# GLOBAL EXPONENTIAL CONVERGENCE TO VARIATIONAL TRAVELING WAVES IN CYLINDERS[*]

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**Abstract.** We prove, under generic assumptions, that the special variational traveling wave that minimizes the exponentially weighted Ginzburg–Landau functional associated with scalar reaction-diffusion equations in infinite cylinders is the long-time attractor for the solutions of the initial value problems with front-like initial data. The convergence to this traveling wave is exponentially fast. The obtained result is mainly a consequence of the gradient flow structure of the considered equation in the exponentially weighted spaces and does not depend on the precise details of the problem. It strengthens our earlier generic propagation and selection result for "pushed" fronts.

**1. Introduction.** One of the most fundamental problems in the theory of reaction-diffusion equations has to do with the long-time asymptotic behavior of solutions of the associated initial value problem on unbounded domains [6, 38, 39]. In its simplest form, it may be formulated for a one-dimensional scalar reaction-diffusion equation

$$(1.1) \qquad u_t = u_{xx} + f(u), \qquad u : \mathbb{R} \times \mathbb{R}^+ \to [0, 1],$$

with an unbalanced bistable nonlinearity $f(u)$, i.e., when $f$ is a smooth function which has precisely three nondegenerate zeros in $[0, 1]$, with

$$(1.2) \qquad f(0) = f(1) = 0, \qquad f'(0) < 0, \quad f'(1) < 0, \qquad \int_0^1 f(u)du > 0,$$

e.g., $f(u) = u(1 - u)(u - \frac{1}{4})$. For such an equation, it was first proved by Kanel' that initial data $u(x, t) = u_0(x)$ with the property that $u_0(x) = 0$ for all $x > b$, $u_0(x) = 1$ for all $x < a$, and $u_0(x)$ is monotone decreasing for $x \in (a, b)$, with some $-\infty < a < b < +\infty$, converges uniformly to a (unique up to translations) traveling wave solution, i.e., a solution $u(x, t) = \bar{u}(x - ct)$ of (1.1), with some uniquely determined speed $c > 0$, connecting monotonically $u = 0$ at $x = +\infty$ with $u = 1$ at $x = -\infty$, in a reference frame moving with speed $c$ [14, 15]. In a subsequent work, Fife and McLeod extended this result to a much wider class of initial data and also showed that the convergence is exponentially fast [7]. Qualitatively, the conclusion of these analyses is that the solution of the considered initial value problem with front-like initial data converges exponentially fast to a traveling front invading the "less stable"

equilibrium $u = 0$ by a "more stable" equilibrium $u = 1$. We note that a similar result was proved for a certain class of monostable nonlinearities [31], but it does *not* hold (in the reference frame moving with constant speed and in the sense of exponential convergence) in the case of the Fisher's equation [17, 35, 4, 16].

In the multidimensional setting, these kinds of results were subsequently obtained for initial boundary value problems for equations in infinite cylindrical domains:

$$(1.3) \qquad u_t = \Delta u + f(u, y), \qquad u(x, 0) = u_0(x),$$

where $u : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$, $\Sigma = \Omega \times \mathbb{R} \subset \mathbb{R}^n$, $\Omega \subset \mathbb{R}^{n-1}$ is a bounded domain with sufficiently smooth boundary and $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is a nonlinear reaction term with either Neumann or Dirichlet boundary conditions. By $x = (y, z) \in \Sigma$, we always denote a point with coordinate $y \in \Omega$ on the cylinder cross section and $z \in \mathbb{R}$ along the cylinder axis. More generally, one can consider either Dirichlet or Neumann boundary conditions on different connected portions of $\partial\Omega$:

$$(1.4) \qquad u\big|_{\partial\Sigma_\pm} = 0, \qquad \nu \cdot \nabla u\big|_{\partial\Sigma_0} = 0,$$

where $\partial\Sigma_\pm = \partial\Omega_\pm \times \mathbb{R}$ and $\partial\Sigma_0 = \partial\Omega_0 \times \mathbb{R}$, allowing for more than one connected component for $\partial\Omega$. (For motivation and further discussion of the boundary conditions, see [24, 25].) Note that transverse advection by a potential flow can also be straightforwardly included in the present treatment, as was done in [24, 25]. For simplicity of presentation, in this paper we do not consider the advection term and we concentrate on pure reaction-diffusion problems.

Without loss of generality, we may assume that $u = 0$ is a trivial solution of (1.3) and consider traveling waves that invade the $u = 0$ equilibrium, i.e., the solutions of (1.3) and (1.4) in the form $u(x, t) = \bar{u}(y, z - ct)$, for some $c > 0$, which converge to zero uniformly as $z \to +\infty$. These solutions satisfy the elliptic equation

$$(1.5) \qquad \Delta\bar{u} + c\bar{u}_z + f(\bar{u}, y) = 0,$$

together with the respective boundary conditions in (1.4). (By a solution, we mean a pair $(c, \bar{u})$, with $\bar{u} \in C^2(\Sigma) \cap C^1(\overline{\Sigma})$ being a classical solution of (1.5) and (1.4).) We refer to [3, 38, 24] and references therein for a comprehensive treatment of the subject of traveling waves. In particular, under certain specific assumptions one obtains uniqueness (up to translations) and global exponential convergence to these solutions for the initial value problem with front-like initial data [22, 29, 30]. (See the end of section 2 for a more detailed discussion and a comparison with the present results.) This property, therefore, indicates the ubiquitous role of the traveling fronts in the behavior of the solutions of (1.3).

Since in general (1.5) may have many solutions, an important question is which of these solutions, if any, can be a long-time limit of the evolution governed by (1.3) for a given class of initial data. As was recently pointed out in [23], in the case of initial data with sufficiently fast exponential decay at $z = +\infty$ the relevant class of traveling wave solutions consists of the so-called variational traveling waves, even for systems of reaction-diffusion equations in which the nonlinearity is a gradient. More recently, we showed that a special class of variational traveling wave solutions that minimize the exponentially weighted Ginzburg–Landau functional (see section 2 for precise definitions and statements) are relevant for the long-time behavior of the initial value problem in the sense of propagation of the leading edge and, in particular, determine the propagation speed for front-like initial data [24]. It is then natural to

ask whether these special traveling fronts are also the long-time attractors for the solutions of (1.3) in the moving reference frame. In this paper, we give a positive answer to this question under a few extra nondegeneracy assumptions to those of [24] which hold generically in the considered class of problems.

Our paper is organized as follows. In section 2, we introduce the variational formulation for the traveling waves of interest, state the main result, and compare it with those available in the literature. In section 3, we list and discuss our assumptions, as well as state a number of auxiliary results used in the paper. In section 4, we perform local stability analysis of the traveling waves of interest in the exponentially weighted Sobolev spaces, and in section 5 we prove convergence to the traveling wave in the large, completing the proof of the main theorem.

**Some notation.** For every $-\infty \leq a < b \leq +\infty$ and $c > 0$, the symbol $L_c^2(\Omega \times (a,b))$ denotes the Hilbert space of all functions $u : \Omega \times (a,b) \to \mathbb{R}$ with $||u||_{L_c^2(\Omega \times (a,b))}^2 = \int_a^b \int_\Omega e^{cz} u^2(y,z) \, dy \, dz$. Likewise, by $L_c^2(\Sigma)$, $H_c^1(\Sigma)$, and $H_c^2(\Sigma)$, we denote the spaces of functions which are square integrable with the above exponential weight, together with their first and second derivatives, respectively, in $\Sigma$. We also use the symbol $C_b(A)$ to denote the space of bounded continuous function on $A$ equipped with the sup-norm. In all statements and proofs the constants are always assumed to implicitly depend on $f$, $\Omega$ and the choice of the boundary conditions. In the proofs the numbers $C, M$, etc., may change from line to line. We will also use the symbol $T_R$ to denote a translation by $R$ along the $z$-axis, i.e., $T_R u(\cdot, z) = u(\cdot, z - R)$.

**2. Variational formulation and main result.** The fact that (1.5) possesses a variational structure in exponentially weighted Sobolev spaces was, to our knowledge, first pointed out by Heinze [11, 12] (see also [7, 38, 29, 19, 9, 24] in the context of (1.3), and [23, 20, 27, 8] in the context of its extensions). As we recently showed in [24], for scalar reaction-diffusion equations considered here the solution of (1.5) which determines the asymptotic speed of propagation with front-like initial data is a special variational traveling wave which is the minimizer of the the exponentially weighted Ginzburg–Landau functional

$$(2.1) \qquad \Phi_c[u] := \int_\Sigma e^{cz} \left( \frac{1}{2} |\nabla u|^2 + V(u,y) \right) dx, \qquad c > 0,$$

where

$$(2.2) \qquad V(u,y) = -\int_0^u f(s,y) \chi_{[0,1]}(s) \, ds, \qquad \chi_{[0,1]}(s) = \begin{cases} 1, & s \in [0,1], \\ 0, & s \notin [0,1], \end{cases}$$

over all functions lying in the exponentially weighted Sobolev space $H_c^1(\Sigma)$. We point out that such a minimizer can exist only for a specific value of $c = c^\dagger > 0$ (see Theorem 2). Under quite general assumptions on the potential $V$, in [24, Theorem 5.8] we proved that the asymptotic speed of propagation of solutions to (1.3) is precisely given by $c^\dagger$, assuming that the initial datum is front-like, i.e., if it stays sufficiently far away from zero as $z \to -\infty$ and decays sufficiently fast to zero as $z \to +\infty$. In this paper, we discuss the local and global stability of such variational traveling waves.

Our main result is contained in the following theorem (for the details of the definitions and hypotheses, see section 3).

THEOREM 1. *Assume hypotheses* (H1)–(H3) *and* (N1)–(N2) *are satisfied, and let* $c^\dagger$, $\bar{u}$, $v$ *be as in Theorem 2. Then there exist* $\alpha > 0$ *and* $\sigma > 0$, *such that if*

$u_0 \in C^0(\overline{\Sigma}) \cap W^{1,\infty}(\Sigma) \cap L^2_{c^\dagger}(\Sigma)$ *satisfies* $0 \le u_0 \le 1$ *and*

$$(2.3) \qquad \liminf_{z \to -\infty} u_0(\cdot, z) \ge v - \alpha \qquad \text{uniformly in } \Omega\,,$$

*there exists* $R_\infty \in \mathbb{R}$, *such that if* $u$ *is the solution of* (1.3) *and* (1.4) *with initial datum* $u_0$, *then*

$$(2.4) \qquad \|\mathrm{T}_{R_\infty - c^\dagger t} u(\cdot, t) - \bar{u}\|_{H^2_{c^\dagger}(\Sigma)} \le C e^{-\sigma t}$$

*for every* $t \ge t_0$ *with arbitrary* $t_0 > 0$ *and some* $C > 0$ *independent of* $t$.

Note that by Proposition 3.1 we know that $u(\cdot, t)$ is bounded in $W^{2,p}(\Omega \times [M, M+1])$ uniformly in $M \in \mathbb{R}$ and $t \in [t_0, +\infty)$ for all $t_0 > 0$ and $p < \infty$. Since this bound also applies to $\bar{u}$, from (2.4) we get the following.

