying (1.2) with $Q(\phi) \in L_c^2(\mathbb{R}^N)$ for some $c > c^\dagger$ and every rotation Q , and that $u(\cdot, t) \rightarrow 1$ locally uniformly as $t \rightarrow \infty$. Then*

$$\lim_{t \rightarrow \infty} \frac{R_\delta^\pm(t)}{t} = c^\dagger. \tag{2.12}$$

Here, as usual, the rotation map Q is defined via $Q(\phi(x)) := \phi(Ax)$ for some $A \in SO(N)$. We note in passing that the same result is well known for $\theta_0 > 0$, or for $\theta_0 = 0$ and $f'(0) > 0$, in the case of compactly supported initial data [2]. In particular, in the latter case the problem exhibits *hair-trigger effect*, i.e., any non-zero initial data gives rise to the solution that converges locally uniformly to 1. Therefore, assuming $f'(0) \leq 0$ throughout our paper is not really a restriction.

Let us note that for $c \geq c^\dagger$ the functional $\Phi_c[u]$ is bounded from below by zero for all $u \in H_c^1(\mathbb{R}^N)$ [36]. Therefore, it would be natural to try to use the monotone decrease of Φ_c evaluated on the solution of (2.5) to establish convergence of solutions of (2.5) to traveling fronts. This is indeed possible in the case $N = 1$, provided that $f'(0) \leq 0$ and $f'(1) < 0$ in addition to (1.3) and (1.5). In this case the solutions of (2.5) with front-like initial data converge exponentially fast to a translate of the one-dimensional non-trivial minimizer of Φ_c [37]. However, for $N > 1$ it is known that solutions of (1.1) with bistable nonlinearities go to zero locally uniformly in the reference frame moving with speed c^\dagger [46, 51].

Remark 2.3. Removing the assumption that $\phi \in L^2_c(\mathbb{R}^N)$ in Theorem 1, one still has

$$\liminf_{t \rightarrow \infty} \frac{R_\delta^\pm(t)}{t} \geq c^\dagger. \tag{2.13}$$

From Theorem 1 and Remark 2.3, one can see that under our assumptions on f the ignition behavior implies propagation for general L^2 initial data. We now consider further implications of propagation for *radial symmetric-decreasing* data.

(SD) The initial condition $\phi(x)$ in (1.2) is radial symmetric-decreasing, i.e., $\phi(x) = g(|x|)$ for some $g(r)$ that is non-increasing for every $r > 0$.

Note that the slight abuse of notation in the definition (SD) is not a problem, since the solution $u(x, t)$ of (1.1) satisfying (1.2) and (SD) is a strictly decreasing function of $|x|$ for all $t > 0$. We will show that for initial data obeying (SD), propagation implies that the energy dissipation rate cannot vanish, which means that ignition always leads to the energy not being bounded from below. In fact, the converse also holds. This leads to the following result which characterizes the ignition scenario via the asymptotic behavior of the energy evaluated on solutions of (1.1).

Theorem 2 (Ignition). *Let (1.3) hold with some $\theta_0 \in [0, 1)$, and let $f'(0) = 0$ if $\theta_0 = 0$, or let (1.5) hold if $\theta_0 > 0$. Assume that $u(x, t)$ is a solution of (1.1) satisfying (1.2) with (SD). Then:*

- (i) *If $u(\cdot, t_n) \rightarrow 1$ locally uniformly in \mathbb{R}^N for some sequence of $t_n \rightarrow \infty$, then $\lim_{t \rightarrow \infty} E[u(\cdot, t)] = -\infty$.*
- (ii) *If $\lim_{t \rightarrow \infty} E[u(\cdot, t)] < 0$, then $u(\cdot, t) \rightarrow 1$ locally uniformly in \mathbb{R}^N as $t \rightarrow \infty$.*

The main implication of Theorem 2 is that it excludes the possibility of the equilibrium $u = 1$ to be the long time limit of solutions of (1.1) with energy bounded from below. Hence, for initial data satisfying (SD) the remaining possibilities are radial non-increasing solutions of (1.6). If v is such a solution, it satisfies an ordinary differential equation in $r = |x|$ and can be parametrized by its value at the origin. More precisely, if $\mu \in [0, 1)$ is such that $\mu = v(0)$, then $v(x) = v_\mu(|x|)$, where $v_\mu \geq 0$ satisfies for all $0 < r < \infty$

$$v''_\mu(r) + \frac{N-1}{r} v'_\mu(r) + f(v_\mu(r)) = 0, \quad v'_\mu(r) \leq 0, \quad v_\mu(0) = \mu, \quad v'_\mu(0) = 0. \tag{2.14}$$

It is easy to see that all solutions of (2.14) are either identically constant (equal to a zero of f), or are strictly decreasing and approaching a zero of f as $r \rightarrow \infty$.

Since ground states in the sense of Definition 2.1 are a particular class of solutions of (1.6) that play a special role for the long time limits of (1.1), we introduce the notation

$$\Upsilon := \{\mu \in (0, 1) : v_\mu(|x|) \text{ is a ground state}\}. \tag{2.15}$$

Recall that in many particular situations the set Υ is generically expected to be a *discrete* set of points, possibly consisting of only a single point, as is the case for the nonlinearities in (1.4) or (1.7). Under this condition, convergence to a ground state becomes full convergence, rather than sequential convergence, as $t \rightarrow \infty$. This conclusion will be seen to remain true for bistable and ignition nonlinearities under the following more general assumption:

(TD) The set Υ is totally disconnected.

By a totally disconnected set, we understand a set whose connected components are one-point sets. We note that verifying (TD) in practice may be rather difficult, in view of the quite delicate structure of the solution set for (1.6) in its full generality. Nevertheless, as was already noted above, this condition is expected to hold generically and allows us to avoid getting into the specifics of the existence theory for the elliptic equation (1.6) and concentrate instead on the evolution problem associated with (1.1).

As a consequence of Theorem 2 and the gradient flow structure of (1.1), we have the following general result about all possible long-time behaviors of solutions of (1.1) with radial symmetric-decreasing initial data in $L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Theorem 3 (Ignition vs. Failure). *Let (1.3) hold with some $\theta_0 \in [0, 1)$, and let $f'(0) = 0$ if $\theta_0 = 0$, or let (1.5) hold if $\theta_0 > 0$. Assume that $u(x, t)$ is a solution of (1.1) satisfying (1.2) with (SD). Then there are two alternatives:*

1. $\lim_{t \rightarrow \infty} u(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u(\cdot, t)] = -\infty$.
2. $\liminf_{t \rightarrow \infty} \sup_{x \in B_R(0)} |u(x, t) - v_\mu(|x|)| = 0$ for every $R > 0$ and every $\mu \in I$, where $I = [a, b]$, with some $0 \leq a \leq b < 1$, $v_\mu(|x|)$ satisfies (2.14) for all $\mu \in I$, and $\lim_{t \rightarrow \infty} E[u(\cdot, t)] \geq 0$.

We note that more precise conclusions for the second alternative in Theorem 3 would need further assumptions on the nonlinearity of the problem, such as those that would yield (TD), or, perhaps, analyticity of $f(u)$ [22, 49]. Apart from the first option, we do not pursue this further in the present paper.

Remark 2.4. It is easy to see that the conclusions of all the above theorems remain true, if one assumes that $f \in C^1[0, \infty)$, $f(0) = f(1) = 0$, $f'(0) \leq 0$, $f(u) \leq 0$ for all $u \geq 1$, and that $u_m = 1$ is the only root of $f(u)$ such that $V(u_m) < 0$.

We now turn our attention to the study of threshold phenomena. We use the notations similar to those in [14]. Let $X := \{\phi(x) : \phi(x) \text{ satisfies (1.2) and (SD)}\}$, and let $\lambda^+ > 0$. We consider a one-parameter family of initial conditions $\phi_\lambda \in X$ with $\lambda \in [0, \lambda^+]$, satisfying the following conditions:

- (P1) The map $\lambda \mapsto \phi_\lambda \in X$ is continuous from $[0, \lambda^+]$ to $L^2(\mathbb{R}^N)$;
- (P2) If $0 < \lambda_1 < \lambda_2$, then $\phi_{\lambda_1} \leq \phi_{\lambda_2}$ and $\phi_{\lambda_1} \neq \phi_{\lambda_2}$ in $L^2(\mathbb{R}^N)$.
- (P3) $\phi_0(x) = 0$ and $E[\phi_{\lambda^+}] < 0$.

We denote by $u_\lambda(x, t)$ the solution of (1.1) with the initial datum ϕ_λ . Clearly, $u_0(x, t) = 0$, and by Theorem 2 we have $u_{\lambda^+}(\cdot, t) \rightarrow 1$ locally uniformly as $t \rightarrow \infty$. Therefore, the solutions corresponding to the endpoints of the interval of $\lambda \in [0, \lambda^+]$ exhibit qualitatively distinct long time behaviors. We wish to characterize all possible behaviors for intermediate values of λ and, in particular, to determine the structure of the threshold set.

To proceed, we consider the cases of bistable, ignition and monostable nonlinearities separately, as they lead to rather different sets of conclusions. We start with the bistable nonlinearity, namely, the nonlinearity f satisfying (1.3) with $\theta_0 > 0$, together with (1.5) and an extra assumption that $f(u) < 0$ for all $u \in (0, \theta_0)$. The key observation is that for these nonlinearities there exists $\theta^* \in (\theta_0, 1)$ such that

$$\int_0^{\theta^*} f(s) ds = 0. \tag{2.16}$$

Furthermore, we have $V(u) > 0$ for all $0 < u < \theta^*$ and $V(u) < 0$ for all $\theta^* < u < \theta^\diamond$, for some $\theta^\diamond \in (1, \infty]$. At the same time, the set of all zeros of f that lie in $[0, 1)$ consists of only two isolated values: $u = 0$ and $u = \theta_0$. Therefore, by Theorem 3, if the solution with initial data satisfying (SD) does not converge locally uniformly to $u = 1$, on sequences of times going to infinity it either converges to $u = 0$, or to a decaying radial symmetric-decreasing solution v of (1.6). Note that for bistable nonlinearities and $N \geq 3$, all positive solutions of (2.14) converging to zero at infinity are ground states (after extension to \mathbb{R}^N), since every decaying solution of (1.6) is subharmonic for $|x| \gg 1$ and, therefore, decays no slower than $|x|^{2-N}$. In this case the statement in (TD) concerns all radial decaying solutions of (1.6).

The next theorem further details the above picture and also establishes the existence of a *unique* threshold between ignition and extinction for monotone families of initial data, under (TD).

Theorem 4 (Threshold for Bistable Nonlinearities). *Let (1.3) hold with some $\theta_0 \in (0, 1)$, let $f(u) < 0$ for all $u \in (0, \theta_0)$ and let (1.5) hold. Assume that $u_\lambda(x, t)$ are solutions of (1.1) with the initial data ϕ_λ satisfying (P1)–(P3). Then, under (TD) there exists $\lambda_* \in (0, \lambda^+)$ such that:*

1. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] = -\infty$ for all $\lambda > \lambda_*$.
2. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 0$ uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] = 0$ for all $\lambda < \lambda_*$.
3. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = v_*$ uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] \geq E_0$, where $v_*(x) = v_{\mu_*}(|x|)$ and v_{μ_*} satisfies (2.14), $\mu_* \in \Upsilon$, and $E_0 := E[v_*] > 0$, for $\lambda = \lambda_*$.

Notice that (TD) is the only assumption on the set of radial symmetric-decreasing solutions of (1.6) that has been made in Theorem 4. The fact that (TD) is sufficient is due to a strong instability of the ground states, which precludes a possibility of an *ordered* family of ground states (for non-degenerate bistable nonlinearities, this fact was spelled out in [8]). It is interesting whether (TD) can be relaxed, so that the sharp threshold result holds even when there is a continuum of ground states.

For ignition nonlinearities, i.e., those that satisfy (1.3) with $\theta_0 > 0$ and having $f(u) = 0$ for all $u \in [0, \theta_0]$, the situation becomes more complicated. Recall that in the considered setting and with $N = 1$ the threshold solution is known to converge to the unstable equilibrium solution $u = \theta_0$ [14, 38, 53]. This happens because in the case $N = 1$ the only symmetric-decreasing solutions of (2.14) that satisfy $0 < v_\mu < 1$ are constant solutions $v_\mu(r) = \mu$, for any $\mu \in (0, \theta_0]$, and by the well known property of the heat equation every solution of (1.1) satisfying (1.2) with $\phi \leq \theta_0$ goes to zero. Hence the solutions of (1.1) cannot converge locally uniformly to any constant solution $0 < v_\mu < \theta_0$. On the other hand, for $N \leq 2$ it is easy to see that (2.14) does not have any non-constant solutions. Thus, the only alternative to ignition and extinction in this case is convergence to $u = \theta_0$. With an extra assumption that $f(u)$ is convex in a neighborhood of $u = \theta_0$, we are then led to the following result.