COROLLARY 2.1. *In the statement of Theorem* 1, *the inequality* (2.4) *may be replaced with*

$$(2.5) \qquad \|\mathrm{T}_{R_\infty - c^\dagger t} u(\cdot, t) - \bar{u}\|_{C^1(\overline{\Omega} \times [z_0, z_1])} \le C e^{-\sigma t}$$

*for all* $z_0 < z_1$, $t \ge t_0 > 0$ *and some* $C > 0$ *independent of* $t$ *and* $z_1$.

Let us point out that the upper bound $u_0 \le 1$ in Theorem 1 can be replaced with the condition $u_0(\cdot, z) \le \bar{v}$ for every $z \in \mathbb{R}$, where $\bar{v} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$(2.6) \qquad \bar{v} > 0, \quad \Delta_y \bar{v} + f(\bar{v}, y) \le 0 \qquad \forall\, y \in \Omega,$$

together with the boundary conditions from (1.4). In this case, the condition $f(1, y) \le 0$ in assumption (H1) below should be replaced by (2.6), the conditions in (H2) should hold for $0 \le u \le \bar{v}$, and the definition of $V$ in (2.2) should be modified accordingly. We note that, in particular, one can choose $\bar{v}$ to be any positive critical point of the energy functional $E$ associated with $\Phi_c$:

$$(2.7) \qquad E[v] := \int_\Omega \left( \frac{1}{2} |\nabla_y v|^2 + V(v, y) \right) dy, \qquad v \in H^1(\Omega), \quad v\big|_{\partial \Omega_\pm} = 0.$$

To each such $\bar{v}$ one can associate a minimizer of $\Phi_c$ in the admissible class of functions that are bounded above by $\bar{v}$. Then, under the assumption that the initial datum approaches $\bar{v}$ uniformly from below as $z \to -\infty$ one can make the conclusion (under generic nondegeneracy assumptions) that the solution of the initial value problem converges exponentially to the corresponding minimizer. Thus, all front-like initial data in a more restricted sense of connecting zero to a critical point $\bar{v}$ of $E$ converges to the minimizer associated with that critical point. More precisely, we have the next corollary.

COROLLARY 2.2. *Under hypotheses* (H1)–(H3), (N1)–(N2), *with the trial function* $u$ *in hypothesis* (H3) *satisfying* $u \le \bar{v}$, *where* $\bar{v} > 0$ *is a critical point of* $E$, *let* $\bar{u}$ *be the unique (up to translations) nontrivial minimizer of* $\Phi_{c^\dagger}$ *over functions* $u \in H^1_{c^\dagger}(\Sigma)$ *satisfying* (1.4) *and* $0 \le u \le \bar{v}$. *Let* $u_0 \in C^0(\overline{\Sigma}) \cap W^{1,\infty}(\Sigma) \cap L^2_{c^\dagger}(\Sigma)$ *satisfy* $0 \le u_0 \le \bar{v}$ *and* $u_0(\cdot, z) \to \bar{v}$ *uniformly in* $\Omega$ *as* $z \to -\infty$. *Then the conclusion of Theorem* 1 *holds.*

An important implication of Corollary 2.2 is that $\bar{v}$ *selects* the attracting variational traveling wave solution in the long time limit. This kind of conclusion was made by us earlier for the propagation speed of the leading edge without the nondegeneracy assumptions of the present paper [24].

We note that the problem of convergence to traveling waves for solutions of (1.3) has been widely considered in the mathematical literature. We refer to [6, 38, 30] and references therein for a general overview on the subject. Specifically, our result should be compared with [30, Theorem 3.7] by Roquejoffre, where, in particular, convergence to variational traveling waves is proved (in our notation) for initial data that approach zero from above as $z \to +\infty$ and a nondegenerate local minimizer $\bar{v} > 0$ of $E$ from below as $z \to -\infty$.

Roquejoffre makes a crucial assumption that there exists a variational traveling wave connecting $\bar{v}$ at $z = -\infty$ with zero at $z = +\infty$. In contrast, our results do not require existence of such a traveling wave. Instead, we require that the initial data decay sufficiently rapidly to zero as $z \to +\infty$ and stay approximately above the local minimizer $v$ of $E$ corresponding to the limit at $z = -\infty$ for the special variational traveling wave $\bar{u}$ given by Theorem 2 as $z \to -\infty$. Under this condition the solution of (1.3) is attracted to a translate of $\bar{u}$ on compacts in the moving reference frame (see Theorem 1 for a precise statement). We note that in the class of frontlike initial data with sufficiently fast exponential decay considered by us global stability of a traveling wave connecting zero to $\bar{v}$ is a simple consequence of Corollary 2.2. Indeed, if there exists a variational traveling wave $u_c$ connecting zero to $\bar{v}$, then by Proposition 3.3 we have $u_c = \bar{u}$, where $\bar{u}$ is as in Corollary 2.2. (Note that in this case hypotheses (H3) and (N2) are unnecessary.) Thus, within the scope of (1.3) and frontlike initial data decaying sufficiently fast, our results are applicable to more general initial data than the ones considered in [30] and, most important, provide a selection criterion for the limit front in terms of the asymptotic behavior of the initial data as $z \to -\infty$. We also point out that our assumptions concerning the nonlinearity $f$ (see (H1)–(H3) below) are quite general compared to the assumptions usually made in the literature [3, 36, 30]. In particular, these assumptions can be readily verified in practice (for examples see [19, 20, 25]).

**3. Preliminaries.** Throughout this paper we assume $\Omega$ to be a bounded domain (connected open set, not necessarily simply connected) with the boundary of class $C^2$. We start by listing the assumptions on the nonlinearity $f$ which we need in Theorem 1. The function $f : [0, 1] \times \overline{\Omega} \to \mathbb{R}$ satisfies

(**H1**)  $f(0, y) = 0$      $f(1, y) \leq 0$      $\forall y \in \Omega$ and

(**H2**)  $f \in C^{0,\gamma}([0, 1] \times \overline{\Omega})$,      $f_u := \dfrac{\partial f}{\partial u} \in C^{0,\gamma}([0, 1] \times \overline{\Omega})$ for some $\gamma \in (0, 1)$.

Hypotheses (H1) and (H2) are needed to guarantee, in particular, existence and basic regularity properties of solutions of (1.3). Indeed, from [24, Proposition 5.1] and [21, Chapter 7] we have the following.

PROPOSITION 3.1. *Under assumptions* (H1) *and* (H2), *let* $u_0 \in C^0(\overline{\Sigma}) \cap W^{1,\infty}(\Sigma)$. *Let also* $u_0$ *satisfy the boundary conditions* (1.4) *and assume* $u_0(x) \in [0, 1]$ *for all* $x \in \Sigma$. *Then there exists a unique solution (using notation of* [5])

$$u \in C_1^2(\Sigma \times (0, \infty)) \cap C^0(\overline{\Sigma} \times [0, +\infty))$$

*of* (1.3) *with boundary conditions* (1.4) *and initial condition* $u(\cdot, 0) = u_0$, *which satisfies* $0 \leq u \leq 1$ *and* $\|\nabla u\|_{C_b(\overline{\Sigma} \times (0, +\infty))} < \infty$. *Moreover, letting* $\Sigma_M := \Omega \times [M, M + 1]$ *for all* $M \in \mathbb{R}$, *we have*

(3.1)          $\|u(\cdot, t)\|_{W^{2,p}(\Sigma_M)} \leq C(t_0, p)$      $\forall t \geq t_0 > 0,\ p > 1.$

*Finally, if $u_0 \in L_c^2(\Sigma)$ for some $c > 0$, we also have*

$$(3.2) \qquad u \in C^\alpha((0, +\infty); H_c^2) \cap C^{1,\alpha}((0, +\infty); L_c^2(\Sigma)) \qquad \forall \alpha \in (0, 1)$$

*and*

$$(3.3) \qquad\qquad u_t \in C((0, +\infty); H_c^1(\Sigma)).$$

We now turn to the assumption which is both necessary and sufficient for the existence of the special variational traveling wave solution considered in this paper [19, 24].

**(H3)** There exist $c > 0$, satisfying $c^2 + 4\nu_0 > 0$, where

$$(3.4) \qquad \nu_0 = \min_{\substack{\psi \in H^1(\Omega) \\ \psi|_{\partial\Omega_\pm} = 0}} \frac{\int_\Omega (|\nabla_y \psi|^2 - f_u(0, y)\psi^2)\, dy}{\int_\Omega \psi^2\, dy}$$

and $u \in H_c^1(\Sigma)$, such that $\Phi_c[u] \leq 0$ and $u \not\equiv 0$.

*Remark* 3.2. As was shown in [24], in the case $\nu_0 \geq 0$ the hypothesis (H3) is equivalent to the condition

$$(3.5) \qquad\qquad \inf_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega_\pm} = 0}} E[v] < 0.$$

Under the above assumptions, we can state the existence result concerning the variational traveling wave which is the minimizer of $\Phi_c$ with a suitably fixed translation.

THEOREM 2. *Under hypotheses* (H1)–(H3), *there exists a unique value of $c^\dagger \geq c$, where $c$ is defined by hypothesis* (H3), *and a unique function $\bar{u} \in C^2(\Sigma) \cap C^1(\overline{\Sigma})$, $\bar{u} \not\equiv 0$, such that $(c^\dagger, \bar{u})$ solve* (1.5) *and* (1.4), *and $\bar{u}$ satisfies $\|\bar{u}(\cdot, 0)\|_{L^\infty(\Omega)} = \frac{1}{2} \sup_{z \in \mathbb{R}} \|\bar{u}(\cdot, z)\|_{L^\infty(\Omega)}$ and minimizes $\Phi_{c^\dagger}$ in $H_{c^\dagger}^1(\Sigma)$. Moreover $\bar{u} \in H_{c^\dagger}^2(\Sigma) \cap W^{1,\infty}(\Sigma)$, $\bar{u}_z \in H_{c^\dagger}^2(\Sigma)$, $\bar{u}_z < 0$ in $\Sigma$, and*

$$(3.6) \qquad \lim_{z \to +\infty} \bar{u}(\cdot, z) = 0, \qquad \lim_{z \to -\infty} \bar{u}(\cdot, z) = v \qquad \text{in } C^1(\overline{\Omega}),$$

*where $v : \Omega \to \mathbb{R}$ is a local minimizer of $E$ defined in* (2.7) *with $E[v] < 0$.*

For the proof see [24, Theorem 3.3] and [20, Proposition 3.3(ii)]. (The latter argument also applies to $\bar{u}_z$ by differentiating (1.5) in $z$.)

Let us point out that the minimizer of Theorem 2 is in some sense the "maximal" variational traveling wave solution. More precisely, we have the following result.

PROPOSITION 3.3. *Let hypotheses* (H1)–(H3) *be satisfied, and let $(c, u)$ solve* (1.5) *and* (1.4) *with $c > 0$, $u \in H_c^1(\Sigma)$, and $0 < u < 1$. Then, if $v$, $c^\dagger$, $\bar{u}$ are as in Theorem* 2, *and*

$$(3.7) \qquad \liminf_{z \to -\infty} u(\cdot, z) \geq v \qquad \text{uniformly in } \Omega,$$

*we have $c = c^\dagger$ and $u = T_R \bar{u}$ for some $R \in \mathbb{R}$. In particular, the inequality in* (3.7) *is, in fact, equality.*

*Proof.* First note that we cannot have $c > c^\dagger$. Indeed, if this inequality were true, by [20, Proposition 3.5] the pair $(c, u)$ can be taken as a trial function in hypothesis (H3), contradicting the conclusion of Theorem 2 that $c^\dagger \geq c$. On the other hand, it

is easy to see that $c < c^\dagger$ is also impossible. Indeed, arguing as in the proof of [24, Proposition 5.5], for any $c' \in (c, c^\dagger)$ there exists a nontrivial minimizer $\bar{u}_{c'}$ of $\Phi_{c'}$ in the class of functions in $H^1_{c'}(\Sigma)$ which stay below $v$ and vanish outside $\Sigma_R = \Omega_\sigma \times (-R, R)$, with $\Omega_\sigma = \{y \in \Omega : \text{dist}(y, \partial\Omega_\pm) > \sigma\}$, where $R > 0$ is large enough and $\sigma > 0$ is small enough.[1] Furthermore, $\bar{u}_{c'}$ is a classical solution of (1.5) with $c = c'$ in $\Sigma_R$ and $\bar{u}_{c'} \leq \max(0, v - \varepsilon)$ for some $\varepsilon > 0$. Therefore, by (3.7) the function $\mathrm{T}_{R'}\bar{u}_{c'} < u$ in $\Sigma$ for some $R' \in \mathbb{R}$ sufficiently large negative, and by parabolic comparison principle [26] we have $\bar{u}_{c'} < \mathrm{T}_{(c-c')t-R'}u$ for all $t > 0$. However, the latter is impossible, since the right-hand side of this inequality converges to zero in $H^1_c(\Sigma)$. Thus $c = c^\dagger$, and hence $u$ is a minimizer by [20, Proposition 3.5], and the result follows from [24, Theorem 3.3(v)].  □

We note that, in particular, the result in Proposition 3.3 allows us to extend the statements about monotonicity and uniqueness of traveling waves established in the classical work of Berestycki and Nirenberg [3] for (1.3) and (1.4) (see also [36, 37]), in the class of variational traveling waves, under only an assumption that the traveling wave approaches a limit from below as $z \to -\infty$ and zero from above as $z \to +\infty$ and is sandwiched between these two limits. Indeed, suppose $(c, u)$ is such a traveling wave with $u(\cdot, z) \to \bar{v}$ as $z \to -\infty$ with $0 < \bar{v} \leq 1$. Then by the argument of [20, Proposition 6.6] $v$ is a critical point of $E$, and by [24, Proposition 3.5] we have $c^2 + 4\nu_0 > 0$. So by Proposition 3.3 this traveling wave is a nontrivial minimizer of $\Phi_c$ in $H^1_c(\Sigma)$ over all positive functions bounded above by $\bar{v}$, and the result follows from [24, Theorem 3.3]. In particular, we do not require any nondegeneracy assumptions for the limits of $u(\cdot, z)$ as $z \to \pm\infty$, as is done in [3]. Thus, we have the next corolllary.

COROLLARY 3.4. *Under hypotheses* (H1) *and* (H2), *let* $c > 0$, *and let* $u \in H^1_c(\Sigma)$ *be a solution of* (1.5) *and* (1.4) *satisfying* $u(\cdot, z) \to \bar{v}$ *uniformly in* $\Omega$ *as* $z \to -\infty$, *where* $0 < \bar{v} \leq 1$ *and* $0 < u < \bar{v}$. *Then* $c^2 + 4\nu_0 > 0$, *the value of* $c$ *is unique*, $u_z < 0$, *and* $u$ *is unique up to translations.*

We now list two additional technical assumptions (see also [30]), which are generically satisfied and are needed to prove global exponential stability of the minimizers of $\Phi_c$ for initial data bounded below by $v$ as $z \to -\infty$.

**(N1)** For $v$ as in Theorem 2 we have

$$(3.8) \qquad \tilde{\nu}_0 = \min_{\substack{\psi \in H^1(\Omega) \\ \psi|_{\partial\Omega_\pm} = 0}} \frac{\int_\Omega (|\nabla_y \psi|^2 - f_u(v, y)\psi^2)\, dy}{\int_\Omega \psi^2\, dy} > 0.$$

**(N2)** For $v < 1$ as in Theorem 2 there is no solution $(c^\dagger, \bar{u})$ of (1.5) and (1.4) with $c^\dagger$ as in Theorem 2, such that $v < \bar{u} < 1$ on $\Sigma$.

Conditions (N1) and (N2) are generic in the sense that the set of nonlinearities $f$ such that (N1) or (N2) do not hold is a meager subset of all $f$'s obeying (H1)–(H3) in the natural topology. (For similar notions related to perturbations of $\Omega$ see [13].) Indeed, condition (N1) is generic, since by the results of [24] we have $\tilde{\nu}_0 \geq 0$, so that (N1) only excludes the degenerate case of $\tilde{\nu}_0 = 0$. Similarly, condition (N2) excludes the nongeneric possibility of existence of a traveling front invading $v$ from above with the same speed $c^\dagger$ as the front invading zero by $v$. To see that the only nontrivial alternative would be to have a front invading $v$ with lower speed, consider the following variational problem. Given $c > 0$ and $h \in H^1_c(\Sigma)$ satisfying (1.4), let

$$(3.9) \qquad \Psi^v_c[h] := \int_\Sigma e^{cz}\left(\frac{|\nabla h|^2}{2} + V(v + h, y) - V(v, y) - V'(v, y)\, h\right) dx,$$

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[1]This choice of $\Sigma_R$ also corrects a minor inaccuracy in the proof of [24, Proposition 5.5].

where we used the notation $V'(s, y) := \partial V(s, y)/\partial s$. Notice that if $\bar{h}$ is a critical point of $\Psi^v_{c^\dagger}$, then $\bar{u} = v + \bar{h}$ is a solution of (1.5) and (1.4). We set

$$(3.10) \qquad c^\dagger_v := \inf \left\{ c > 0 : \ \Psi^v_c[h] \geq 0 \ \forall h \geq 0 \right\}.$$

Then the following result concerning $c^\dagger_v$ holds.

LEMMA 3.5. *The functional $\Psi^v_c$ is weakly sequentially lower semicontinuous and coercive in $H^1_c(\Sigma)$ for all $c > c^\dagger_v$, and $c^\dagger_v \leq c^\dagger$, where $c^\dagger$ is as in Theorem* 2. *Moreover, under hypothesis* (N2) *we have $c^\dagger_v < c^\dagger$.*

In other words, under hypothesis (N2) it is only possible to have such a system of stacked waves [38] invading zero that the front connecting zero with $v$ moves faster than the front invading $v$ from above.

*Proof of Lemma* 3.5. First, reasoning as in [20, Proposition 5.5] and using the fact that $\tilde{\nu}_0 \geq 0$, where $\tilde{\nu}_0$ is defined in (3.8) [24, Theorem 3.3(iv)], one can see that $\Psi^v_c$ is weakly lower semicontinuous in $H^1_c(\Sigma)$ for all $c > 0$, so the results of [24] apply to $\Psi^v_c$. Moreover, reasoning as in the proof of [20, Proposition 6.9], we also get that $\Psi^v_c$ is coercive in $H^1_c(\Sigma)$ for all $c > c^\dagger_v$.

Let us now prove that $c^\dagger_v \leq c^\dagger$. Assume by contradiction that there exists $w \geq v$ such that $\Psi^v_{c^\dagger}[w - v] < 0$. Slightly perturbing $w$, we can ensure that $w = v$ for $z \geq z_0$, with $z_0 \in \mathbb{R}$ big enough. Let $\bar{u}$ be the minimizer of $\Phi_{c^\dagger}$ given by Theorem 2, and let $\varepsilon \leq -\Psi^v_{c^\dagger}[w - v]/2$. Since $\bar{u}(\cdot, z) \to v$ in $H^1(\Omega)$ as $z \to -\infty$, up to a suitable translation we can perturb $\bar{u}$ into a function $\tilde{u} \in H^1_{c^\dagger}(\Sigma)$ such that $\tilde{u} = v$ for $z \leq z_0$ and $\Phi_{c^\dagger}[\tilde{u}] \leq \varepsilon$. Define $\hat{u} \in H^1_{c^\dagger}(\Sigma)$ as

$$\hat{u}(y, z) := \begin{cases} w(y, z) & \text{if } z \leq z_0, \\ \tilde{u}(y, z) & \text{if } z > z_0. \end{cases}$$

Letting $h = w - v \in H^1_{c^\dagger}(\Sigma)$ and satisfying (1.4), after an integration by parts and using the Euler–Lagrange equation for $E$ satisfied by $v$, we get

$$\begin{aligned} \Phi_{c^\dagger}[\hat{u}] &= \int_{-\infty}^{z_0} \int_\Omega e^{c^\dagger z} \left( \frac{|\nabla(v + h)|^2}{2} + V(v + h, y) \right) dy\, dz \\ &\quad + \int_{z_0}^{+\infty} \int_\Omega e^{c^\dagger z} \left( \frac{|\nabla \tilde{u}|^2}{2} + V(\tilde{u}, y) \right) dy\, dz \\ &= \int_{-\infty}^{z_0} \int_\Omega e^{c^\dagger z} \left( \frac{|\nabla h|^2}{2} + V(v + h, y) - V(v, y) - V'(v, y)h \right) dy\, dz \\ &\quad + \int_{-\infty}^{z_0} \int_\Omega e^{c^\dagger z} \left( \frac{|\nabla_y v|^2}{2} + V(v, y) \right) dy\, dz \\ &\quad + \int_{z_0}^{+\infty} \int_\Omega e^{c^\dagger z} \left( \frac{|\nabla \tilde{u}|^2}{2} + V(\tilde{u}, y) \right) dy\, dz \\ &= \Psi^v_{c^\dagger}[h] + \Phi_{c^\dagger}[\tilde{u}] \leq \frac{\Psi^v_{c^\dagger}[h]}{2} < 0, \end{aligned}$$

which contradicts the minimizing property $\Phi_{c^\dagger}[\bar{u}] = 0$ of $\bar{u}$ [20, Proposition 3.2].

To conclude the proof, it remains to prove that $c^\dagger_v < c^\dagger$ under hypothesis (N2). If $c^\dagger_v = c^\dagger$, then for every $c \in (0, c^\dagger)$, there exists a function $h_c \not\equiv 0$ such that $\Psi^v_c[h_c] < 0$. Hence, the analog of hypothesis (H3) holds for $\Psi^v_c$, and, therefore, there exists a nontrivial minimizer $\bar{h}$ of $\Psi^v_{\hat{c}}$ for some $\hat{c} \geq c^\dagger$. On the other hand, by the argument of

[24, Proposition 5.5], we have $\hat{c} \leq c_v^\dagger$. So $\hat{c} = c^\dagger$, and since $\hat{u} = v + \bar{h} > v$ is a solution of (1.5) and (1.4) with $c = \hat{c}$, this violates assumption (N2). $\square$

Finally, we note that if either (N1) or (N2) is violated, one would no longer expect exponential stability of $\bar{u}$ in the reference frame moving with speed $c^\dagger$. Therefore, in some sense these conditions are also necessary for the results obtained by us.