Theorem 5 (Threshold for Ignition Nonlinearities: Low Dimensions). *Let $N \leq 2$, let (1.3) hold with some $\theta_0 \in (0, 1)$, let $f(u) = 0$ for all $u \in [0, \theta_0]$ and let $f(u)$ be convex in some neighborhood of $u = \theta_0$. Assume that $u_\lambda(x, t)$ are solutions of (1.1) with the initial data ϕ_λ satisfying (P1)–(P3). Then there exists $\lambda_* \in (0, \lambda^+)$ such that:*

1. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] = -\infty$ for all $\lambda > \lambda_*$.
2. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 0$ uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] = 0$ for all $\lambda < \lambda_*$.
3. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = \theta_0$ locally uniformly in \mathbb{R}^N , and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] \geq 0$, if $\lambda = \lambda_*$.

On the other hand, for $N \geq 3$ the situation becomes much more complicated, since in this case many solutions of (2.14) exist. In fact, by the results of Berestycki and Lions [5], for every $v^\infty \in [0, \theta_0)$ there exists a solution of (2.14) such that $v_\mu(\infty) = v^\infty$. Also, there may exist non-constant solutions of (2.14) with $v^\infty = \theta_0$, even continuous families of such solutions. Take, for instance, $f(u) = (u - \theta_0)^{ps}$ for all $\theta_0 < u < \theta_1$ for some $\theta_1 \in (\theta_0, 1)$. Dealing with all these situations would lead us away from the main subject of the paper, so instead we give a rather general sufficient condition for our results to hold. Namely, we assume that all non-constant solutions of (2.14) that converge to θ_0 at infinity are ground states for the problem with the nonlinearity shifted by θ_0 .

- (V) If v is a radial symmetric-decreasing solution of (1.6), then it satisfies all the properties of a ground state, except $v(x) \rightarrow \theta_0$ as $|x| \rightarrow \infty$.

Under this assumption, we are able to exclude all solutions of (2.14) with $v^\infty > 0$ as potential long time limits of (1.1).

Theorem 6 (Threshold for Ignition Nonlinearities: High Dimensions). *Let $N \geq 3$ and let (1.3) hold with some $\theta_0 \in (0, 1)$, let $f(u) = 0$ for all $u \in [0, \theta_0]$ and assume (V). Assume also that $u_\lambda(x, t)$ are solutions of (1.1) with the initial data ϕ_λ satisfying (P1)–(P3). Then, under (TD) there exists $\lambda_* \in (0, \lambda^+)$ such that:*

1. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] = -\infty$ for all $\lambda > \lambda_*$.
2. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 0$ uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] = 0$ for all $\lambda < \lambda_*$.
3. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = v_*$ uniformly in \mathbb{R}^N and $\lim_{t \rightarrow \infty} E[u_\lambda(\cdot, t)] \geq E_0$, where $v_*(x) = v_{\mu_*}(|x|)$ and v_{μ_*} satisfies (2.14), $\mu_* \in \Upsilon$, and $E_0 := E[v_*] > 0$, for $\lambda = \lambda_*$.

One can see that this situation is more reminiscent of the bistable case, with ground states taking over the role as the limits of the threshold solutions. In particular, uniqueness of the ground state would imply that it attracts the threshold solution uniformly as $t \rightarrow \infty$. Note that no assumption on convexity of the nonlinearity near $u = \theta_0$ is needed in this case.

Finally, we turn to monostable nonlinearities, i.e., when f satisfies (1.3) with $\theta_0 = 0$. Here, once again, one needs to distinguish the cases $N \leq 2$ and $N \geq 3$. Just as in the case $N = 1$ [38], there are no solutions of (2.14) when $f(u) > 0$ for any $\mu \in (0, 1)$ and $N = 2$. Hence, the threshold behavior becomes particularly simple.

Theorem 7 (Threshold for Monostable Nonlinearities: Low Dimensions). *Let $N \leq 2$ and let (1.3) hold with $\theta_0 = 0$. Assume that $u_\lambda(x, t)$ are solutions of (1.1) with the initial data ϕ_λ satisfying (P1)–(P3). Then there exists $\lambda_* \in [0, \lambda^+)$ such that:*

1. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N for all $\lambda > \lambda_*$.
2. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 0$ uniformly in \mathbb{R}^N for all $\lambda \leq \lambda_*$.

The possibility of $\lambda_* = 0$ corresponds to the hair-trigger effect and is always realized when $f'(0) > 0$ [2]. Similarly, hair-trigger effect is still observed in the case

$f'(0) = 0$ when $f(u) \geq cu^{p_F}$ for some $c > 0$ and all sufficiently small u (see [24] and references therein). At the same time, as was already mentioned in the introduction, the statement of the theorem becomes non-trivial when $f(u) = o(u^{p_F})$ for $u \ll 1$, in the sense that $\lambda_* > 0$ for some choices of families of initial data. Note that here and in the remaining theorems we did not pursue the limit behavior of the energy, since its analysis for monostable nonlinearities becomes rather tricky and, at the same time, its consequences may not be very informative.

The situation becomes considerably more delicate for $N \geq 3$, since in this case many radial, symmetric-decreasing and decaying solutions of (1.6) can exist, and their existence and properties depend quite sensitively on the behavior of $f(u)$ near zero and the dimension N (for an extensive discussion in the case of pure power nonlinearities, see [45]). Here our ability to characterize sharp threshold behaviors relies on the assumption that all the decaying solutions of (1.6) be ground states (in the sense of Definition 2.1). The fact that $|\nabla v| \in L^2(\mathbb{R}^N)$ for a ground state v gives rise to a strong instability of v , which we also used to establish sharp threshold results for bistable and ignition nonlinearities. At the same time, it is known that in the case $N \geq 11$ and pure power nonlinearities $f(u) = u^p$ with $p \geq p_{JL}$, where $p_{JL} := 1 + 4/(N - 4 - 2\sqrt{N - 1})$ is the Joseph-Lundgren critical exponent, the radial, symmetric-decreasing and decaying solutions become *stable* in a certain sense and form a monotone increasing continuous family [24]. This family of solutions of (1.6) clearly produces a counterexample for the expected sharp threshold behavior for monotone families of data that do not lie in $L^2(\mathbb{R}^N)$.

We give two results in which sharp threshold behavior is established for monostable nonlinearities for $N \geq 3$. We begin with the first case, in which we assume that there are no solutions of (2.14) with $\mu \in (0, 1)$. This situation takes place, for example, when $f(u) \geq cu^p$ for some $c > 0$ and $p \leq p_{sg}$ for all $u \ll 1$, where $p_{sg} := N/(N - 2)$ is the Serrin critical exponent. In this situation, (1.6) is known to have no positive solutions below $u = 1$ [13, 45].

Theorem 8 (Threshold for Monostable Nonlinearities: High Dimensions, Simple).

Let $N \geq 3$, let (1.3) hold with $\theta_0 = 0$ and assume that (2.14) has no solutions with $\mu \in (0, 1)$. Then, if $u_\lambda(x, t)$ are solutions of (1.1) with the initial data ϕ_λ satisfying (P1)–(P3), the conclusion of Theorem 7 holds true.

Once again, the result of the theorem is non-trivial, for example, if $f(u) \simeq cu^p$ with some $c > 0$ and $p_F < p \leq p_{sg}$ for all $u \ll 1$.

Remark 2.5. The assumptions of Theorem 8 also hold, for example, for $f(u) = u^p - u^q$ with any $p_{sg} < p \leq p_S$ and $q > p$ [7, Theorem 3].

On the other hand, by [50, Theorem 2] the set Υ for the nonlinearity in Remark 2.5 consists of a single point for all $p_S < p < q$, despite the existence of a continuous family of positive radial symmetric-decreasing decaying solutions of (1.6). We suspect that in this case, apart from zero, the unique ground state may still be the only attractor of the threshold solutions.

We now proceed to the second case. As we already noted, the situation is quite complex to make detailed conclusions about the sharp threshold behavior without any further assumptions on f and N in the monostable case. Here, as in the case of ignition nonlinearities for $N \geq 3$ we chose to give a rather general sufficient condition in terms of the properties of positive decaying solutions of (2.14), namely, that they consist only of ground states (however, for an example of nonlinearities for which this is false, see [30]). This assumption may be verified with the knowledge

of the asymptotic decay of solutions of (2.14) at infinity. Note that existence of ground states for (1.6) is known in the case when $f(u) = o(u^{ps})$ for $N \geq 3$ [5].

Theorem 9 (Threshold Monostable Nonlinearities: High Dimensions, Complex). *Let $N \geq 3$, let (1.3) hold with $\theta_0 = 0$ and suppose that every non-constant radial symmetric-decreasing solution of (1.6) is a ground state in the sense of Definition 2.1. Assume that $u_\lambda(x, t)$ are solutions of (1.1) with the initial data ϕ_λ satisfying (P1)–(P3). Then, under (TD) there exists $\lambda_* \in [0, \lambda^+)$ such that:*

1. $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N for all $\lambda > \lambda_*$.
2. If $\lambda_* > 0$, then $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 0$ uniformly in \mathbb{R}^N for all $\lambda < \lambda_*$.
3. For $\lambda = \lambda_*$, there are two alternatives:
 - (a) $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = 0$ uniformly in \mathbb{R}^N .
 - (b) $\lim_{t \rightarrow \infty} u_\lambda(\cdot, t) = v_*$ uniformly in \mathbb{R}^N , where $v_*(x) = v_{\mu_*}(|x|)$ and v_μ satisfies (2.14) with $\mu_* \in \Upsilon$.

Assuming that $\lambda_* > 0$, i.e., that the hair-trigger effect does not occur, the main point of the above theorem is that under (TD) the threshold is sharp. Yet, we note that one can imagine rather complex behaviors of the threshold solutions as $t \rightarrow \infty$, if (TD) does not hold. For example, taking $f(u) = u^{ps}$ for all $u \leq \frac{1}{2}$ and $f(u)/u^{ps}$ decreasing to 0 on $[\frac{1}{2}, 1]$, it follows from [7, Theorem 3] that all solutions of (2.14) with $\mu \in (0, 1)$ are ground states and are given by (1.8) with $\lambda \in [2^{2/(N-2)}, \infty)$. Hence, our Theorem 9 does not apply, while Theorem 3 does. Here it is not a priori clear whether one could rule out a threshold solution which approaches the family in (1.8) with $\lambda = 1/g(t)$ for some function $g : [0, \infty) \rightarrow (0, 2^{-2/(N-2)})$ that slowly oscillates (with increasing period) between the two endpoints of its range, approaching zero on a sequence of times going to infinity. Note, however, that the more exotic behaviors discussed in [43] cannot occur in our setting, since we consider L^2 initial data.

3. Preliminaries. We start with a basic existence result for (1.1) with initial data from (1.2). Since we want to take advantage of the variational structure associated with Φ_c in (2.7), we also provide an existence result for initial data that lie in the exponentially weighted spaces. Throughout the rest of the paper, (1.3) is always assumed to hold, unless stated otherwise.

Proposition 3.1. *There exists a unique $u \in C^2_1(\mathbb{R}^N \times (0, \infty)) \cap L^\infty(\mathbb{R}^N \times (0, \infty))$ satisfying (1.1) and (1.2) (using the notations from [15]), with*

$$u \in C([0, \infty); L^2(\mathbb{R})) \cap C((0, \infty); H^2(\mathbb{R}^N)), \tag{3.1}$$

and $u_t \in C((0, \infty); H^1(\mathbb{R}^N))$. Furthermore, if there exists $c > 0$ such that the initial condition $\phi \in L^2_c(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then the solution of (1.1) and (1.2) satisfies

$$u \in C([0, \infty); L^2_c(\mathbb{R})) \cap C((0, \infty); H^2_c(\mathbb{R}^N)), \tag{3.2}$$

with $u_t \in C((0, \infty); H^1_c(\mathbb{R}^N))$. In addition, small variations of the initial data in $L^2(\mathbb{R}^N)$ result in small changes of solution in $H^1(\mathbb{R}^N)$ at any $t > 0$, and if ϕ satisfies (SD), then so does $u(\cdot, t)$ for all $t > 0$.