**4. Local stability in $L_{c^\dagger}^2(\Sigma)$.** In this section we prove stability of the variational traveling wave $\bar{u}$ minimizing $\Phi_{c^\dagger}$ in the reference frame moving with speed $c^\dagger$ up to perturbations which are small in the $L_{c^\dagger}^2$-norm and stay approximately above $v$ behind the front.

THEOREM 3. *Assume hypotheses* (H1)–(H3) *and* (N1)–(N2) *hold, and let $\bar{u}$ and $c^\dagger$ be as in Theorem 2. Then there exist $\alpha > 0$ and $\sigma > 0$ such that for every $u_0$ as in Theorem 1 and for every $\omega > 0$ there exists $\varepsilon > 0$ such that if*

$$(4.1) \qquad ||u_0 - \bar{u}||_{L_{c^\dagger}^2(\Sigma)} \leq \varepsilon,$$

*the solution $u(x,t)$ of*

$$(4.2) \qquad u_t = \Delta u + c^\dagger u_z + f(u, y)$$

*with boundary conditions in* (1.4) *and $u(x, 0) = u_0(x)$ satisfies*

$$(4.3) \qquad ||u(\cdot, t) - \mathrm{T}_{R_\infty} \bar{u}||_{L_{c^\dagger}^2(\Sigma)} \leq \omega e^{-\sigma t}, \qquad |R_\infty| \leq \omega,$$

*for some $R_\infty \in \mathbb{R}$.*

We note that our approach differs somewhat from the conventional approach to the studies of front stability [32, 2, 28, 29] in the way we treat translations along the cylinder axis. We track the front position by minimizing the $L_{c^\dagger}^2$-distance between the solution of (4.2) and a translate of $\bar{u}$. As a consequence, the deviation between the solution and the closest translate of $\bar{u}$ is automatically orthogonal to the null-space of the linearization operator, allowing us to readily establish the exponential decay of the $L_{c^\dagger}^2$-distance. Thus, our method is more variational. Let us also point out that, in contrast to the usual approach, our initial data do not need to be close to $\bar{u}$ in $L^\infty$ in the whole cylinder; they may be significantly larger than $\bar{u}$ at large negative $z$.

Throughout the rest of this section, hypotheses (H1)–(H3) and (N1)–(N2) are assumed to hold, and $c^\dagger$, $\bar{u}$, $v$ always refer to the minimizer in Theorem 2. We begin with the following basic lemma concerning the linearization around $\bar{u}$.

LEMMA 4.1. *There exists $K > 0$ such that*

$$(4.4) \qquad \int_\Sigma e^{c^\dagger z} \left( |\nabla w|^2 - f_u(\bar{u}, y) w^2 \right) dx \geq K \int_\Sigma e^{c^\dagger z} w^2 dx$$

*for all $w \in H_{c^\dagger}^1(\Sigma)$ satisfying $\int_\Sigma e^{c^\dagger z} w \bar{u}_z dx = 0$.*

*Proof.* First, observe that by choosing $R_1$ and $R_2$ sufficiently large, we have

$$(4.5) \qquad \frac{\int_{-\infty}^{-R_1} \int_\Omega e^{c^\dagger z} \left( |\nabla w|^2 - f_u(\bar{u}, y) w^2 \right) dy\, dz}{\int_{-\infty}^{-R_1} \int_\Omega e^{c^\dagger z} w^2\, dy\, dz} \geq K_1 > 0,$$

$$(4.6) \qquad \frac{\int_{R_2}^{+\infty} \int_\Omega e^{c^\dagger z} \left( |\nabla w|^2 - f_u(\bar{u}, y) w^2 \right) dy\, dz}{\int_{R_2}^{+\infty} \int_\Omega e^{c^\dagger z} w^2\, dy\, dz} \geq K_2 > 0$$

for all $w \in H^1_{c^\dagger}(\Sigma)$. Indeed, if $z$ is large enough negative, then $\bar{u}(\cdot, z)$ is sufficiently close in $L^\infty(\Omega)$ to $v$. Hence, (4.5) holds in view of (3.8). On the other hand, by the estimate of [20, Lemma 5.1], we have

$$\int_{R_2}^{+\infty} \int_\Omega e^{c^\dagger z} \left( |\nabla w|^2 - f_u(0, y)w^2 \right) dy \, dz$$

$$\geq \int_{R_2}^{+\infty} \int_\Omega e^{c^\dagger z} \left( \frac{c^{\dagger 2}}{4} w^2 + |\nabla_y w|^2 + f_u(0, y)w^2 \right) dy \, dz$$

$$\geq \left( \frac{c^{\dagger 2}}{4} + \nu_0 \right) \int_{R_2}^{+\infty} \int_\Omega e^{c^\dagger z} w^2 \, dy \, dz.$$

So, by hypothesis (H3) and (3.6), the inequality in (4.6) holds for some $K_2 > 0$ and $R_2$ large enough.

Let us now show that the inequality in (4.4) holds with $K = 0$ for all $w \in H^1_{c^\dagger}(\Sigma)$ and that equality holds if and only if $w$ is a multiple of $\bar{u}_z$. (The proof essentially follows the ideas of concentration compactness principle in the case of exponentially weighted Sobolev spaces [18, 33] and relies on the maximum principle.) Indeed, denote by $H[w]$ the left-hand side of (4.4) and let $(w_n)$ be a minimizing sequence for $H$ subject to the constraint $||w_n||_{L^2_{c^\dagger}(\Sigma)} = 1$. By coercivity of $H$ on the constraint, ensured by hypothesis (H2), we have $w_n \rightharpoonup w_0$ in $H^1_{c^\dagger}(\Sigma)$. In fact, $w_0 \neq 0$, since otherwise $w_n \to 0$ in $L^2_{\mathrm{loc}}(\Sigma)$, and so $\int_{-\infty}^{-R_1} \int_\Omega e^{c^\dagger z} w_n^2 dx + \int_{R_2}^{+\infty} \int_\Omega e^{c^\dagger z} w_n^2 dx \geq 1 - \varepsilon$ for any $\varepsilon > 0$ and large enough $n$. Therefore, by (4.5) and (4.6) we would have $H[w_n] \geq K > 0$. However, this contradicts the fact (first pointed out in [1]) that $\bar{u}_z$ is an eigenfunction associated with zero eigenvalue of the linearization of (1.5) around $\bar{u}$ (related to the translational symmetry in the $z$-direction [1, 32, 2, 28]), which can be seen by differentiating (1.5) with respect to $z$ and noting that $\bar{u}_z \in H^2_{c^\dagger}(\Sigma)$ by Theorem 2.

In view of lower semicontinuity of $H$ with respect to the weak convergence in $H^1_{c^\dagger}(\Sigma)$, which follows from [20, Proposition 5.5], hypothesis (H3), and Theorem 2, we have $H[w_0] \leq \liminf_{n\to\infty} H[w_n] \leq ||\bar{u}_z||^{-2}_{L^2_{c^\dagger}(\Sigma)} H[\bar{u}_z] = 0$. Then, since $w_0 \neq 0$, the function $\bar{w} = ||w_0||^{-1}_{L^2_{c^\dagger}(\Sigma)} |w_0| \geq 0$ is a minimizer of the considered constrained minimization problem. In fact, $H[\bar{w}] = 0$, since otherwise $\bar{w}$ must be orthogonal to $\bar{u}_z$, which is impossible because $\bar{u}_z < 0$ by Theorem 2. So $H[w] \geq 0$ for all $w \in H^1_{c^\dagger}(\Sigma)$. Moreover, $H[w] = 0$ implies that $w$ is a multiple of $\bar{u}_z$ (compare also with [2, 28, 30]). If not, there exists a minimizer $w'$ which is orthogonal to $\bar{u}_z$ in $L^2_{c^\dagger}(\Sigma)$ and, therefore, changes sign. But $|w'|$ is also a minimizer, hence both $w'$ and $|w'|$ satisfy the linearized version of (1.5) in the classical sense, thanks to hypothesis (H2) and Theorem 2. So by strong maximum principle $|w'| = 0$, leading to a contradiction.

To complete the proof of the lemma, suppose, to the contrary of its statement, there exists a sequence $(w_n)$ with the properties that $||w_n||_{L^2_{c^\dagger}(\Sigma)} = 1$, $\int_\Sigma e^{c^\dagger z} \bar{u}_z w_n \, dx = 0$ and $H[w_n] \to 0$ as $n \to \infty$. Hence $w_n$ is a minimizing sequence and converges to a nontrivial multiple of $\bar{u}_z$ weakly in $H^1_{c^\dagger}(\Sigma)$. But this contradicts the orthogonality of $w_n$ to $\bar{u}_z$, which is preserved in the limit as $n \to \infty$.  $\square$

Let us note that one may naturally think that the result of Lemma 4.1 may be used to show that the minimizer $\bar{u}$ is, in fact, a strict minimizer of $\Phi_{c^\dagger}$ on a suitable subset of $H^1_{c^\dagger}(\Sigma)$. This, however, proves difficult, since the functional $\Phi_c[u]$ is not a priori twice continuously differentiable in $H^1_c(\Sigma)$. We will get back to this question after Proposition 4.4.

Our next result shows that if a solution to (4.2) with initial datum satisfying (2.3) with $\alpha$ sufficiently small is close enough to a suitable translate of $\bar{u}$ in $L^2_{c^\dagger}(\Sigma)$, then it is also close in $L^\infty$ on some growing portion of $\Sigma$, provided that $\varepsilon$ is small enough. More precisely, let $R : [0, \infty) \to \mathbb{R}$. For a given $\delta > 0$, we define $z_\delta : [0, \infty) \to \overline{\mathbb{R}}$ as

$$(4.7) \qquad z_\delta(t) := \sup \left\{ z \in \mathbb{R} : \|u(\cdot, z, t) - \bar{u}(\cdot, z - R(t))\|_{L^\infty(\Omega)} > \delta \right\} \qquad \forall t \geq 0.$$

Then the following result holds true.

PROPOSITION 4.2. *There exists $b > 0$ such that for every $\delta > 0$ sufficiently small there exist $\alpha = \alpha(\delta) > 0$, $a = a(\delta) > 0$, $\bar{z}_0 = \bar{z}_0(\delta, u_0) \in \mathbb{R}$, and $\eta = \eta(\delta, u_0) > 0$ such that for every $z_0 \leq \bar{z}_0$ there exists $\varepsilon = \varepsilon(\delta, z_0) > 0$ such that for all $T > 0$*

$$(4.8) \qquad z_\delta(t) \leq z_0 + a - bt \qquad \forall t \in [0, T]$$

*whenever*

$$(4.9) \qquad |R(t)| \leq \delta \ \text{ and } \ \|u(\cdot, t) - \mathrm{T}_{R(t)}\bar{u}\|_{L^2_{c^\dagger}(\Sigma)} \leq \eta \qquad \forall t \in [0, T],$$

*where $u_0$, $u$, $\alpha$, $\varepsilon$ are as in Theorem 3.*

*Proof.* By (4.9) and the uniform Lipschitz continuity of $u(\cdot, t)$ in $\Sigma$, reasoning as in the proof of [20, Proposition 3.3(iii)], we have the following $L^\infty$-estimate:

$$(4.10) \qquad \|u(\cdot, t) - \mathrm{T}_{R(t)}\bar{u}\|^{n+2}_{C_b(\overline{\Omega} \times [z_0, +\infty))} \leq C\eta^2 e^{-c^\dagger z_0}$$

for any $z_0 \in \mathbb{R}$, any $t \in [0, T]$, and some $C > 0$ depending on $\|\nabla u\|_{C_b(\overline{\Sigma} \times (0, +\infty))}$ (see Proposition 3.1). On the other hand, by Theorem 2 for any $\alpha > 0$ there exists $\bar{z}_0 \in \mathbb{R}$ such that

$$(4.11) \qquad \|\bar{u}(\cdot, z - R(t)) - v\|_{C^0(\overline{\Omega})} \leq \alpha \qquad \forall z \leq \bar{z}_0 \ \forall t \in [0, T].$$

Recalling (2.3) and possibly reducing $\bar{z}_0$, we can also assume that

$$(4.12) \qquad u_0(\cdot, z) \geq v - 2\alpha \qquad \forall z \leq \bar{z}_0.$$

Now, choosing $\eta > 0$ sufficiently small, the right-hand side of (4.10) can be bounded by $\alpha^{n+2}$ at $z_0 = \bar{z}_0$, so we have

$$(4.13) \qquad \|u(\cdot, t) - \mathrm{T}_{R(t)}\bar{u}\|_{C_b(\overline{\Omega} \times [\bar{z}_0, +\infty))} \leq \alpha \qquad \forall t \in [0, T].$$

Therefore,

$$(4.14) \qquad \|u(\cdot, z, t) - \bar{u}(\cdot, z - R(t))\|_{C^0(\overline{\Omega})} \leq \delta \qquad \forall z \geq \bar{z}_0 \ \forall t \in [0, T]$$

as long as $\alpha \leq \delta$, so that $z_\delta(t) \leq \bar{z}_0$ for all $t \in [0, T]$.

It remains to show that the inequality in (4.14) also holds for $z \in [z_0 + a - bt, \bar{z}_0]$ for some positive $a$ and $b$ for small enough $\alpha$ and $\varepsilon$. We proceed by constructing explicit upper and lower barriers for (4.2) in $\Omega \times (-\infty, \bar{z}_0] \times [0, T]$.

*Subsolution.* First, consider the case of $\partial\Omega_\pm = \varnothing$, i.e., pure Neumann boundary conditions in (1.4). Then, it is straightforward to verify that by hypotheses (H1)–(H2) and (N1) the function $v_\delta^- = v - C\delta\tilde{\psi}_0$, where $\tilde{\psi}_0 > 0$ is an eigenfunction associated with $\tilde{\nu}_0$ in (3.8) and $C = \|\tilde{\psi}_0\|^{-1}_{C^0(\overline{\Omega})}$, is the desired subsolution, provided that $\delta$ is sufficiently small.

The construction is more delicate in the presence of Dirichlet boundary conditions, since we do not wish to put any restrictions on the derivative of the initial data near the Dirichlet portion of the boundary. So, let us now assume that $\partial\Omega_\pm \neq \varnothing$, implying, in particular, that $v < 1$ in $\overline{\Omega}$. We construct a subsolution in the form of a nonnegative local minimizer of $E$ that lies sufficiently close to and below $v$ and vanishes identically within some small distance to $\partial\Omega_\pm$.

We proceed in the usual way by introducing the modified energy $\tilde{E}$, given by (2.7) in which $V$ is replaced by $\tilde{V}(u, y) = -\int_0^u \tilde{f}(s, y)ds$, where $\tilde{f}$ is obtained from $f$ by the odd extension for $u < 0$ and the $C^1$ linear extrapolation for $|u - v| > \delta$ for some fixed $0 < \delta \ll 1$ and each $y \in \Omega$. We note that $\tilde{f}(u, y) = f(u, y)$ whenever $|u - v| \leq \delta$ and $u \geq 0$. Now, by hypotheses (H2) and (N1) the energy $\tilde{E}$ is strictly convex for all functions vanishing on $\partial\Omega_\pm$ and, hence, admits a unique minimizer $v_\delta^- \in H^1(\Omega)$ in the class of functions vanishing outside $\Omega_\sigma = \{y \in \Omega : \mathrm{dist}(y, \partial\Omega_\pm) > \sigma\}$ with $\sigma > 0$ sufficiently small. Moreover, we have $|v_\delta^- - v| = O(\sigma)$ in $\Omega_\sigma$. Indeed, testing $\tilde{E}$ with $\tilde{v} = \max(0, v - C\sigma)$ for $C > 0$ so large that $\tilde{v} \equiv 0$ in $\Omega \backslash \Omega_\sigma$ and using coercivity of $\tilde{E}$ and the fact that $v$ satisfies the Euler–Lagrange equation for $\tilde{E}$ in the whole of $\Omega$, we obtain that $\|v_\delta^- - v\|_{L^2(\Omega)} = O(\sigma)$. Therefore, by elliptic regularity theory [10] and possibly reducing $\sigma$, we have $\|v_\delta^- - v\|_{L^\infty(\Omega)} = O(\sigma) \leq \delta$, and so $v_\delta^-$ satisfies the Euler–Lagrange equation for the original energy $E$ whenever $v_\delta^- > 0$.

In fact, $v_\delta^- \geq 0$ in $\Omega$ and is strictly positive in $\Omega_\sigma$. Indeed, by its definition the function $\tilde{V}(u, \cdot)$ is even, whenever $|u - v| \leq \delta$. Hence, if $v_\delta^-$ is a minimizer satisfying the latter inequality, so is $|v_\delta^-|$. But by uniqueness the two must be equal. On the other hand, this implies that $v_\delta^-$ is a critical point of the original energy $E$. Therefore, by strong maximum principle we have $v_\delta^- > 0$ in $\Omega_\sigma$. Similarly, we must have $v_\delta^- < v$ in $\Omega$, since $\bar{v} = v + a\tilde{\psi}_0$ is a strict supersolution for any $0 < a \ll 1$ and, therefore, cannot touch $v_\delta^-$ from above. Thus, we constructed a function $v_\delta^-$ which is a nonnegative subsolution of the Euler–Lagrange equation for $E$ and $0 \leq v_\delta^- \leq v$. In particular, by construction

$$(4.15) \qquad v - \delta \leq v_\delta^- \leq \max(0, v - 2\alpha)$$

in $\Omega$ for $\alpha$ sufficiently small, depending only on $\delta$. Finally, extending this function to $\Sigma \times [0, T]$ by defining $u^-(y, z, t) := v_\delta^-(y)$, we obtain a subsolution on the desired domain.

*Supersolution.* Let $\sigma > 0$ be sufficiently small. By the same type of argument as in the construction of $v_\delta^-$ above, there exists a local minimizer $v_\delta^+$ of $E$ such that $v_\delta^+(y) = \sigma$ for all $y \in \partial\Omega_\pm$, and we have $v + \beta \leq v_\delta^+ \leq v + \frac{1}{4}\delta$ for some $\beta > 0$.

Now, let $c \in (c_v^\dagger, c^\dagger)$, and consider $\Psi_c^{v_\delta^+}$ defined in (3.9) with $v_\delta^+$ in place of $v$. Then, by an extension of the argument of Lemma 3.5 it is not difficult to see that there exists a minimizer $\bar{h}$ of $\Psi_c^{v_\delta^+}$ in the set

$$X := \Big\{ h \in H_c^1(\Sigma) : 0 \leq h \leq 1 - v_\delta^+, \, h = 1 - v_\delta^+ \text{ in } \Omega \times (-\infty, 0],$$

$$h(y, z) = (1 - v_\delta^+(y))\eta(z) \text{ for } (y, z) \in \partial\Omega_\pm \times \mathbb{R} \Big\},$$

where $\eta \in C^\infty(\mathbb{R})$ is a cutoff function with the property that $\eta(z) = 1$ for all $z < 0$ and $\eta(z) = 0$ for all $z > 1$. Indeed, semicontinuity and coercivity of $\Psi_c^{v_\delta^+}$ depend only on the behavior of the functional for large values of $z$. Since $v_\delta^+$ is still a local minimizer of $E$, the functional $\Psi_c^{v_\delta^+}$ is lower semicontinuous by [20, Proposition 5.5].

Furthermore, by hypothesis (H2) and the Taylor formula

$$\Psi_c^{v_\delta^+}[h] = \Psi_c^v[h] + \int_\Sigma \int_0^h e^{cz}\Big(f_u(v+s,y) - f_u(v_\delta^+ + s, y)\Big)(h-s)ds\,dx$$

$$\geq \Psi_c^v[h] - C\|v_\delta^+ - v\|_{L^\infty(\Sigma)}^\gamma \int_\Sigma \int_0^h e^{cz}(h-s)ds\,dx$$

$$\geq \Psi_c^v[h] - C\delta^\gamma\|h\|_{L_c^2(\Sigma)}^2$$

for some $C > 0$, implying coercivity of $\Psi_c^{v_\delta^+}$ for small enough $\delta$ by the argument of [20, Proposition 6.9]. So the minimizer $\bar{h}$ of $\Psi_c^{v_\delta^+}$ exists, and $\bar{h}(\cdot, z) \to 0$ uniformly in $\Omega$, as $z \to +\infty$ (indeed, the convergence is exponential by [20, Proposition 3.3(iii)]). Therefore, there exists $a > 0$ such that $\bar{h}(\cdot, z) \leq \frac{1}{4}\delta$ for all $z \geq a$.