Proof. The proof follows as in [31, Chapter 7] and [37, Proposition 3.1], noting that the function $\bar{u}(x, t) = \max\{1, \|\phi\|_{L^\infty(\mathbb{R}^N)}\}$ is a global supersolution. The symmetric decreasing property of u follows, e.g., from [45, Proposition 52.17]. \square

Note that the regularity of solutions in Proposition 3.1 guaranties that $E[u(\cdot, t)]$ (resp. $\Phi_c[\tilde{u}(\cdot, t)]$) is finite, continuously differentiable and satisfies (2.2) on solutions of (1.1) (resp. (2.9) on solutions of (2.5)), for any $t > 0$.

We next recall the classical regularity properties of bounded solutions of (1.1). Let $D_1 = Q_1 \times [t_1, t_2]$ be an $(N + 1)$ -dimensional cylindrical domain in (x, t) , where $Q_1 \subset \mathbb{R}^N$ is open and $t_1 \geq 0$. Let $Q_2 \subset Q_1$ be open, and assume that there exists $\varepsilon > 0$ such that

$$\bigcup_{x \in Q_2} B_\varepsilon(x) \subset Q_1. \tag{3.3}$$

Moreover, let $D_2 = Q_2 \times [t_1 + \varepsilon, t_2]$. Then, if $u(x, t)$ is a solution of (1.1), by Schauder estimates (see, e.g., [21]), there exists $C > 0$, depending on ε but independent of u and D_1 , such that

$$\|u_t\|_{L^\infty(D_2)} + \sum_{1 \leq i \leq N} \|\partial_{x_i} u\|_{L^\infty(D_2)} + \sum_{1 \leq i, j \leq N} \|\partial_{x_i} \partial_{x_j} u\|_{L^\infty(D_2)} \leq C \|u\|_{L^\infty(D_1)}. \tag{3.4}$$

We will refer to this boundedness as “standard parabolic regularity”. We note that the estimate in (3.4) also holds for all classical solutions of (1.6), since they can be trivially considered as time-independent solutions of (1.1).

Corollary 3.2. *Let $u(x, t)$ be a solution of (1.1) satisfying (1.2). Then*

$$\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq 1. \tag{3.5}$$

Proof. By Proposition 3.1 and standard parabolic regularity, for every $T > 0$ the solution $u(x, T)$ is bounded and converges uniformly to zero as $|x| \rightarrow \infty$. Therefore, if $a > 0, b > 0$ and

$$\bar{u}(x, t) := 1 + \frac{a}{[4\pi(t - T + b)]^{N/2}} \exp\left\{-\frac{|x|^2}{4(t - T + b)}\right\}, \tag{3.6}$$

then $\bar{u}(x, t)$ is a supersolution for (1.1) for all $t \geq T$, and it is possible to choose a and b in such a way that $u(x, T) \leq \bar{u}(x, T)$ for all $x \in \mathbb{R}^N$. The result then follows by comparison principle. \square

We now turn to a useful property of solutions of (1.1) whose energy remains bounded for all time. Because of the gradient flow structure of (1.1), one should expect that such solutions exhibit “slowing down” while approaching stationary solutions on sequences of times going to infinity. More is true, however, namely, that a solution with bounded energy also remains close to the limit stationary solution on a sequence of growing temporal intervals. This conclusion is a consequence of uniform Hölder continuity of $u(x, t)$ in t for each $x \in \mathbb{R}^N$ whenever $\lim_{t \rightarrow \infty} E[u(\cdot, t)] \neq -\infty$ that we establish below. The result is a basic generalization of the one in [38, Proposition 2.8] obtained for $N = 1$.

Proposition 3.3. *Let $u(x, t)$ be a solution of (1.1) satisfying (1.2). If $E[u(\cdot, t)]$ is bounded from below, then $u(x, \cdot) \in C^\alpha[T, \infty)$ with $\alpha = \frac{1}{2(N+1)}$, for each $x \in \mathbb{R}^N$ and $T > 0$. Moreover, the corresponding Hölder constant of $u(x, \cdot)$ converges to 0 as $T \rightarrow \infty$ uniformly in x .*

Proof. By Proposition 3.1, we have $E[u(\cdot, t)] < +\infty$ for all $t > 0$, and by (2.2) we have that $E[u(\cdot, t)]$ is a non-increasing function of t . We define $E_\infty := \lim_{t \rightarrow \infty} E[u(\cdot,$

$t)$] and observe that by our assumptions $E_\infty \in \mathbb{R}$. Then, for any $x_0 \in \mathbb{R}^N$ and $t_2 > t_1 \geq T$ we have

$$\begin{aligned} \int_{B_1(x_0)} |u(x, t_2) - u(x, t_1)| dx &\leq \int_{t_1}^{t_2} \int_{B_1(x_0)} |u_t(x, t)| dx dt \\ &\leq C_N \sqrt{t_2 - t_1} \left(\int_{t_1}^{t_2} \int_{B_1(x_0)} u_t^2(x, t) dx dt \right)^{1/2} \\ &\leq C_N \sqrt{t_2 - t_1} \left(\int_T^\infty \int_{\mathbb{R}^N} u_t^2(x, t) dx dt \right)^{1/2} \\ &= C_N \sqrt{(t_2 - t_1)(E[u(\cdot, T)] - E_\infty)}. \end{aligned} \tag{3.7}$$

Here and in the rest of the proof $C_N > 0$ is a constant depending only on N that changes from line to line.

On the other hand, by standard parabolic regularity there exists $M > 0$ such that

$$\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq M, \quad \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \forall t \geq T. \tag{3.8}$$

Without loss of generality we can further assume that $u(x_0, t_2) - u(x_0, t_1) \in (0, M]$. Then, for every $x \in B_r(x_0)$, where

$$r := \frac{u(x_0, t_2) - u(x_0, t_1)}{2M} \in (0, 1), \tag{3.9}$$

we have

$$u(x, t_2) \geq u(x_0, t_2) - M|x - x_0| \geq u(x_0, t_1) + M|x - x_0| \geq u(x, t_1). \tag{3.10}$$

This implies that

$$\begin{aligned} &\int_{B_1(x_0)} |u(x, t_2) - u(x, t_1)| dx \\ &\geq \int_{B_r(x_0)} (u(x, t_2) - u(x, t_1)) dx \\ &\geq \int_{B_r(x_0)} (u(x_0, t_2) - u(x_0, t_1) - 2M|x - x_0|) dx \\ &= |u(x_0, t_2) - u(x_0, t_1)| \int_{B_r(x_0)} (1 - r^{-1}|x - x_0|) dx \\ &= C_N M^{-N} |u(x_0, t_2) - u(x_0, t_1)|^{N+1}. \end{aligned} \tag{3.11}$$

Combining this with (3.7) yields

$$|u(x_0, t_2) - u(x_0, t_1)| \leq C_N \left(M^{2N} (E[u(\cdot, T)] - E_\infty) \right)^{\frac{1}{2(N+1)}} (t_2 - t_1)^{\frac{1}{2(N+1)}}, \tag{3.12}$$

i.e., we have $u(x, \cdot) \in C^\alpha[T, \infty)$ for $\alpha = \frac{1}{2(N+1)}$. Moreover, the limit of the Hölder constant in (3.12) is

$$\lim_{T \rightarrow \infty} C_N \left(M^{2N} (E[u(\cdot, T)] - E_\infty) \right)^{\frac{1}{2(N+1)}} = 0, \tag{3.13}$$

which completes the proof. \square

We will need a technical result about the ground states for (1.6), namely, that all these solutions exhibit a strong instability with respect to the dynamics governed by (1.1). In the case $f'(0) < 0$ such a result for all positive solutions of (1.6) that

decay at infinity is well known (see, e.g., [6, Theorem IV.I] or [4, Theorem 5.4]). Here we provide a generalization for nonlinearities that might exhibit a degeneracy near $u = 0$ (for closely related results, see [9, 12, 48]). The key assumption for the lemma below to hold is that the solution of (1.6) has square-integrable first derivatives.

Lemma 3.4. *Let $f \in C^1[0, \infty)$ and let v be a ground state in the sense of Definition 2.1. Then there exists $\phi_0^R \geq 0$ with $\text{supp}(\phi_0^R) = B_R(0)$ for some $R > 0$ such that $\bar{v}^\varepsilon(x, t) := v(x) - \varepsilon\phi_0^R(x)$ is a supersolution, and $\underline{v}^\varepsilon(x, t) = v(x) + \varepsilon\phi_0^R(x)$ is a subsolution, respectively, for (1.1), for all $\varepsilon > 0$ sufficiently small.*

Proof. Consider the Schrödinger-type operator:

$$\mathfrak{L} = -\Delta + \mathcal{V}(x), \quad \mathcal{V}(x) := -f'(v(x)), \tag{3.14}$$

and the associated Rayleigh quotient (for technical background, see, e.g., [29, Chapter 11]):

$$\mathfrak{R}(\phi) := \frac{\int_{\mathbb{R}^N} (|\nabla\phi|^2 + \mathcal{V}(x)\phi^2) dx}{\int_{\mathbb{R}^N} \phi^2 dx}. \tag{3.15}$$

To study the minimization problem for \mathfrak{R} , we also consider

$$\tilde{\mathfrak{L}} = -\Delta + \tilde{\mathcal{V}}(x), \quad \tilde{\mathcal{V}}(x) := \mathcal{V}(x) + f'(0) = -(f'(v(x)) - f'(0)), \tag{3.16}$$

with the associated Rayleigh quotient

$$\tilde{\mathfrak{R}}(\phi) = \mathfrak{R}(\phi) + f'(0). \tag{3.17}$$

Since $\tilde{\mathcal{V}}(x) \in L^\infty(\mathbb{R})$ and vanishes at infinity, by [29, Theorem 11.5] there exists a function $\phi_0 \in H^1(\mathbb{R}^N)$ such that $\phi_0 \neq 0$ and ϕ_0 minimizes $\tilde{\mathfrak{R}}$, provided

$$\mathfrak{E}_0 := \inf\{\tilde{\mathfrak{R}}(\phi) : \phi \in H^1(\mathbb{R}^N), \phi \neq 0\} < 0. \tag{3.18}$$

Moreover, by [29, Theorem 11.8], if there exists a minimizer $\phi_0 \in H^1(\mathbb{R}^N)$, $\phi_0 \neq 0$, then ϕ_0 can be chosen to be a strictly positive function, and ϕ_0 is unique up to a constant factor.

Now, differentiating (1.6) with respect to x_i , $i = 1, \dots, N$, by boundedness of f' on the range of v the function $v_i := \partial v / \partial x_i$ satisfies

$$\Delta v_i + f'(v)v_i = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.19}$$

Hence, by elliptic regularity we also have $v_i \in H^2(\mathbb{R}^N)$ [23]. Thus, each v_i is an admissible test function in (3.15), and by (3.19) we have

$$\tilde{\mathfrak{R}}(v_i) = f'(0). \tag{3.20}$$

Existence of a ground state v implies that $f'(0) \leq 0$ (otherwise there is hair-trigger effect [2]). Moreover, since v_i changes sign, we know that v_i is not a minimizer of $\tilde{\mathfrak{R}}$, so $\mathfrak{E}_0 < \tilde{\mathfrak{R}}(v_i) \leq 0$, and there exists a positive function $\phi_0 \in H^1(\mathbb{R}^N)$ that minimizes $\tilde{\mathfrak{R}}$, with

$$\min\{\tilde{\mathfrak{R}}(\phi) : \phi \in H^1(\mathbb{R}^N), \phi \neq 0\} = \tilde{\mathfrak{R}}(\phi_0) < f'(0). \tag{3.21}$$

Note that ϕ_0 also minimizes \mathfrak{R} , with

$$\min\{\mathfrak{R}(\phi) : \phi \in H^1(\mathbb{R}^N), \phi \neq 0\} = \mathfrak{R}(\phi_0) =: \nu_0 < 0. \tag{3.22}$$

Approximating ϕ_0 by a function with compact support and using it as a test function, we can then see that

$$\min\{\mathfrak{R}(\phi) : \phi \in H^1(\mathbb{R}^N), \text{supp}(\phi) \subseteq B_R(0), \phi \neq 0\} =: \nu_0^R < 0 \tag{3.23}$$

as well for a sufficiently large $R > 0$. In this case, there exists a minimizer ϕ_0^R of the problem in (3.23) whose restriction to $B_R(0)$ is positive and belongs to $H_0^1(B_R(0)) \cap C^2(\bar{B}_R(0))$. Furthermore, we have

$$\mathfrak{L}(\phi_0^R) = \nu_0^R \phi_0^R \quad \text{in } B_R(0). \tag{3.24}$$

Finally, for $\varepsilon > 0$ define $\underline{w}(x, t) := \varepsilon \phi_0^R(x)$. Then, using the fact that $f(v + \underline{w}) - f(v) = f'(\tilde{v})\underline{w}$ for some $v \leq \tilde{v} \leq v + \underline{w}$, we have for all $x \in B_R(0)$ and all $\varepsilon > 0$ sufficiently small

$$\begin{aligned} \underline{w}_t - \Delta \underline{w} - f'(\tilde{v})\underline{w} &= -\Delta \underline{w} - f'(v)\underline{w} + (f'(v) - f'(\tilde{v}))\underline{w} \\ &= \nu_0^R \underline{w} + (f'(v) - f'(\tilde{v}))\underline{w} \\ &\leq \frac{\nu_0^R}{2} \underline{w} \\ &\leq 0. \end{aligned} \tag{3.25}$$

It is then easy to see that $\underline{v}^\varepsilon = v + \underline{w}$ is a subsolution for (1.1), since v is a solution of (1.6).

The case of \bar{v}^ε is treated analogously. □

We note that as a corollary to this result, we have that for nonlinearities of one sign near the origin there are no *ordered* collections of ground states. Once again, this result is well known in the case when $f'(0) < 0$ (see, e.g., [8, Lemma 3.2]). More generally, we have the following statement (for a related result, see [12, Theorem 6.1.4]).

Corollary 3.5. *Let $f \in C^1[0, \infty)$, and assume that there exists $\alpha > 0$ such that f does not change sign on $(0, \alpha)$. Let v_1 and v_2 be two ground states in the sense of Definition 2.1 such that $v_1 \leq v_2$. Then $v_1 = v_2$.*

Proof. By strong maximum principle, either $v_1 = v_2$ or $v_1 < v_2$ in all of \mathbb{R}^N . We argue by contradiction and assume the latter. Let $\underline{v}_1^\varepsilon$ be the corresponding subsolution from Lemma 3.4 obtained from v_1 , and choose $\varepsilon > 0$ so small that $\underline{v}_1^\varepsilon < v_2$. Denote by $\underline{u}_1^\varepsilon$ the classical solution of (1.1) with $\underline{v}_1^\varepsilon$ as initial datum. Then by comparison principle we have $\underline{u}_1^\varepsilon(x, t) < v_2(x)$ for all $x \in \mathbb{R}^N$ and $t \geq 0$. Existence of such a solution is standard. Furthermore, we claim that $\underline{u}_1^\varepsilon(\cdot, t) - v_1 \in H^1(\mathbb{R}^N)$ for each $t \geq 0$, and (2.2) holds for $\underline{u}_1^\varepsilon$. Indeed, let $w^\varepsilon := \underline{u}_1^\varepsilon - v_1$. Then $w^\varepsilon(\cdot, 0) \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and $w^\varepsilon(x, t)$ solves

$$w_t^\varepsilon = \Delta w^\varepsilon + f(v_1 + w^\varepsilon) - f(v_1). \tag{3.26}$$

Therefore, $w^\varepsilon(x, t)$ satisfies the first half of the conclusions of Proposition 3.1 (cf. [31, Chapter 7]). In particular, (3.26) is an L^2 gradient flow generated by the energy

$$\tilde{E}[w] := \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla w|^2 + V(v_1 + w) - V(v_1) - V'(v_1)w \right) dx, \tag{3.27}$$

easily seen to be well defined for all $w \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and we have

$$\frac{d\tilde{E}[w^\varepsilon(\cdot, t)]}{dt} = - \int_{\mathbb{R}^N} |w_t^\varepsilon|^2 dx. \tag{3.28}$$

We claim that

$$E[v_1 + w^\varepsilon(\cdot, t)] = E[v_1] + \tilde{E}[w^\varepsilon(\cdot, t)], \tag{3.29}$$

for each $t \geq 0$, and thus by (3.28) equation (2.2) holds for $\underline{u}_1^\varepsilon$. Caution is needed here, since with our general assumptions on f we have very little information about

the decay of the ground states as $|x| \rightarrow \infty$. In particular, it is not a priori clear if $E[v_1 + w]$ is well defined for all $w \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Hence we need to use some minimal regularity (at infinity) possessed by the ground states in the sense of Definition 2.1.

As was established in the proof of Lemma 3.4, we have $\nabla v \in H^2(\mathbb{R}^N; \mathbb{R}^N)$ for every ground state v . Hence, by (1.6) we also have $f(v) \in L^2(\mathbb{R}^N)$. Then, using Taylor formula it is easy to see that the integral of $V(v_1 + w)$ makes sense. Thus, we can write

$$E[v_1 + w] - E[v_1] = \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla w|^2 + \nabla v_1 \cdot \nabla w + V(v_1 + w) - V(v_1) \right) dx. \tag{3.30}$$

Again, using the fact that $\nabla v_1 \in H^2(\mathbb{R}^N; \mathbb{R}^N)$, we can integrate the second term in the right-hand side of (3.30) by parts and use (1.6) to arrive at (3.29).

Now, since $\underline{v}_1^\varepsilon$ is a subsolution of (1.1) and by construction is a strict subsolution in $B_R(0)$, the function $\underline{u}_1^\varepsilon(x, t)$ is strictly monotonically increasing in t for each $x \in \mathbb{R}^N$. In particular, $v_1 < \underline{u}_1^\varepsilon(\cdot, t)$ for each $t > 0$. Since $\underline{u}_1^\varepsilon(x, t)$ is bounded above for each $x \in \mathbb{R}^N$, by standard parabolic regularity $\underline{u}_1^\varepsilon(\cdot, t)$ converges to a solution v_3 of (1.6) strongly in $C^1(\mathbb{R}^N)$ as $t \rightarrow \infty$. Again, by comparison principle $v_1 < v_3 < v_2$.

We now show that v_3 is also a ground state. Indeed, by the decrease of energy we have for any $R > 0$

$$\begin{aligned} +\infty > E[\underline{u}_1^\varepsilon(\cdot, 0)] &\geq \int_{B_R(0)} \left(\frac{1}{2} |\nabla \underline{u}_1^\varepsilon(x, t)|^2 + V(\underline{u}_1^\varepsilon(x, t)) \right) dx \\ &\quad + \int_{\mathbb{R}^N \setminus B_R(0)} V(\underline{u}_1^\varepsilon(x, t)) dx. \end{aligned} \tag{3.31}$$

In view of the fact that $\underline{u}_1^\varepsilon(x, t) < v_2(x)$ for all $x \in \mathbb{R}^N$, for every $R_0 > 0$ large enough we have $V(\underline{u}_1^\varepsilon(x, t)) < \alpha$ for all $t \geq 0$ and all $|x| > R_0$. Recall that by our assumptions the function $V(u)$ is monotone for all $u \in (0, \alpha)$. Therefore, the last term in (3.31) can be bounded from below as follows:

$$\int_{\mathbb{R}^N \setminus B_R(0)} V(\underline{u}_1^\varepsilon(x, t)) dx \geq \min \left\{ 0, \int_{\mathbb{R}^N \setminus B_{R_0}(0)} V(v_2) dx \right\}, \quad \forall R > R_0. \tag{3.32}$$

Then, passing to the limit as $t \rightarrow \infty$ in (3.31), we obtain

$$\int_{B_R(0)} \left(\frac{1}{2} |\nabla v_3|^2 + V(v_3) \right) dx \leq E[\underline{u}_1^\varepsilon(\cdot, 0)] + \int_{\mathbb{R}^N \setminus B_{R_0}(0)} |V(v_2)| dx, \tag{3.33}$$

for all $R > R_0$. Furthermore, since $|V(v_3(x))| \leq |V(v_2(x))|$ for all $|x| > R_0$, passing to the limit as $R \rightarrow \infty$ in (3.33) and using Lebesgue monotone convergence theorem in the first term and Lebesgue dominated convergence theorem in the second term, we get $|\nabla v_3| \in L^2(\mathbb{R}^N)$ and $V(v_3) \in L^1(\mathbb{R}^N)$, so that v_3 is also a ground state.

Finally, by Lemma 3.4 there exists $\delta > 0$ sufficiently small and a supersolution $\bar{v}_3^\delta(x, t)$ such that $\underline{v}_1^\varepsilon < \bar{v}_3^\delta < v_3$. Therefore, by comparison principle we have $\underline{u}_1^\varepsilon(x, t) < \bar{v}_3^\delta(x)$ for every $x \in \mathbb{R}^N$ and $t > 0$. But this contradicts the fact that $\underline{u}_1^\varepsilon(\cdot, t) \rightarrow v_3$ uniformly as $t \rightarrow \infty$. \square

We also establish strict positivity of the energy of ground states, using a kind of Hamiltonian identity for (1.6) (see a related discussion in [10]).

Lemma 3.6. *Let $f \in C^1[0, \infty)$ and let v be a ground state in the sense of Definition 2.1. Then*

$$E[v] = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2 dx > 0. \tag{3.34}$$

Proof. Let $v_i := \partial v / \partial x_i, i = 1, \dots, N$, and for $R > 0$ let $\chi_R \in C_c^\infty(\mathbb{R}^N)$ be a cutoff function such that $0 \leq \chi_R \leq 1, \|\nabla \chi_R\|_{L^\infty(\mathbb{R}^N)} \leq C$ for some $C > 0$ independent of $R, \chi_R(x) = 1$ for all $|x| < R$ and $\chi_R(x) = 0$ for all $|x| > R + 1$. We multiply (1.6) by $\chi_R v_i$ and integrate over all $x \in \mathbb{R}^N$ such that $x_i < \xi$, for a fixed $\xi \in \mathbb{R}$. With the help of the fact that $\nabla v \in H^2(\mathbb{R}^N; \mathbb{R}^N)$ demonstrated in the proof of Lemma 3.4, this yields, after a number of integrations by parts,

$$\begin{aligned} 0 &= \int_{\{x_i=\xi\}} \chi_R (v_i^2 - V(v)) d\mathcal{H}^{N-1}(x) \\ &\quad - \int_{\{x_i<\xi\}} \chi_R \nabla v \cdot \nabla v_i dx - \int_{\{x_i<\xi\}} v_i \nabla \chi_R \cdot \nabla v dx + \int_{\{x_i<\xi\}} V(v) \frac{\partial \chi_R}{\partial x_i} dx \\ &= \int_{\{x_i=\xi\}} \chi_R \left(v_i^2 - \frac{1}{2} |\nabla v|^2 - V(v) \right) d\mathcal{H}^{N-1}(x) \\ &\quad - \int_{\{x_i<\xi\}} v_i \nabla \chi_R \cdot \nabla v dx + \int_{\{x_i<\xi\}} \left(\frac{1}{2} |\nabla v|^2 + V(v) \right) \frac{\partial \chi_R}{\partial x_i} dx. \end{aligned} \tag{3.35}$$

Since $\nabla \chi_R$ is uniformly bounded and supported on $\mathbb{R}^N \setminus B_R(0)$, by integrability of $|\nabla v|^2$ and $V(v)$ the last line in (3.35) goes to zero when $R \rightarrow \infty$. Therefore, by Fubini’s theorem and Lebesgue dominated convergence theorem we have for a.e. $\xi \in \mathbb{R}$

$$\int_{\{x_i=\xi\}} v_i^2 d\mathcal{H}^{N-1}(x) = \int_{\{x_i=\xi\}} \left(\frac{1}{2} |\nabla v|^2 + V(v) \right) d\mathcal{H}^{N-1}(x). \tag{3.36}$$

Finally, integrating (3.36) over all $\xi \in \mathbb{R}$, we get, again, by Fubini’s theorem,

$$E[v] = \int_{\mathbb{R}^N} v_i^2 dx. \tag{3.37}$$

In view of the fact that this identity holds for each $i = 1, \dots, N$, summing up over all i yields the statement. \square

Remark 3.7. We note that by the argument in the proof of Lemma 3.6, for every ground state v the function $\varphi(n) := \|n \cdot \nabla v\|_{L^2(\mathbb{R}^N)}$ is independent of n , for every $n \in \mathbb{S}^{N-1}$. This is consistent with radial symmetry of solutions of (1.6) known for many specific choices of f .

We will need the following simple non-existence result.