We finally let $u^+(y, z, t) := v_\delta^+(y) + \bar{h}(y, z - z_0 + bt)$ with $b := c^\dagger - c > 0$, which is a supersolution for (4.2) on $\Sigma \times [0, T]$. Notice that

$$(4.16) \qquad u^+(\cdot, 0) = 1 \qquad \text{on } \Omega \times (-\infty, z_0]$$

and

$$(4.17) \qquad u^+(\cdot, z, t) \leq v + \frac{\delta}{2} \qquad \text{on } \Omega \qquad \forall t \geq 0 \ \forall z \geq z_0 + a - bt.$$

*Comparison.* From (4.12) and (4.15) for $\alpha$ small enough we have

$$u^-(\cdot, 0) \leq u_0 \qquad \text{on } \Omega \times (-\infty, \bar{z}_0].$$

Also, by (4.11), (4.13), and (4.15) for $\eta$ small enough we have

$$(4.18) \qquad u^-(\cdot, \bar{z}_0, t) \leq u(\cdot, \bar{z}_0, t) \qquad \text{on } \Omega \qquad \forall t \in [0, T].$$

Therefore, by parabolic comparison principle [26] we obtain

$$(4.19) \qquad u^- \leq u \qquad \text{on } \Omega \times (-\infty, \bar{z}_0] \times [0, T].$$

In particular, by (4.15) and the fact that by Theorem 2 we have $\bar{u}(\cdot, z) < v$ for every $z \in \mathbb{R}$, it follows that

$$(4.20) \qquad u(\cdot, z, t) \geq \bar{u}(\cdot, z - R(t)) - \delta \qquad \forall z \leq \bar{z}_0 \ \forall t \geq 0.$$

On the other hand, in view of (4.16), the fact that $u^+ \geq v + \beta$, (4.10) with $\eta$ replaced by $\varepsilon$ at $t = 0$ due to (4.1), and the fact that $|R(0)| \leq \delta$, for every $z_0$ it is possible to choose $\varepsilon$ small enough so that

$$(4.21) \qquad u_0 \leq u^+(\cdot, 0) \qquad \text{on } \Sigma.$$

Then, by the parabolic comparison principle we have

$$(4.22) \qquad u \leq u^+ \qquad \text{on } \Sigma \times [0, +\infty),$$

and, possibly reducing $\bar{z}_0$ to ensure that $\bar{u}(\cdot, \bar{z}_0 + a + \delta) \geq v - \frac{1}{2}\delta$, in view of monotonicity of $\bar{u}(\cdot, z)$ by Theorem 2, for every $z_0 \leq \bar{z}_0$ we obtain

$$(4.23) \qquad u(\cdot, z, t) \leq \bar{u}(\cdot, z - R(t)) + \delta \qquad \forall z \in [z_0 + a - bt, \bar{z}_0 + a] \ \forall t \geq 0.$$

Finally, combining (4.20) with (4.23) and (4.14), we get (4.8). $\qquad\square$

We now prove a technical lemma that will be useful in the proof of Proposition 4.4.

LEMMA 4.3. *There exist $0 < C_1 < C_2$ such that for all $|R| \leq 1$ we have*

$$(4.24) \qquad C_1|R| \leq ||T_R\bar{u} - \bar{u}||_{L^2_{c^\dagger}(\Sigma)} \leq C_2|R|.$$

*Furthermore,*

$$(4.25) \qquad ||T_R\bar{u} - \bar{u}||_{L^2_{c^\dagger}(\Sigma)} \geq C_1$$

*for all $|R| \geq 1$.*

*Proof.* Let us first prove the upper bound. Notice that thanks to Theorem 2, the function $\bar{u}$ belongs to $H^1_{c^\dagger}(\Sigma)$; hence in particular the map $\eta \mapsto T_\eta\bar{u}$ defines a differentiable curve in $L^2_{c^\dagger}(\Sigma)$. A direct computation then gives

$$||T_R\bar{u} - \bar{u}||_{L^2_{c^\dagger}(\Sigma)} = \left\|\int_0^R T_\eta\bar{u}_z \, d\eta\right\|_{L^2_{c^\dagger}(\Sigma)} \leq \int_{-|R|}^{|R|} ||T_\eta\bar{u}_z||_{L^2_{c^\dagger}(\Sigma)} \, d\eta$$

$$\leq \left(\int_{-|R|}^{|R|} e^{\frac{c^\dagger}{2}\eta} \, d\eta\right) ||\bar{u}_z||_{L^2_{c^\dagger}(\Sigma)} \leq C|R|,$$

where we used the identity $||T_\eta\bar{u}_z||_{L^2_{c^\dagger}(\Sigma)} = e^{\frac{c^\dagger\eta}{2}}||\bar{u}_z||_{L^2_{c^\dagger}(\Sigma)}$.

To obtain the lower bound in (4.24), we observe that for any $\Sigma_0 \Subset \Sigma$ compact, we have by the Taylor formula

$$||T_R\bar{u} - \bar{u}||^2_{L^2_{c^\dagger}(\Sigma)} \geq \int_{\Sigma_0} e^{c^\dagger z}(\bar{u}(y, z - R) - \bar{u}(y, z))^2 dx$$

$$(4.26) \qquad = R^2 \int_{\Sigma_0} e^{c^\dagger z}\bar{u}_z^2(y, z - \tilde{R}(y, z)) \, dx$$

for some $0 < |\tilde{R}(y, z)| < |R|$. The lower bound then follows from the fact that $\bar{u}_z < 0$ in $\Sigma$ and, hence, $|u_z(y, z - \tilde{R}(y, z))|$ is bounded away from zero in $\Sigma_0$, as long as $|R| \leq 1$. Finally, to get (4.25) we observe that $||T_R\bar{u} - \bar{u}||^2_{L^2_{c^\dagger}(\Sigma)}$ is monotonically increasing in $|R|$. $\square$

We now look for a suitable translation of $\bar{u}$ which serves as the best approximation, in some sense, to the solution of (4.2). For a given $u \in H^1_{c^\dagger}(\Sigma)$ and $R \in \mathbb{R}$, we define the function $h$ as

$$(4.27) \qquad h(u, R) := \frac{1}{2}\int_\Sigma e^{c^\dagger z}(u(y, z) - \bar{u}(y, z - R))^2 dx \geq 0.$$

In the following proposition, we show that the optimal approximation to $u$ can be naturally introduced by minimizing $h$ in (4.27) with respect to $R$.

PROPOSITION 4.4. *For any $\delta > 0$ sufficiently small there exists $\varepsilon > 0$ such that for any $u \in H^1_{c^\dagger}(\Sigma)$ satisfying $\|u - \bar{u}\|_{L^2_{c^\dagger}(\Sigma)} \leq \varepsilon$ the function $h(u, \cdot)$ attains its global minimum. Furthermore, this minimum is unique and is contained in $(-\delta, \delta)$ and there are no other critical points in $(-\delta, \delta)$.*

*Proof.* First, observe that $\eta \mapsto \bar{u}(y, z - \eta)$ is a twice differentiable curve in $L^2_{c^\dagger}$, thanks to Theorem 2. By assumption

$$(4.28) \qquad \inf_{R \in \mathbb{R}} h(u, R) \leq h(u, 0) \leq \varepsilon^2.$$

Furthermore, from the lower bound in Lemma 4.3 we get that

$$C \min\{1, |R|\} \leq ||T_R \bar{u} - \bar{u}||_{L^2_{c^\dagger}(\Sigma)}$$

(4.29)
$$\leq ||u - \bar{u}||_{L^2_{c^\dagger}(\Sigma)} + \sqrt{2h(u, R)} \leq \varepsilon + \sqrt{2h(u, R)}$$

for some $C > 0$. Therefore, by continuity of $h(u, \cdot)$ its minimum is attained and lies in $(-\delta, \delta)$ for any $\delta > 0$, provided that $\varepsilon$ is sufficiently small.

We now calculate the first and the second derivative of $h(u, \cdot)$ with respect to $R$:

(4.30)
$$h'(u, R) = \int_\Sigma e^{c^\dagger z}(u(y, z) - \bar{u}(y, z - R))\bar{u}_z(y, z - R) \, dx,$$

(4.31)
$$h''(u, R) = c^\dagger h'(u, R) + \int_\Sigma e^{c^\dagger z} u_z(y, z)\bar{u}_z(y, z - R) \, dx,$$

where from now on the prime denotes the derivative with respect to $R$. Recalling Lemma 4.3, for all $|R| \leq \delta \leq 1$ we have

$$|h'(u, R)| \leq C ||u - T_R \bar{u}||_{L^2_{c^\dagger}(\Sigma)}$$

$$\leq C \left( ||u - \bar{u}||_{L^2_{c^\dagger}(\Sigma)} + ||\bar{u} - T_R \bar{u}||_{L^2_{c^\dagger}(\Sigma)} \right)$$

(4.32)
$$\leq C (\varepsilon + |R|)$$

for some $C > 0$. Now, observe that upon integration by parts we have

$$\int_\Sigma e^{c^\dagger z} \bar{u}_z(y, z - R)(u_z(y, z) - \bar{u}_z(y, z - R)) \, dx$$

$$= -\int_\Sigma e^{c^\dagger z}(\bar{u}_{zz}(y, z - R) + c^\dagger \bar{u}_z(y, z - R))(u(y, z) - \bar{u}(y, z - R)) \, dx.$$

Therefore, since $\bar{u}_z \in H^1_{c^\dagger}(\Sigma)$ by Theorem 2, applying the Cauchy–Schwarz inequality we obtain

$$\left| \int_\Sigma e^{c^\dagger z} \bar{u}_z(y, z - R)(u_z(y, z) - \bar{u}_z(y, z - R)) \, dx \right|$$

(4.33)
$$\leq Ce^{c^\dagger R/2} ||u - T_R \bar{u}||_{L^2_{c^\dagger}(\Sigma)}$$

for some $C > 0$. Applying this estimate to (4.31) and combining it with the estimates in (4.32), we obtain

(4.34)
$$h''(u, R) \geq e^{c^\dagger R} ||\bar{u}_z||^2_{L^2_{c^\dagger}(\Sigma)} - C(\varepsilon + |R|)$$

for some constant $C > 0$. This implies that $h''(u, R) \geq M$ for some $M > 0$ and all $|R| \leq \delta$, provided that $\delta$ and $\varepsilon$ are small enough. Hence $h(u, \cdot)$ is a strictly convex function on $[-\delta, \delta]$, and the minimum of $h(u, \cdot)$ is the unique critical point in $(-\delta, \delta)$. $\quad\square$

Recalling the comment following Lemma 4.1, we can now formulate a nonlinear analog of the result of that lemma.

*Remark* 4.5. Suppose that $u$ is sufficiently close to $\bar{u}$ in $L^2_{c^\dagger}(\Sigma) \cap L^\infty(\Sigma)$. Then by Proposition 4.4 there exists $R_0 \in \mathbb{R}$ such that the function $h(u, R)$ in (4.27) is

minimized with respect to $R$ at $R = R_0$. Therefore, we have that $u - T_{R_0}\bar{u}$ is orthogonal to $T_{R_0}\bar{u}$ in $L^2_{c^\dagger}(\Sigma)$, and so by Lemma 4.1, hypothesis (H2), and the minimizing property of $\bar{u}$ we have $\Phi_{c^\dagger}[u] \geq \frac{1}{2}K||u - T_{R_0}\bar{u}||^2_{L^2_{c^\dagger}(\Sigma)}$, where $K > 0$ is as in Lemma 4.1. Hence $\bar{u}$ is, in fact, a strict local minimizer of $\Phi_{c^\dagger}$ in the above sense.

We now conclude the proof of Theorem 3.

*Proof of Theorem* 3. Let $\delta > 0$ be sufficiently small so that Proposition 4.4 applies with $u = u_0$ and all $0 < \varepsilon \leq \varepsilon_0$, for some $\varepsilon_0 > 0$. Then by Propositions 3.1 and 4.4 there exists $T_0 > 0$ such that there exists a minimizer $R(t)$ of $h(u(\cdot, t), R)$ in $R$ for each $t \in [0, T_0]$. Furthermore, $R(t)$ is the unique critical point of $h(u(\cdot, t), R)$ in $(-\delta, \delta)$. In fact, $R(t)$ is a continuously differentiable function of $t$ on $[0, T_0]$. Indeed, since $R(t)$ minimizes $h(u(\cdot, t), R)$ in $R$, we have

$$(4.35) \qquad h'(u(\cdot, t), R) = \int_\Sigma e^{c^\dagger z}(u(y, z, t) - \bar{u}(y, z - R))\bar{u}_z(y, z - R)\, dx = 0$$

whenever $R = R(t)$. In view of the continuity of $u_t(\cdot, t)$ in $L^2_{c^\dagger}(\Sigma)$ guaranteed by Proposition 3.1, as well as Theorem 2 and Lemma 4.3, the function in (4.35) is continuously differentiable in $R$ and $t$ in some small neighborhood of the origin. Then, arguing as in Proposition 4.4 one can see that $h''(u(\cdot, t), R) > 0$ there, so we can apply the implicit function theorem to (4.35). Furthermore, after some algebra we obtain

$$(4.36) \qquad \frac{dR(t)}{dt} = -\frac{\int_\Sigma e^{c^\dagger z} u_t(y, z, t)\bar{u}_z(y, z - R(t))\, dx}{\int_\Sigma e^{c^\dagger z} u_z(y, z, t)\bar{u}_z(y, z - R(t))\, dx}.$$

For $t \in (0, T_0]$ and $u$ solving (4.2) we define

$$(4.37) \qquad w(y, z, t) := u(y, z, t) - \bar{u}(y, z - R(t)).$$

The function $w$ satisfies the equation

$$(4.38) \qquad w_t = \Delta w + c^\dagger w_z + \frac{dR}{dt}\bar{u}_z + f_u(\tilde{u}, y)w$$

for some $\tilde{u}$, with $|\tilde{u} - T_{R(t)}\bar{u}| \leq |w|$. Also, by construction we have

$$(4.39) \qquad \int_\Sigma e^{c^\dagger z} w(y, z, t)\bar{u}_z(y, z - R(t))\, dx = 0.$$

We now introduce

$$m(t) := \int_\Sigma e^{c^\dagger z} w^2(x, t)\, dx, \qquad t \geq 0,$$

so that $m(0) \leq \varepsilon^2$. Multiplying (4.38) with $e^{c^\dagger z} w$ and integrating over $\Sigma$, we obtain

$$(4.40) \qquad \frac{dm(t)}{dt} = -2\int_\Sigma e^{c^\dagger z}\left(|\nabla w|^2 - f_u(\tilde{u}, y)w^2\right)\, dx,$$

where we used (4.39) to erase the term multiplying $dR/dt$. By Lemma 4.1 we have
(4.41)

$$\int_\Sigma e^{c^\dagger z}\left(|\nabla w|^2 - f_u(\tilde{u}, y)w^2\right)\, dx = \int_\Sigma e^{c^\dagger z}\left(|\nabla w|^2 - f_u(\bar{u}, y)w^2\right)\, dx$$

$$+ \int_\Sigma e^{c^\dagger z}\left(f_u(\bar{u}, y) - f_u(\tilde{u}, y)\right)w^2\, dx$$

$$\geq Km(t) + \int_\Sigma e^{c^\dagger z}\left(f_u(\bar{u}, y) - f_u(\tilde{u}, y)\right)w^2\, dx.$$

Possibly reducing $\delta$ and recalling assumption (H2), we have

$$\int_{\{|w|<\delta\}} e^{c^\dagger z} \, |f_u(\bar{u}, y) - f_u(\tilde{u}, y)| \, w^2 dx \leq \frac{K}{2} m(t),$$

which, combined with (4.41), gives

$$\int_\Sigma e^{c^\dagger z} \left(|\nabla w|^2 - f_u(\tilde{u}, y) w^2\right) dx \geq \frac{K}{2} m(t) - \int_{\{|w|\geq\delta\}} e^{c^\dagger z} \, |f_u(\bar{u}, y) - f_u(\tilde{u}, y)| \, w^2 dx$$

$$\geq \frac{K}{2} m(t) - \int_{-\infty}^{z_\delta(t)} \int_\Omega e^{c^\dagger z} \, |f_u(\bar{u}, y) - f_u(\tilde{u}, y)| \, w^2 \, dy \, dz$$

$$\geq \frac{K}{2} m(t) - C e^{c^\dagger z_\delta(t)}$$

for some $C > 0$, where $z_\delta(t)$ is defined in (4.7).

We can now apply Proposition 4.2 with $z_0 = \frac{4}{c^\dagger} \log \omega$, which yields some $\varepsilon > 0$ and $\eta > 0$, with $\eta$ independent of $\omega$, and $T \in (0, T_0]$ depending on $\eta$. We then get
(4.42)
$$\int_\Sigma e^{c^\dagger z} \left(|\nabla w|^2 - f_u(\tilde{u}, y) w^2\right) dx \geq \frac{K}{2} m(t) - C e^{c^\dagger(z_0 - bt)} \geq \frac{K}{2} m(t) - C\omega^4 e^{-c^\dagger bt}$$

for some $b > 0$ and $C > 0$ and all $t \in [0, T]$. From (4.40) and (4.42) we therefore obtain

$$(4.43) \qquad \frac{dm(t)}{dt} \leq -Km(t) + 2C\omega^4 e^{-c^\dagger bt} \qquad \forall t \in [0, T],$$

which gives

$$(4.44) \qquad ||w(\cdot, t)||_{L^2_{c^\dagger}(\Sigma)} \leq M\omega^2 e^{-\sigma t} \qquad \forall t \in [0, T]$$

for some $\sigma > 0$ and $M > 0$, provided that $\varepsilon$ is small enough.

To estimate the behavior of $R(t)$, we substitute $u_t = w_t - \bar{u}_z dR/dt$ into (4.36) and take into account (4.38) and (1.5) differentiated in $z$, noting that $\bar{u}_z \in H_c^2(\Sigma)$ by Theorem 2. After a few integrations by parts we obtain

$$(4.45) \quad \frac{dR(t)}{dt} = \frac{\int_\Sigma e^{c^\dagger z}(f_u(\bar{u}(y, z - R(t)), y) - f_u(\tilde{u}, y)) w(y, z, t) \bar{u}_z(y, z - R(t)) dx}{\int_\Sigma e^{c^\dagger z}(\bar{u}_z(y, z - R(t)) + w_z(y, z, t)) \bar{u}_z(y, z - R(t)) dx}.$$

By the same argument as the one leading to (4.33), we have

$$(4.46) \qquad \left| \int_\Sigma e^{c^\dagger z} \bar{u}_z(y, z - R(t)) w_z(y, z, t) \, dx \right| \leq C e^{c^\dagger R(t)/2} ||w(\cdot, t)||_{L^2_{c^\dagger}(\Sigma)}.$$

With hypothesis (H2), this leads to the estimate for $dR/dt$

$$(4.47) \qquad \left| \frac{dR(t)}{dt} \right| \leq \frac{C||w(\cdot, t)||_{L^2_{c^\dagger}(\Sigma)}}{\widetilde{C} e^{c^\dagger R(t)/2} - ||w(\cdot, t)||_{L^2_{c^\dagger}(\Sigma)}}$$

for some constants $C, \widetilde{C} > 0$, provided that $\omega$ is so small that by (4.44) the denominator in (4.47) is positive for all $t \in [0, T]$. Then we have

$$(4.48) \qquad \left| \frac{dR(t)}{dt} \right| \leq C\omega^2 e^{-\sigma t} \qquad \forall t \in [0, T],$$

and hence

$$(4.49) \qquad |R(t)| \le \widetilde{M}\omega^2 \le \omega \qquad \forall t \in [0, T]$$

for some $\widetilde{M} > 0$ and $\omega$ small enough. Moreover, since $\eta$ and $\delta$ are independent of $\omega$, (4.44) also implies that (4.9) holds uniformly in $T$ for $\omega$ small enough, whence $T = T_0$. Indeed, if $T_1 < T_0$ is the maximum value of $T$ for which Proposition 4.4 can be applied, then by (4.44) the left-hand side of (4.9) is bounded by $\eta/2$ at $t = T_1$, provided that $\omega$ is sufficiently small. Therefore, by continuity of $w(\cdot, t)$ in $L^2_{c^\dagger}(\Sigma)$ guaranteed by Proposition 3.1, the inequality in (4.9) also holds for some interval beyond $T_1$, contradicting the maximality of $T_1$.

Moreover, by (4.49) the function $R(t)$ is in fact defined and continuously differentiable for all $t \ge 0$. Indeed, let us take $T_0$ to be the largest possible value for which $||u(\cdot, t) - \bar{u}||_{L^2_{c^\dagger}(\Sigma)} \le \varepsilon_0$ for all $t \in [0, T_0]$, so that Proposition 4.4 still applies. In view of Lemma 4.3, (4.44), and (4.49), we have

$$(4.50) \qquad \begin{aligned} ||u(\cdot, t) - \bar{u}||_{L^2_{c^\dagger}(\Sigma)} &\le ||u(\cdot, t) - \mathrm{T}_{R(t)}\bar{u}||_{L^2_{c^\dagger}(\Sigma)} \\ &+ ||\bar{u} - \mathrm{T}_{R(t)}\bar{u}||_{L^2_{c^\dagger}(\Sigma)} \\ &\le M\omega^2 \qquad \forall t \in [0, T_0] \end{aligned}$$

for some $M > 0$. Therefore, choosing $\omega$ so small that the right-hand side of (4.50) is bounded by $\frac{1}{2}\varepsilon_0$ and, once again, taking into account continuity of $w(\cdot, t)$ in $L^2_{c^\dagger}(\Sigma)$, we can then make sure that the assumptions of Proposition 4.4 are satisfied on some interval beyond $T_0$, contradicting maximality of $T_0$. We thus prove that we can take an arbitrarily large $T_0 > 0$ in all the arguments above.

Finally, using (4.44) and (4.47) again and keeping in mind that by (4.49) the denominator in (4.47) is bounded away from zero, we finally obtain that the limit $R_\infty := \lim_{t \to +\infty} R(t)$ exists, and recalling Lemma 4.3 we have

$$(4.51) \qquad ||u(\cdot, t) - \mathrm{T}_{R_\infty}\bar{u}||_{L^2_{c^\dagger}(\Sigma)} \le \omega e^{-\sigma t} \qquad \forall t \ge 0$$

for some $\sigma > 0$, provided that $\omega$ is small enough, yielding the thesis of the theorem. $\square$

**5. Proof of the main result.** We will prove Theorem 1 in the reference frame moving with speed $c^\dagger$, that is, we will prove that if $u$ is the solution of (4.2) with the initial datum satisfying the assumptions of Theorem 1, then it converges in $H^2_{c^\dagger}(\Sigma)$ to $\mathrm{T}_{R_\infty}\bar{u}$ for some $R_\infty$ as $t \to \infty$. The result then follows by noting that $\mathrm{T}_{c^\dagger t}u$ solves (1.3) with the same initial condition, upon applying $\mathrm{T}_{-R_\infty}$.