Lemma 3.8. *Let $f \in C^1[0, \infty)$ and suppose that there exist $0 \leq \alpha < \beta$ such that $f(u) \geq 0$ for all $u \in (\alpha, \beta)$. Then (1.6) has no non-constant radial symmetric-decreasing solutions with range in (α, β) whenever $N \leq 2$.*

Proof. The proof is elementary via the ordinary differential equation in (2.14). Let v_μ be a solution of (2.14) satisfying $\alpha < v_\mu < \beta$. If $N = 1$, then $v_\mu(r)$ is concave for all $r > 0$. Since it is also strictly decreasing, we will necessarily have $v_\mu(r_0) = \alpha$ for some $r_0 > 0$, contradicting our assumption that $v_\mu(r)$ solves the equation for all $r > 0$ with $\alpha < v_\mu(r) < \beta$.

If, on the other hand, $N = 2$, then with $s = \ln r$ as a new variable the solution of (2.14) obeys (with a slight abuse of notation, we still denote the solution as $v_\mu(s)$)

$$v''_\mu(s) + e^{2s}f(v_\mu(s)) = 0, \quad v'_\mu(s) \leq 0, \quad \alpha < v_\mu(s) < \beta, \quad -\infty < s < +\infty. \tag{3.38}$$

Once again, $v_\mu(s)$ is concave and strictly decreasing, which is a contradiction. \square

To conclude this section, we state the Poincaré type inequality characterizing the exponentially weighted Sobolev spaces, which is a straightforward generalization of [36, Lemma 2.2] to the whole space.

Lemma 3.9. *Let $c > 0$ and let $u \in H^1_c(\mathbb{R}^N)$. Then for every open set $\Omega \subseteq \mathbb{R}^{N-1}$ there holds*

$$\int_R^\infty \int_\Omega e^{cz} u_z^2 \, dy \, dz \geq \frac{c^2}{4} \int_R^\infty \int_\Omega e^{cz} u^2 \, dy \, dz, \tag{3.39}$$

for every $R \in [-\infty, +\infty)$.

4. Propagation: Proof of Theorems 1, 2 and 3. We begin with the proof of Theorem 2, which uses an adaptation of the arguments from [35, 36, 38] to the problem in \mathbb{R}^N . The key notion used in the proof is that of a wave-like solution.

Definition 4.1. We call the solution $u(x, t)$ of (1.1) and (1.2) *wave-like*, if $u(\cdot, T) \in H^1_c(\mathbb{R}^N)$ and $\Phi_c[u(\cdot, T)] < 0$ for some $c > 0$ and $T \geq 0$.

We want to show that for radial symmetric-decreasing solutions of (1.1) the wave-like property implies propagation whenever $f'(0) \leq 0$.

Our first lemma connects the wave-like property of solutions with the sign of their energy.

Lemma 4.2. *Let $u(x, t)$ be the solution of (1.1) satisfying (1.2). Suppose that $\phi \in L^2_{c_0}(\mathbb{R}^N)$ for some $c_0 > 0$ and suppose that there exists $T \geq 0$ such that $E[u(\cdot, T)] < 0$. Then $u(x, t)$ is wave-like.*

Proof. This lemma is a multidimensional extension of [38, Lemma 3.2] to the general nonlinearities in (1.3). By Proposition 3.1 we have $u(\cdot, T) \in H^1(\mathbb{R}^N) \cap H^1_{c_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, so, in view of (1.3), for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{\{|x|>R \cap \{z>0\}\}} e^{c_0 z} \left(\frac{1}{2} |\nabla u(x, T)|^2 + V^+(u(x, T)) \right) dx < \frac{\varepsilon}{4}, \tag{4.1}$$

$$\int_{\{|x|>R \cap \{z<0\}\}} \left(\frac{1}{2} |\nabla u(x, T)|^2 + V^+(u(x, T)) \right) dx < \frac{\varepsilon}{4}, \tag{4.2}$$

$$\tag{4.3}$$

where $V^+(u) := \max\{V(u), 0\}$. Hence

$$\int_{\{|x|>R\}} e^{cz} \left(\frac{1}{2} |\nabla u(x, T)|^2 + V^+(u(x, T)) \right) dx < \frac{\varepsilon}{2}, \tag{4.4}$$

for every $c \in (0, c_0)$.

Possibly increasing the value of R , we also have

$$\int_{\{|x|<R\}} \left(\frac{1}{2} |\nabla u(x, T)|^2 + V(u(x, T)) \right) dx < -\varepsilon, \tag{4.5}$$

provided that ε is small enough. Therefore, it is possible to choose $c \in (0, c_0)$ sufficiently small such that

$$\int_{\{|x|<R\}} e^{cz} \left(\frac{1}{2} |\nabla u(x, T)|^2 + V(u(x, T)) \right) dx < -\frac{\varepsilon}{2}. \tag{4.6}$$

Combining this with (4.4) yields $\Phi_c[u(\cdot, T)] < 0$, proving the claim. □

Remark 4.3. If $u(x, t)$ is a solution of (1.1) satisfying (1.2) that is wave-like, then we also have $\Phi_c[u(\cdot, t)] < 0$ for all $t \geq T$.

Proof. Since $u(\cdot, T) \in H_c^1(\mathbb{R}^N)$ and $\Phi_c[u(\cdot, T)] < 0$, by Proposition 3.1 we have that (2.9) holds. Hence, if \tilde{u} is defined in (2.4), we have $\Phi_c[u(\cdot, t)] = e^{c^2 t} \Phi_c[\tilde{u}(\cdot, t)] \leq e^{c^2 t} \Phi_c[\tilde{u}(\cdot, T)] = e^{c^2(t-T)} \Phi_c[u(\cdot, T)] < 0$. □

We next show that the level sets of radial symmetric-decreasing wave-like solutions propagate with positive speed. For such solutions, the leading and the trailing edges defined in (2.10) and (2.11) coincide: $R_\delta^+(t) = R_\delta^-(t) =: R_\delta(t)$. We use the convention that $R_\delta(t) := 0$ if $u(x, t) < \delta$ for all $x \in \mathbb{R}^N$.

Lemma 4.4. *Let $f'(0) \leq 0$ and assume that (1.5) holds. Let $u(x, t)$ be the solution of (1.1) satisfying (1.2) and (SD), and suppose $u(\cdot, T) \in H_c^1(\mathbb{R}^N)$ and $\Phi_c[u(\cdot, T)] < 0$ for some $c > 0$ and $T \geq 0$. Then for every $\delta \in (0, 1)$ and every $c' \in (0, c)$ there is $R_0 \in \mathbb{R}$ such that*

$$R_\delta(t) > c't + R_0, \tag{4.7}$$

for all $t \geq 0$.

Proof. Generalizing the definition in (2.16), let

$$\theta^* := \inf \{ u > 0 : V(u) < 0 \}, \tag{4.8}$$

and observe that by our assumptions $\theta^* \in [\theta_0, 1)$. Next, define

$$\theta_c := \inf \left\{ u > 0 : V(u) + \frac{c^2 u^2}{8} < 0 \right\}. \tag{4.9}$$

We claim that $\theta_c \in (\theta^*, 1)$. In particular, we have $\theta_c > \theta_0$. Indeed, clearly $\theta_c > \theta^*$ if $\theta^* > 0$. At the same time, since $f'(0) \leq 0$, we have $\theta_c > 0$. Furthermore, by Lemma 3.9 and Remark 4.3 there holds

$$0 > \Phi_c[u(\cdot, t)] \geq \int_{\mathbb{R}^N} e^{cz} \left(\frac{c^2 u^2(x, t)}{8} + V(u(x, t)) \right) dx, \tag{4.10}$$

for all $t \geq T$. Therefore, passing to the limit $t \rightarrow \infty$ in (4.10) and using Corollary 3.2, in view of (1.3) we conclude that $\theta_c < 1$.

Now, by (2.9) we have for any $t > T$

$$e^{-c(R_{\theta_c}(t)-ct)} \Phi_c[\tilde{u}(\cdot, T)] \geq e^{-c(R_{\theta_c}(t)-ct)} \Phi_c[\tilde{u}(\cdot, t)] = e^{-cR_{\theta_c}(t)} \Phi_c[u(\cdot, t)], \tag{4.11}$$

where \tilde{u} is defined in (2.4). Again, using Lemma 3.9, (SD) and noting that $\min_{u \geq 0} V(u) = V(1) < 0$ by (1.5), we obtain

$$\begin{aligned} \Phi_c[u(\cdot, t)] &\geq \int_{\{|y| < R_{\theta_c(t)}\} \cap \{z < R_{\theta_c(t)}\}} e^{cz} V(u(x, t)) \, dx \\ &\quad + \int_{\{|y| < R_{\theta_c(t)}\} \cap \{z > R_{\theta_c(t)}\}} e^{cz} \left(\frac{c^2 u^2(x, t)}{8} + V(u(x, t)) \right) \, dx \\ &\quad + \int_{\{|y| > R_{\theta_c(t)}\}} e^{cz} \left(\frac{c^2 u^2(x, t)}{8} + V(u(x, t)) \right) \, dx \\ &\geq \frac{C_N V(1)}{c} R_{\theta_c}^{N-1}(t) e^{cR_{\theta_c}(t)}, \end{aligned} \tag{4.12}$$

for some $C_N > 0$ depending only on N . Thus, for every $t > T$ we have

$$e^{-c(R_{\theta_c}(t) - ct)} \Phi_c[\tilde{u}(\cdot, T)] \geq \frac{C_N V(1)}{c} R_{\theta_c}^{N-1}(t). \tag{4.13}$$

Dividing this inequality by a negative quantity $\Phi_c[\tilde{u}(\cdot, T)]$ and taking the logarithm of both sides, we obtain

$$R_{\theta_c}(t) + \frac{N-1}{c} \ln R_{\theta_c}(t) \geq ct + \frac{1}{c} \ln \frac{c \Phi_c[\tilde{u}(\cdot, T)]}{C_N V(1)}. \tag{4.14}$$

As $t \rightarrow \infty$, the right-hand side of (4.14) goes to positive infinity, which implies that $\lim_{t \rightarrow \infty} R_{\theta_c}(t) = \infty$. Then $R_{\theta_c}(t)$ dominates in the left-hand side of (4.14) and, therefore, for any $c' \in (0, c)$ we have $R_{\theta_c}(t) > c't$ for any sufficiently large t . This proves the desired result for all $\delta \in (0, \theta_c]$.

To complete the proof, we need to show that (4.7) also holds for all $\delta \in (\theta_c, 1)$. We note that by (SD) we have $\hat{u}(x, t) > \theta_c$ for all $x \in B_R(0)$ and $t \geq T'$, with some $T' > 0$ sufficiently large depending on $R > 0$, where $\hat{u}(y, z, t) := u(y, z + c't, t)$. At the same time, since $\theta_c > \theta_0$, we have that $\underline{u} : B_R(0) \times [T', \infty) \rightarrow [0, 1)$ solving

$$\underline{u}_t = \Delta \underline{u} + c' \underline{u}_z + f(\underline{u}), \quad (x, t) \in B_R(0) \times [T', \infty), \tag{4.15}$$

$$\underline{u}(x, T') = \theta_c, \quad x \in B_R(0), \tag{4.16}$$

$$\underline{u}(x, t) = \theta_c, \quad (x, t) \in \partial B_R(0) \times [T', \infty), \tag{4.17}$$

is a monotonically increasing in t subsolution for $\hat{u}(x, t)$ in $B_R(0) \times [T', \infty)$. In particular, by standard parabolic regularity we have $\underline{u}(x, t) \rightarrow \underline{u}_R^\infty(x)$ from below as $t \rightarrow \infty$ uniformly in $x \in B_R(0)$, where $\underline{u}_R^\infty(x) < 1$ is a stationary solution of (4.15) and (4.17). By standard elliptic regularity [23], the latter, in turn, constitute an increasing family of solutions of (4.15) that converge locally uniformly to a limit solution $\underline{u}^\infty(x) > \theta_c$ in all of \mathbb{R}^N as $R \rightarrow \infty$. At the same time, from the fact that $f(u) > 0$ for all $u \in [\theta_c, 1)$ we conclude that $\underline{u}^\infty(x) = 1$ (use the solution of $u_t = f(u)$ with $u(x, 0) = \theta_c$ as a subsolution). Hence, by comparison principle, we have $\lim_{t \rightarrow \infty} u(y, z + c't, t) = 1$, yielding the claim in view of (SD). \square

Corollary 4.5. *Let $f'(0) \leq 0$ and assume that (1.5) holds. Let $u(x, t)$ be a wave-like solution satisfying (SD). Then $\lim_{t \rightarrow \infty} u(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N .*

Finally, using a truncation argument similar to the one in [38, Lemma 3.4], we can construct a wave-like subsolution for a solution of (1.1) whose energy becomes negative at some t . Then, applying Corollary 4.5 to that subsolution and using comparison principle, we arrive at the following result.

Lemma 4.6. *Let $f'(0) \leq 0$ and assume that (1.5) holds. Let $u(x, t)$ be the solution of (1.1) satisfying (1.2) and (SD), and suppose that there exists $T \geq 0$ such that $E[u(\cdot, T)] < 0$. Then $\lim_{t \rightarrow \infty} u(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N .*

Lemma 4.6 essentially constitutes the statement of part (ii) of Theorem 2. The proof of part (i) then comes from the following lemma that estimates the energy dissipation rate for radial symmetric-decreasing solutions that propagate.

Lemma 4.7. *Let $f'(0) \leq 0$ and assume that (1.5) holds. Let $u(x, t)$ be the solution of (1.1) satisfying (1.2) and (SD), and suppose that $u(\cdot, t_n) \rightarrow 1$ locally uniformly in \mathbb{R}^N for some sequence of $t_n \rightarrow \infty$. Then $\lim_{t \rightarrow \infty} E[u(\cdot, t)] = -\infty$.*

Proof. We argue by contradiction. Suppose that $\lim_{t \rightarrow \infty} u(\cdot, t) = 1$ locally uniformly in \mathbb{R}^N and $E[u(\cdot, t)]$ is bounded below. Fix $\varepsilon \in (0, 1 - \theta^*)$, where θ^* is defined in (4.8), and $R > 0$, and consider

$$\phi_{\varepsilon, R}(x) = \begin{cases} 1 - \varepsilon, & |x| < R, \\ (1 - \varepsilon)(R + 1 - |x|), & R \leq |x| \leq R + 1, \\ 0, & |x| > R + 1. \end{cases} \tag{4.18}$$

It is easy to see that there exists $R = R_\varepsilon$ such that $E[\phi_{\varepsilon, R_\varepsilon}] < 0$. Now let $u_\varepsilon(x, t)$ be the solution of (1.1) and (1.2) with $\phi = \phi_{\varepsilon, R_\varepsilon}$. By Lemma 4.2, $u_\varepsilon(x, t)$ is wave-like. Therefore, by Lemma 4.4 we have $R_{\theta_c}^\varepsilon(t) > c't + R_0$ for all $c' \in (0, c)$ and all $t \geq 0$, with some $R_0 \in \mathbb{R}$ independent of t , where θ_c is defined in (4.9) and $R_\delta^\varepsilon(t)$ is the leading edge of $u_\varepsilon(x, t)$.

Since $u(\cdot, t_n) \rightarrow 1$ as $t \rightarrow \infty$ locally uniformly in \mathbb{R}^N , there exists $T_\varepsilon \geq 0$ such that $u(x, T_\varepsilon) \geq u_\varepsilon(x, 0)$ for all $x \in \mathbb{R}^N$. Therefore, $u_\varepsilon(x, t - T_\varepsilon)$ is a subsolution for $u(x, t)$ for all $t \geq T_\varepsilon$ and, consequently, by comparison principle we have $R_{\theta_c}(t) > ct/2$ for all $t \geq t_0$, for some $t_0 \geq T_\varepsilon$. This implies that $u(x, t) \geq \theta_c$ for all $t \geq t_0$ and $x \in \mathbb{R}^N$ such that $|x| \leq ct/2$.

By (2.2), for any $\alpha > 0$ there exists $t_\alpha \geq 0$ such that

$$\int_{t_\alpha}^\infty \int_{\mathbb{R}^N} u_t^2(x, t) \, dx \, dt \leq \alpha^2. \tag{4.19}$$

Let us take

$$\alpha = 2^{-(3N+4)/2} c^{N/2} \theta_c |B_1(0)|^{1/2}. \tag{4.20}$$

We also take $t_1 \geq 0$ sufficiently large such that $t_1 \geq \max\{t_\alpha, t_0\}$ and $r_0 = R_{\theta_c/2}(t_1) > 1$. In addition, we take T sufficiently large such that $T > \max\{t_0, 1\}$ and $r_0 < cT/4$. Finally, we take $t_2 = t_1 + T$ and $r = cT/2$. Since $t_2 > t_1 \geq t_\alpha$, by Cauchy-Schwarz inequality we have

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\{|x| < r\}} |u_t(x, t)| \, dx \, dt \\ & \leq \sqrt{r^N |B_1(0)| (t_2 - t_1)} \left(\int_{t_1}^{t_2} \int_{|x| < r} u_t^2(x, t) \, dx \, dt \right)^{1/2} \\ & \leq \sqrt{r^N |B_1(0)| (t_2 - t_1)} \left(\int_{t_\alpha}^\infty \int_{\mathbb{R}^N} u_t^2(x, t) \, dx \, dt \right)^{1/2} \\ & \leq \alpha \sqrt{r^N |B_1(0)| (t_2 - t_1)} \end{aligned}$$

$$= \frac{\theta_c |B_1(0)|}{4} \left(\frac{c}{4}\right)^N T^{(N+1)/2}. \tag{4.21}$$

On the other hand, we also have

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\{|x|<r\}} |u_t(x,t)| dx dt &\geq \int_{t_1}^{t_2} \int_{\{cT/4 < |x| < cT/2\}} |u_t(x,t)| dx dt \\ &\geq \int_{\{cT/4 < |x| < cT/2\}} (u(x,t_2) - u(x,t_1)) dx. \end{aligned} \tag{4.22}$$

Since $t_2 > T > t_0$, we have $u(x, t_2) \geq \theta_c$ for $|x| \leq cT/2$, and by the definition of r_0 and T we have $u(x, t_1) < \theta_c/2$ for $cT/4 < |x| < cT/2$. So we have

$$\int_{t_1}^{t_2} \int_{\{|x|<r\}} |u_t(x,t)| dx dt \geq \frac{\theta_c |B_1(0)|}{2} \left(\frac{c}{4}\right)^N T^N, \tag{4.23}$$

which contradicts (4.21), because $T > 1$ and $N \geq 1$. □

Proof of Theorem 2. We just need to verify that the assumptions of Lemma 4.6 and Lemma 4.7 are satisfied. If $\theta_0 = 0$, then by (1.3) we must have $f(u) > 0$ for all $u \in (0, 1)$, and so (1.5) is clearly satisfied. On the other hand, if $\theta_0 > 0$, then by (1.3) we must have $f'(0) \leq 0$. □

Proof of Theorem 3. By Theorem 2, either $\lim_{t \rightarrow \infty} E[u(\cdot, t)] = -\infty$ or $\lim_{t \rightarrow \infty} E[u(\cdot, t)] \geq 0$. Indeed, if $\lim_{t \rightarrow \infty} E[u(\cdot, t)] \in (-\infty, 0)$, then by Theorem 2(ii) we have $u(\cdot, t) \rightarrow 1$ locally uniformly in \mathbb{R}^N . However, this contradicts Theorem 2(i), since in this case one would have $\lim_{t \rightarrow \infty} E[u(\cdot, t)] = -\infty$.

If $\lim_{t \rightarrow \infty} E[u(\cdot, t)] = -\infty$, then there exists $T > 0$ such that $E[u(\cdot, T)] < 0$. Hence $u(\cdot, t) \rightarrow 1$ locally uniformly in \mathbb{R}^N by Theorem 2(ii). This establishes the first alternative. In the second alternative, we have $E[u(\cdot, t)] \geq 0$ for all $t \geq 0$. Therefore, by (2.2) there exists a sequence of $t_n \in [n, n + 1)$ such that $u_t(\cdot, t_n) \rightarrow 0$ in $L^2(\mathbb{R}^N)$. In turn, by standard parabolic regularity one can extract a subsequence t_{n_k} from this sequence such that $u(\cdot, t_{n_k}) \rightarrow v$ in $C^1_{loc}(\mathbb{R}^N)$ as $k \rightarrow \infty$. Following the usual argument for gradient flows, from (2.3) we then conclude that v solves (1.6) distributionally and, hence, classically [23]. Furthermore, v is radial symmetric-decreasing, and, taking into account Corollary 3.2, we have $v(x) = v_\mu(|x|)$ for some $\mu \in [0, 1]$ and all $x \in \mathbb{R}^N$, where v_μ solves (2.14). Note that all radial symmetric-decreasing solutions of (1.6) can be parametrized by $\mu = v(0)$. In particular, by continuous dependence of the solutions of (2.14) on μ in the $C^1_{loc}(\mathbb{R})$ topology the set of all μ 's for which the solution of (2.14) exists is closed.

By Theorem 2(i), $\mu = 1$ is impossible when $E[u(\cdot, t)] \geq 0$ for all $t \geq 0$. Hence the set of all μ 's corresponding to the limits of $u(\cdot, t_{n_k})$ is contained in $[0, 1)$. Denoting by $\omega(\phi) \subseteq [0, 1)$ the set of all limits of $u(\cdot, t_{n_k})$ in $C^1_{loc}(\mathbb{R}^N)$ parametrized by $\mu = v(0)$, which coincides with the ω -limit set of $u(x, t)$ (cf. Proposition 3.3 and standard parabolic regularity), by the usual properties of ω -limit sets we have that $\omega(\phi) = [a, b]$ for some $0 \leq a \leq b < 1$, i.e., that $\omega(\phi) \subseteq [0, 1)$ is closed and connected. Thus

$$\lim_{n \rightarrow \infty} \inf_{\mu \in \omega(\phi)} \sup_{x \in B_R(0)} |u(x, t_n) - v_\mu(|x|)| \rightarrow 0, \tag{4.24}$$

for any $R > 0$. In view of Proposition 3.3, this completes the proof of the statement in the second alternative. □

Proof of Theorem 1. Since for $N = 1$ the result was established in [38, Proposition 2.4], in the rest of the proof we assume that $N \geq 2$. By rotational symmetry,

the upper bound on $R_\delta^+(t)$ follows exactly as in [36, Proposition 5.2], noting that $u(\cdot, t) \in H^2(\mathbb{R}^N) \cap H_{c_0}^2(\mathbb{R}^N)$ for each $t > 0$. To prove the matching lower bound, for each $c \in (0, c^\dagger)$ we construct a test function $u_c \in H_c^1(\mathbb{R}^N)$ which is radial symmetric-decreasing and satisfies $\Phi_c[u_c] < 0$. Then the result follows by Lemma 4.4.

Let $\bar{u}_c(z)$ be a one-dimensional minimizer from [38, Proposition 2.3], which, e.g., by simple phase plane arguments is non-increasing in z . For $R > 0$, we define

$$u_c^R(x) := \bar{u}_c(|x| - R), \quad x \in \mathbb{R}^N. \tag{4.25}$$

In particular, $\text{supp}(u_c^R) = \bar{B}_R(0)$, and u_c^R satisfies (SD). We also note that by the definition of \bar{u}_c and boundedness of \bar{u}_c and \bar{u}'_c there exists $K > 0$ such that for all $R > 0$ sufficiently large we have

$$m_{c,R} := \int_0^\infty e^{cz} \left(\frac{1}{2} |\bar{u}'_c(z - R)|^2 + V(\bar{u}_c(z - R)) \right) dz < -K e^{cR}. \tag{4.26}$$

Let us now evaluate $\Phi_c[u_c]$. Passing to the spherical coordinates, we obtain

$$\begin{aligned} \Phi_c[u_c] &= \int_{\mathbb{R}^N} e^{cz} \left(\frac{1}{2} |\nabla u_c|^2 + V(u_c) \right) dx \\ &= |\mathbb{S}^{N-2}| \int_0^\infty \int_0^\pi e^{cr \cos \theta} \left(\frac{1}{2} |\bar{u}'_c(r - R)|^2 + V(\bar{u}_c(r - R)) \right) r^{N-1} \sin^{N-2} \theta \, d\theta \, dr \\ &= |\mathbb{S}^{N-2}| R^{N-1} m_{c,R} \int_0^\pi e^{-cR(1-\cos \theta)} \sin^{N-2} \theta \, d\theta + |\mathbb{S}^{N-2}| \int_0^\infty \int_0^\pi e^{cr \cos \theta} \\ &\quad \times \left(\frac{1}{2} |\bar{u}'_c(r - R)|^2 + V(\bar{u}_c(r - R)) \right) (r^{N-1} - R^{N-1}) \sin^{N-2} \theta \, d\theta \, dr \\ &\quad + |\mathbb{S}^{N-2}| R^{N-1} \int_0^\infty \int_0^\pi e^{cr} \left(e^{-cr(1-\cos \theta)} - e^{-cR(1-\cos \theta)} \right) \\ &\quad \times \left(\frac{1}{2} |\bar{u}'_c(r - R)|^2 + V(\bar{u}_c(r - R)) \right) \sin^{N-2} \theta \, d\theta \, dr. \end{aligned} \tag{4.27}$$

We proceed to estimate, using the fact that $e^{-cr(1-\cos \theta)} \simeq e^{-cr\theta^2/2}$ for $\theta \ll 1$ (the details are left to the reader):

$$e^{-cR} R^{-\frac{N-1}{2}} \Phi_c[u_c] \leq -CK + C'R^{-1}, \tag{4.28}$$

for some $C, C' > 0$ independent of R . Therefore, choosing R sufficiently large yields the claim. \square

5. Bistable nonlinearities: Proof of Theorem 4. We now proceed to the sharp threshold results. For bistable nonlinearities satisfying (1.5), we establish existence of a sharp threshold in Theorem 4 under (TD). We define

$$\Sigma_0 := \{ \lambda \in [0, \lambda^+] : u_\lambda(\cdot, t) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ uniformly in } \mathbb{R}^N \}, \tag{5.1}$$

$$\Sigma_1 := \{ \lambda \in [0, \lambda^+] : u_\lambda(\cdot, t) \rightarrow 1 \text{ as } t \rightarrow \infty \text{ locally uniformly in } \mathbb{R}^N \}, \tag{5.2}$$

$$\Sigma_* := [0, \lambda^+] \setminus (\Sigma_0 \cup \Sigma_1). \tag{5.3}$$

Our goal is to prove that the threshold set Σ_* is a single point and to characterize the threshold solution.

Proof of Theorem 4. By Theorem 3, for every $\lambda \in [0, \lambda^+]$ we have either $E[u_\lambda(\cdot, T)] < 0$ for some $T > 0$, or $E[u_\lambda(\cdot, t)] \geq 0$ for all $t > 0$. In the first case, we have $u_\lambda(\cdot, t) \rightarrow 1$ locally uniformly as $t \rightarrow \infty$ and, therefore, $\lambda \in \Sigma_1$. In the second case, we have $u_\lambda(\cdot, t) \not\rightarrow 1$ locally uniformly, so $\lambda \notin \Sigma_1$.

Consider the case of $\lambda \in \Sigma_1$. Note that by (P3), the set Σ_1 is non-empty. As was mentioned in the preceding paragraph, we have $E[u_\lambda(\cdot, T)] < 0$ for some $T > 0$. Then by continuous dependence of the solution on the initial data we also have $E[u_{\lambda'}(\cdot, T)] < 0$ for all $\lambda' \in (0, \lambda^+]$ in a sufficiently small neighborhood of λ . Furthermore, by (P2) and comparison principle, for every $0 < \lambda_1 < \lambda_2 < \lambda^+$ we have $u_{\lambda_1}(\cdot, t) < u_{\lambda_2}(\cdot, t)$ for all $t > 0$. Therefore, if $\lambda_1 \in \Sigma_1$, then so is λ_2 . This means that there exists $\lambda_*^+ \in (0, \lambda^+)$ such that $\Sigma_1 = (\lambda_*^+, \lambda^+]$.

At the same time, we know that $\lambda \in \Sigma_0$ if and only if there exists $T > 0$ such that $u_\lambda(0, T) < \theta_0$ (use the solution of $u_t = f(u)$ with $u(x, 0) = u_0 \in (0, \theta_0)$ as a supersolution). Again, by continuous dependence of solutions on the initial data, if $u_\lambda(0, T) < \theta_0$ for some $\lambda \in [0, \lambda^+)$, then $u_{\lambda'}(0, T) < \theta_0$ as well for all $\lambda' \in [0, \lambda^+)$ sufficiently close to λ , and by comparison principle we have $u_{\lambda_1}(\cdot, t) \rightarrow 0$ uniformly for all $0 < \lambda_1 < \lambda_2 < \lambda^+$, whenever $u_{\lambda_2}(\cdot, t) \rightarrow 0$, as $t \rightarrow \infty$. Thus, again, by (P3) there exists $\lambda_*^- \in (0, \lambda^+)$ such that $\Sigma_0 = [0, \lambda_*^-)$.

Now, let $\lambda \in \Sigma_* = [\lambda_*^-, \lambda_*^+] \neq \emptyset$. We claim that under our assumptions $u_\lambda(\cdot, t) \rightarrow v$ uniformly as $t \rightarrow \infty$, where v is a ground state. Indeed, for a sequence of $t_n \rightarrow \infty$ as in the proof of Theorem 3, let $u(\cdot, t_{n_k}) \rightarrow v$ in $C_{loc}^1(\mathbb{R}^N)$ for some $n_k \rightarrow \infty$, with $v(x) = v_\mu(|x|)$ and v_μ solving (2.14) with $\mu = v(0)$. Note that since $\lambda \notin \Sigma_0$, by comparison principle we have $u_\lambda(0, t_{n_k}) > \theta_0$ for all $k \in \mathbb{N}$ and, hence, $\mu \geq \theta_0$. Furthermore, for every k there is a unique $R_k > 0$ such that $u_\lambda(x, t_{n_k}) = \theta_0/2$ for $|x| = R_k$. Also, there exists $R_0 \geq 0$ such that $\theta_0 < v_\mu(R_0) < \theta^*$, where θ^* is defined via (2.16), with the convention that $R_0 = 0$ if this inequality has no solution. Then, by (SD) there exists $k_0 \in \mathbb{N}$ such that $u_\lambda(x, t_{n_k}) < \theta^*$ for all $|x| > R_0$ and all $k \geq k_0$. In turn, this means that $V(u_\lambda(x, t_{n_k})) \geq 0$ and $R_k > R_0$ for all $k \geq k_0$ and $|x| > R_0$.

By (2.2), for any $T \in (0, 1)$ and $k \geq k_0$ we have for any $R \geq R_0$

$$\begin{aligned} +\infty &> E[u_\lambda(\cdot, T)] \geq E[u_\lambda(\cdot, t_{n_k})] \\ &\geq \int_{B_R(0)} \left(\frac{1}{2} |\nabla u_\lambda(x, t_{n_k})|^2 + V(u_\lambda(x, t_{n_k})) \right) dx \\ &\quad + \int_{\mathbb{R}^N \setminus B_R(0)} V(u_\lambda(x, t_{n_k})) dx. \end{aligned} \tag{5.4}$$

On the other hand, since $V(u_\lambda(x, t_{n_k})) \geq V_0 > 0$ for all $R_0 < |x| < R_k$ and $k \geq k_0$, where $V_0 := \min\{V(\theta_0/2), V(v_\mu(R_0))\}$, from (5.4) with $R = R_0$ we get for all $k \geq k_0$:

$$+\infty > E[u_\lambda(\cdot, T)] \geq E[u_\lambda(\cdot, t_{n_k})] \geq \int_{B_{R_0}(0)} V(u_\lambda(x, t_{n_k})) dx + C_N(R_k^N - R_0^N)V_0, \tag{5.5}$$

for some $C_N > 0$ depending only on N . Thus, if $v \geq \theta_0$, we would have $R_k \rightarrow \infty$ as $k \rightarrow \infty$, contradicting (5.5).

We just demonstrated that $v(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then, passing to the limit in (5.4) as $k \rightarrow \infty$, we get

$$+\infty > E[u_\lambda(\cdot, T)] \geq \int_{B_R(0)} \left(\frac{1}{2} |\nabla v|^2 + V(v) \right) dx, \tag{5.6}$$

for all $R > R_0$ and $t > 0$. Then, sending $R \rightarrow \infty$, by Lebesgue dominated convergence theorem and monotonicity of $E[u_\lambda(\cdot, t)]$ we obtain that $E[u(\cdot, t)] \geq E[v]$. On the other hand, since $V(v(x)) > 0$ for all $|x| > R_0$, this implies that $V(v) \in L^1(\mathbb{R}^N)$ and $|\nabla v| \in L^2(\mathbb{R}^N)$. Thus, v is a ground state.

By the arguments above, every limit of $u_\lambda(\cdot, t_{n_k})$ is a ground state independently of the choice of n_k . Therefore, from Theorem 3 we get $I \subseteq \Upsilon$. On the other hand, by (TD) this means that $a = b \in (0, 1)$. In particular, the limit v is independent of n_k . Thus, $u_\lambda(\cdot, t_n) \rightarrow v$ as $n \rightarrow \infty$, and by Proposition 3.3 we have $u_\lambda(\cdot, t) \rightarrow v$ uniformly as $t \rightarrow \infty$. By standard parabolic regularity, this convergence is also in $C^1(\mathbb{R}^N)$.

Finally, we claim that Σ_* consists of only a single point, i.e., that $\lambda_*^- = \lambda_*^+ =: \lambda_*$. We argue by contradiction. Suppose, to the contrary, that $\lambda_*^- < \lambda_*^+$. Since $E[u_\lambda(\cdot, t)] \geq 0$ for all $t > 0$ and all $\lambda \in \Sigma_*$, there exists a sequence $n_k \in \mathbb{N}$, $n_k \rightarrow \infty$ as $k \rightarrow \infty$, and two sequences, $t_{n_k}^- \in [n_k, n_k + 1)$ and $t_{n_k}^+ \in [n_k, n_k + 1)$, such that $u_{\lambda_*^-}(\cdot, t_{n_k}^-) \rightarrow v^-$ (resp. $u_{\lambda_*^+}(\cdot, t_{n_k}^+) \rightarrow v^+$) in $C_{loc}^1(\mathbb{R}^N)$ as $k \rightarrow \infty$, where $v^-(x) = v_{\mu^-}(|x|)$ (resp. $v^+(x) = v_{\mu^+}(|x|)$) and v_{μ^-} (resp. v_{μ^+}) solve (2.14) for some $\mu^- \in [\theta_0, 1)$ (resp. $\mu^+ \in [\theta_0, 1)$; cf. the arguments in the proof of Theorem 3). Furthermore, by comparison principle we have $\mu^- \leq \mu^+$. Then, by Proposition 3.3 and standard parabolic regularity we also have $u_{\lambda_*^-}(\cdot, t_{n_k}) \rightarrow v^-$ (resp. $u_{\lambda_*^+}(\cdot, t_{n_k}) \rightarrow v^+$) in $C_{loc}^1(\mathbb{R}^N)$ for $t_{n_k} = n_k + 2$, as $k \rightarrow \infty$. Therefore, since $v^- \leq v^+$ and both v^- and v^+ are ground states, by Corollary 3.5 we have $v^- = v^+ =: v_*$.

Let us show that this gives rise to a contradiction. Let $w(x, t) := u_{\lambda_*^+}(x, t) - u_{\lambda_*^-}(x, t) > 0$ for all $x \in \mathbb{R}^N$ and $t > 0$. By (1.1), $w(x, t)$ solves

$$w_t = \Delta w + f'(\tilde{u})w, \tag{5.7}$$

for some $u_{\lambda_*^-}(x, t) < \tilde{u}(x, t) < u_{\lambda_*^+}(x, t)$. On the other hand, since both $u_{\lambda_*^-}(x, t)$ and $u_{\lambda_*^+}(x, t)$ converge uniformly to v_* , we can use the arguments leading to (3.25) in the proof of Lemma 3.4 to show that $w(x, t) > \varepsilon \phi_0^R(x)$, where ϕ_0^R is as in (3.24), for all $x \in \mathbb{R}^N$ and $t > 0$ sufficiently large, contradicting the fact that $\|w(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$ as $t \rightarrow \infty$.

It remains to establish the asymptotic behavior of the energy as $t \rightarrow \infty$. If $\lambda \in \Sigma_1$, then the statement is contained in Theorem 3. On the other hand, if $\lambda \in \Sigma_0$, then we know that $u_\lambda(\cdot, t) \rightarrow 0$ in $L^2(\mathbb{R}^N)$ (use the solution of the heat equation as a supersolution for large t). Hence by (3.1) we also have $u_\lambda(\cdot, t+1) \rightarrow 0$ in $H^1(\mathbb{R}^N)$, which implies that $E[u_\lambda(\cdot, t)] \rightarrow 0$ as $t \rightarrow \infty$ in this case. Finally, if $\lambda \in \Sigma_*$, then $u_\lambda(\cdot, t) \rightarrow v_*$ for some ground state v_* and $E[u_\lambda(\cdot, t)] \geq E[v_*] > 0$ by (5.6) and Lemma 3.6. \square

6. Ignition nonlinearities: Proof of Theorems 5 and 6.

Proof of Theorem 5. As in the proof of Theorem 4, we can define the sets Σ_0, Σ_1 and Σ_* , and by the same argument we have $\Sigma_1 = (\lambda_*^+, \lambda^+]$ for some $\lambda_*^+ \in (0, \lambda^+)$. Similarly, every solution in Σ_0 satisfies the linear heat equation for all sufficiently large t and, therefore, we have $\Sigma_0 = [0, \lambda_*^-)$ for some $\lambda_*^- \in (0, \lambda^+)$. Thus $\Sigma_* = [\lambda_*^-, \lambda_*^+] \neq \emptyset$.

Let now $\lambda \in \Sigma_*$, and notice that since $\lambda \notin \Sigma_0$, by comparison principle we have $u_\lambda(0, t_{n_k}) > \theta_0$ for all k . Then by the same arguments as in the proof of Theorem 4, there exists a sequence of $t_n \in [n, n + 1]$ and a sequence of $n_k \rightarrow \infty$ such that $u_\lambda(\cdot, t_{n_k}) \rightarrow v$ as $k \rightarrow \infty$ for some $v(x) = v_\mu(|x|)$, where v_μ solves (2.14) with $\mu = v(0) \geq \theta_0$. By Lemma 3.8, if $N \leq 2$ then $\mu = \theta_0$. Hence, in view of the uniqueness of the limit v independently of n_k , we have $u_\lambda(\cdot, t) \rightarrow \theta_0$ locally uniformly as $t \rightarrow \infty$. The bound on the energy is contained in Theorem 3.

The proof is completed by showing that $\lambda_*^- = \lambda_*^+$, which can be done as in the proof of [38, Theorem 9]. The latter relies on [53, Lemma 4], which is valid in \mathbb{R}^N for all $N \geq 1$. \square

Proof of Theorem 6. The proof proceeds as that of Theorem 5 up to the point when $u_\lambda(\cdot, t_{n_k}) \rightarrow v$ for $\lambda \in \Sigma_*$. However, in contrast to lower dimensions, for $N \geq 3$ there exist many solutions of (2.14), including a continuous family of non-constant solutions with $v^\infty := v_\mu(\infty) \in [0, \theta_0)$ [5, 6]. In particular, there is at least one ground state [5].

We claim that v is a ground state. Indeed, a priori we have $v^\infty \in [0, \theta_0]$. Consider first the case $v^\infty \in (0, \theta_0)$. Since $u_\lambda(\cdot, t_{n_k}) \not\rightarrow 0$, we have $u_\lambda(0, t_{n_k}) > \theta_0$ for all k . Therefore, there exists a unique $R_k > 0$ such that $u_\lambda(x, t_{n_k}) = v^\infty$ for $|x| = R_k$. Also, there exists a unique $R_0 > 0$ which solves $v^\infty < v_\mu(R_0) < \theta_0$. Since $u_\lambda(\cdot, t_{n_k}) \rightarrow v$ in $C^1(B_{R_0}(0))$, by (SD) we also have $u_\lambda(\cdot, t_{n_k}) < \theta_0$ for all $|x| > R_0$ and all $k \geq k_0$, with some $k_0 \in \mathbb{N}$ large enough, and, hence, for all $k \geq k_0$ we have $R_k > R_0$ and $V(u_\lambda(x, t_{n_k})) = 0$ for all $|x| > R_0$.

By (2.2), for any $T \in (0, 1)$, $R > R_0$ and $k \geq k_0$ we have

$$\begin{aligned}
 +\infty > E[u_\lambda(\cdot, T)] &\geq E[u_\lambda(\cdot, t_{n_k})] = \int_{B_R(0)} \left(\frac{1}{2} |\nabla u_\lambda(x, t_{n_k})|^2 + V(u_\lambda(x, t_{n_k})) \right) dx \\
 &\quad + \frac{1}{2} \int_{\mathbb{R}^N \setminus B_R(0)} |\nabla u_\lambda(x, t_{n_k})|^2 dx. \tag{6.1}
 \end{aligned}$$

On the other hand, by [5, Radial Lemma A.III] we can estimate the right-hand side of (6.1) from below as

$$+\infty > E[u_\lambda(\cdot, T)] \geq \int_{B_{R_0}(0)} V(u_\lambda(x, t_{n_k})) dx + C_N R_k^{N-2} |v^\infty|^2, \tag{6.2}$$

for all $k \geq k_0$, where $C_N > 0$ depends only on N . Therefore, if $v^\infty \in (0, \theta_0)$, we would have $R_k \rightarrow \infty$ as $k \rightarrow \infty$, contradicting (6.2).

Thus, we have either $v^\infty = 0$ or $v^\infty = \theta_0$. Let us consider the first case. Passing to the limit in (6.1) as $k \rightarrow \infty$, we get

$$+\infty > E[u_\lambda(\cdot, T)] \geq \int_{B_R(0)} \left(\frac{1}{2} |\nabla v|^2 + V(v) \right) dx, \tag{6.3}$$

for all $R \geq R_0$. Then, sending $R \rightarrow \infty$, by Lebesgue dominated convergence theorem we obtain that $E[u(\cdot, T)] \geq E[v]$. On the other hand, since $V(v(x)) = 0$ for all $|x| > R_0$, this implies that $V(v) \in L^1(\mathbb{R}^N)$ and $|\nabla v| \in L^2(\mathbb{R}^N)$. Thus, v is a ground state. Furthermore, we claim that if v' is a limit of $u_\lambda(\cdot, t_{n'_k})$ for another choice $t_{n'_k}$ of a subsequence of t_n , then $v' = v$. Indeed, since $v_\mu(R_1) = \theta_0$ for some $R_1 > 0$, by continuous dependence of solutions of (2.14) on μ we have $v_{\mu'}(\infty) = 0$ as well for all μ' in some small neighborhood of μ , whenever $v_{\mu'}$ exists. Therefore,

by (TD) the set $\Upsilon \cup \{v'(0)\}$ is disconnected, contradicting part 2 of the statement of Theorem 3, unless $v' = v$.

Consider now the case $v^\infty = \theta_0$. Define $R_k > 0$ to be such that $u(x, t_{n_k}) = \theta_0/2$ for $|x| = R_k$, and observe that $R_k \rightarrow \infty$ as $k \rightarrow \infty$. Arguing as in the preceding paragraph, we have

$$\begin{aligned}
 +\infty > E[u_\lambda(\cdot, T)] &\geq E[u_\lambda(\cdot, t_{n_k})] \geq \int_{B_{R_k}(0)} \left(\frac{1}{2} |\nabla u_\lambda(x, t_{n_k})|^2 + V(u_\lambda(x, t_{n_k})) \right) dx \\
 &\quad + C_N R_k^{N-2} |\theta_0|^2. \tag{6.4}
 \end{aligned}$$

Therefore, for every $M > 0$ there exists $k_0 \in \mathbb{N}$ and $R_0 := R_{k_0}$ such that $R_k \geq R_0$ for all $k \geq k_0$ and

$$E_0[u_\lambda(x, t_{n_{k_0}})] := \int_{B_{R_0}(0)} \left(\frac{1}{2} |\nabla u_\lambda(x, t_{n_{k_0}})|^2 + V(u_\lambda(x, t_{n_{k_0}})) \right) dx \leq -M. \tag{6.5}$$

We now take $\underline{u}_\lambda(x, t)$ to be the solution of (1.1) on $B_{R_0}(0)$ for $t > t_{n_{k_0}}$ with $\underline{u}_\lambda(x, t_{n_{k_0}}) = u_\lambda(x, t_{n_{k_0}})$ for all $x \in B_{R_0}(0)$ and $\underline{u}_\lambda(x, t) = \theta_0/2$ for all $x \in \partial B_{R_0}(0)$ and $t > t_{n_{k_0}}$. Possibly increasing the value of k_0 , we also have that $u_\lambda(x, t) > \theta_0/2$ for all $x \in \partial B_{R_0}(0)$ and $t > t_{n_{k_0}}$. Indeed, if not, then there is a sequence of $t'_n \rightarrow \infty$ such that $u_\lambda(x, t'_n) \leq \theta_0$ for all $|x| \geq R_0$ as $n \rightarrow \infty$. However, by the preceding arguments this would imply that $u_\lambda(\cdot, t'_n)$ converges to a ground state in $C^1_{loc}(\mathbb{R}^N)$ as $n \rightarrow \infty$, which contradicts our assumption of $v^\infty = \theta_0$. Thus, $\underline{u}_\lambda(x, t)$ is a subsolution for $u_\lambda(x, t)$ for all $x \in B_{R_0}(0)$ and $t > t_{n_{k_0}}$, and by comparison principle we have $\underline{u}_\lambda(x, t) < u_\lambda(x, t)$ in $B_{R_0}(0)$. In addition, from the gradient flow structure of (1.1) on $B_{R_0}(0)$ we have $E_0[\underline{u}_\lambda(\cdot, t_{n_k})] \leq E_0[\underline{u}_\lambda(\cdot, t_{n_{k_0}})] = E_0[u_\lambda(\cdot, t_{n_{k_0}})]$ for all $k \geq k_0$. Therefore, since $u_\lambda(\cdot, t_{n_k}) \rightarrow v$ in $C^1(B_{R_0}(0))$ as $k \rightarrow \infty$ and $V(u)$ is a non-increasing function of u , by (6.5) we conclude that

$$\begin{aligned}
 -M &\geq \lim_{k \rightarrow \infty} E_0[\underline{u}_\lambda(\cdot, t_{n_k})] \geq \lim_{k \rightarrow \infty} \int_{B_{R_0}(0)} V(u_\lambda(x, t_{n_k})) dx \\
 &= \int_{B_{R_0}(0)} V(v) dx \geq -\|V(v)\|_{L^1(\mathbb{R}^N)}. \tag{6.6}
 \end{aligned}$$

However, by (V) this is a contradiction when M is sufficiently large. Thus, $v^\infty = \theta_0$ is impossible.

We thus established that v is a ground state and that v is the full limit of $u_\lambda(\cdot, t)$ as $t \rightarrow \infty$ for any $\lambda_* \in \Sigma_*$. The remainder of the proof follows as in the proof of Theorem 4. \square

7. Monostable nonlinearities: Proof of Theorems 7, 8 and 9. In view of the hair-trigger effect for $f'(0) > 0$ [2], in which case the statements of all the theorems trivially holds true with $\lambda_* = 0$, it is sufficient to assume $f'(0) = 0$ in all the proofs.

Proof of Theorem 7 and Theorem 8. Once again, we define the sets Σ_1, Σ_0 and Σ_* and note that by the same argument as in the proofs of the preceding theorems we have $\Sigma_1 = (\lambda_*^+, \lambda_+]$ for some $\lambda_*^+ \in (0, \lambda_+)$. At the same time, by Lemma 3.8 there are no positive solutions of (2.14) for any $\mu \in (0, 1)$ and $N \leq 2$. Similarly, by the assumption of Theorem 8 there are no positive solutions of (2.14) for any $\mu \in (0, 1)$ and $N \geq 3$. Thus, $\Sigma_* = \emptyset$, and we have $\Sigma_0 = [0, \lambda_*^+]$. \square

Proof of Theorem 9. The proof proceeds in the same fashion as before, establishing that $\Sigma_1 = (\lambda_*^+, \lambda_+]$ for some $\lambda_*^+ \in (0, \lambda_+)$. If $\lambda_*^+ = 0$, we are done. Otherwise, suppose that $\lambda_*^+ > 0$. If there exists $\lambda \in (0, \lambda_*^+]$ such that $u_\lambda(\cdot, t) \rightarrow 0$ uniformly as $t \rightarrow \infty$, then by comparison principle $[0, \lambda] \subseteq \Sigma_0$. Let $\lambda_*^- \leq \lambda_*^+$ be the supremum of all such values of λ . Then either $\Sigma_0 = [0, \lambda_*^-)$ or $\Sigma_0 = [0, \lambda_*^-]$. In the second case and with $\lambda_*^- = \lambda_*^+$ we are done once again. Otherwise $\Sigma_* \neq \emptyset$. Finally, by our assumptions and the arguments in the proofs of the preceding theorems, for every $\lambda \in \Sigma_*$ we have $u_\lambda(\cdot, t) \rightarrow v$ uniformly as $t \rightarrow \infty$, where v is a ground state. Then by the arguments in the proof of Theorem 4 we have $\lambda_*^- = \lambda_*^+$. \square

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