From now on, $u$ always refers to the solution of (4.2). We divide the proof into five steps.

*Step* 1. We begin by constructing an appropriate pair of barrier solutions of (4.2) to ensure that the solution of the initial value problem for (4.2) does not move too far toward the ends of the cylinder. The barriers are obtained by considering the solutions $\bar{u}^\pm$ of (4.2) with the initial data

$$(5.1) \qquad \bar{u}_0^-(y, z) = \min\{u_0(y, z - R), \bar{u}(y, z)\},$$
$$(5.2) \qquad \bar{u}_0^+(y, z) = \max\{u_0(y, z + R), \bar{u}(y, z)\},$$

where $R > 0$ is so big that both $\bar{u}_0^\pm$ satisfy the assumptions of Theorem 3, provided that $\alpha$ in (2.3) is small enough. Indeed, by definition and (3.6) the assumption in

(2.3) is satisfied for both $\bar{u}_0^{\pm}$. Moreover, for $\bar{u}_0^+ - \bar{u} = \max(\mathrm{T}_{-R}u_0 - \bar{u}, 0)$ we have

$$(5.3) \qquad 0 \leq \bar{u}_0^+ - \bar{u} \leq \mathrm{T}_{-R}u_0 \to 0 \ \text{ in } L_{c^\dagger}^2(\Sigma) \ \text{ as } R \to +\infty,$$

so $\bar{u}_0^+ \to \bar{u}$ in $L_{c^\dagger}^2(\Sigma)$ as $R \to +\infty$. By a similar argument for $\bar{u} - \bar{u}_0^- = \max(\bar{u} - \mathrm{T}_R u_0, 0)$ we have

$$(5.4) \qquad 0 \leq \bar{u} - \bar{u}_0^- \leq \bar{u} \to 0 \ \text{ in } L_{c^\dagger}^2(\Omega \times (M, +\infty)) \ \text{ as } M \to +\infty$$

uniformly in $R$. At the same time, by boundedness of $\bar{u}_0^-$ and $\bar{u}$ we have $\|\mathrm{T}_R u_0 - \bar{u}\|_{L_{c^\dagger}^2(\Omega \times (-\infty, -M))} \to 0$ as $M \to +\infty$, again uniformly in $R$. Finally, in view of (2.3), (3.6), and the Hopf lemma, for every $M > 0$ and $R > 0$ large enough we have $|\{x \in \Omega \times (-M, M) : \bar{u}_0^-(x) < \bar{u}(x)\}| \leq C\alpha$ with some $C = C(M) > 0$ for small enough $\alpha$. Therefore, it is possible to choose $M$ large enough, then $R$ large enough, and then $\alpha$ small enough so that $\|\bar{u} - \bar{u}_0^-\|_{L_{c^\dagger}^2(\Sigma)}$ can be made as small as desired.

Note that both functions $\bar{u}_0^{\pm}$ obtained above satisfy (2.3) uniformly in $R$.

We now claim that $\bar{u}^{\pm}(y, z \mp R, t)$, i.e., the solutions of (4.2) with initial data $\bar{u}_0^{\pm}(y, z \mp R)$ are the appropriate barrier solutions. Indeed, by construction the initial data $u_0$ is sandwiched between $\bar{u}^{\pm}(y, z \mp R, t)$ at $t = 0$; hence by parabolic comparison principle [26] the solution of (4.2) will remain so for all times. By Theorem 3 we know that there exist $R_\infty^{\pm}$ such that

$$(5.5) \qquad \|\bar{u}^{\pm}(y, z \mp R, t) - \bar{u}(y, z \mp R_\infty^{\pm})\|_{L_{c^\dagger}^2(\Sigma)} \leq e^{-\sigma t}$$

for some $\sigma > 0$, provided that $\alpha$ is sufficiently small and $R$ is sufficiently large.

*Step* 2. We now use the functional $\Phi_{c^\dagger}$ as a Lyapunov functional to establish existence of a sequence $t_n \to +\infty$ on which $u(\cdot, t_n)$ converges to a translate of $\bar{u}$. Indeed, multiplying (4.2) by a test function $\varphi \in C_0^\infty(\mathbb{R}^n)$ vanishing on $\partial\Sigma_{\pm}$ and integrating over $\Sigma$, we can write (4.2) in the weak form as

$$(5.6) \qquad \int_\Sigma e^{c^\dagger z} \varphi \, u_t \, dx = -\int_\Sigma e^{c^\dagger z} \left(\nabla u \cdot \nabla \varphi - f(u, y)\varphi\right) dx,$$

where the integral in the right-hand side is the Gâteaux derivative of $\Phi_{c^\dagger}$ at $u(\cdot, t)$ in the direction of $\varphi$. Therefore, (4.2) is the gradient flow generated by $\Phi_{c^\dagger}$ in $L_{c^\dagger}^2(\Sigma)$, and in view of (3.3) for all $t_2 \geq t_1 > 0$, we have

$$(5.7) \qquad \Phi_{c^\dagger}[u(\cdot, t_1)] - \Phi_{c^\dagger}[u(\cdot, t_2)] = \int_{t_1}^{t_2} \|u_t(\cdot, t)\|_{L_{c^\dagger}^2(\Sigma)}^2 dt.$$

Letting $t_2 \to +\infty$ and recalling that $\Phi_{c^\dagger}[u]$ is bounded below by Theorem 2, from (5.7) it follows that there exists a sequence $t_n \to +\infty$ such that

$$(5.8) \qquad \lim_{n \to +\infty} \|u_t(\cdot, t_n)\|_{L_{c^\dagger}^2(\Sigma)} = 0.$$

Also note that since $0 \leq u(y, z, t) \leq \bar{u}^+(y, z - R, t)$ for all $(y, z) \in \Sigma$ and $t \geq 0$, and $\bar{u}^+(\cdot, t)$ is uniformly bounded in $L_{c^\dagger}^2(\Sigma)$ by Theorem 3, in view of hypotheses (H1)–(H2) we have that

$$(5.9) \qquad \|u\|_{H_{c^\dagger}^1(\Sigma)}^2 \leq 2\Phi_{c^\dagger}[u] + C\|u\|_{L_{c^\dagger}^2(\Sigma)}^2 \leq M(t_0) \qquad \forall t \geq t_0 > 0$$

for some constants $C, M(t_0) > 0$.

From (5.8) and (5.9), up to a possible subsequence, we can pass to the limit in (5.6) and get that $u(\cdot, t_n)$ converges to a critical point of $\Phi_{c^\dagger}$ weakly in $H^1_{c^\dagger}(\Sigma)$. In fact, the limit must be a nontrivial critical point $u_\infty$ of $\Phi_{c^\dagger}$ in view of Step 1 and hence a translate of $\bar{u}$ by [20, Propositions 3.2 and 3.5] and Theorem 2.

*Step* 3. We now prove that $u(\cdot, t_n) \to u_\infty$ in $L^2_{c^\dagger}(\Sigma)$. Notice first that since both $u$ and $u_\infty$ are uniformly bounded, for a given $\varepsilon > 0$ we can find $M$ such that

$$(5.10) \qquad \|u(\cdot, t_n) - u_\infty\|_{L^2_{c^\dagger}(\Omega \times (-\infty, -M])} \leq \varepsilon.$$

Moreover, since $u(y, z, t) \leq \bar{u}^+(y, z - R, t)$, from (5.5) it also follows that

$$
\begin{aligned}
(5.11) \qquad \|u(\cdot, t_n)\|_{L^2_{c^\dagger}(\Omega \times (M, +\infty))} &\leq \|\bar{u}^+(\cdot, t_n)\|_{L^2_{c^\dagger}(\Omega \times (M, +\infty))} \\
&\leq \|\bar{u}(y, z - R^+_\infty)\|_{L^2_{c^\dagger}(\Omega \times (M, +\infty))} \\
&\quad + \|\bar{u}^+(\cdot, t_n) - \bar{u}(y, z - R^+_\infty)\|_{L^2_{c^\dagger}(\Sigma)} \\
&\leq \varepsilon
\end{aligned}
$$

for $M$ big enough and all $n \geq N$ for some $N = N(M) \in \mathbb{N}$. Recalling that $H^1_{c^\dagger}(\Sigma)$ compactly embeds into $L^2_{c^\dagger}(\Omega \times (-M, M))$, from Proposition 3.1, (5.10), and (5.11) we obtain that

$$u(\cdot, t_n) \to u_\infty \qquad \text{in } L^2_{c^\dagger}(\Sigma).$$

*Step* 4. Take $n$ big enough so that

$$(5.12) \qquad \|u(\cdot, t_n) - u_\infty\|_{L^2_{c^\dagger}(\Sigma)} \leq \varepsilon \qquad \forall t \geq t_n\,,$$

where $\varepsilon$ is the same as the one corresponding to $\omega = 1$ in Theorem 3. On the other hand, for every $\alpha' > 0$ it is possible to choose $\delta \leq \alpha'$ in Proposition 4.2 such that the subsolution $u^-$ constructed there satisfies $u^-(\cdot, z, t) \geq v - \alpha'$ for all $z \leq \bar{z}_0$ with some $\bar{z}_0 \in \mathbb{R}$ independent of $\varepsilon$ and $R$ in the definition of $\bar{u}^-$ and all $t \geq 0$ if $\alpha$ is sufficiently small. Therefore, we have $\bar{u}^- \geq u^-$ in $\Omega \times (-\infty, \bar{z}_0] \times [0, +\infty)$, and since $\bar{u}^- \leq u$ for all $t \geq 0$, the same inequality holds for $u$. So we can apply Theorem 3 to $u(\cdot, t_n)$ in place of $u_0$ (also applying suitable translations in $z$ and $t$) and obtain

$$(5.13) \qquad \|u(\cdot, t) - u_\infty\|_{L^2_{c^\dagger}(\Sigma)} \leq e^{-\sigma(t - t_n)}$$

for some $\sigma > 0$ independent of $u_0$ and all $t \geq t_n$.

*Step* 5. We now demonstrate that the exponential convergence of (5.13) also holds in spaces of higher regularity. We show this first for $H^1_{c^\dagger}(\Sigma)$ and then for $H^2_{c^\dagger}(\Sigma)$. In the following, we denote by $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to L^2_{c^\dagger}(\Sigma)$ the sectorial operator $\mathcal{A} = \Delta + c^\dagger \partial_z$ with domain $\mathcal{D}(\mathcal{A}) = H^2_{c^\dagger}(\Sigma)$ dense in $L^2_{c^\dagger}(\Sigma)$ (see also [24, 21]).

Letting $w(\cdot, t) := u(\cdot, t) - u_\infty$, we have

$$(5.14) \qquad w_t = \mathcal{A}w + g(x, t)w,$$

where $g(y, z, t) = f_u(\tilde{u}(y, z, t), y)$ for some $\tilde{u}$ such that $|\tilde{u} - u_\infty| \leq |w|$, i.e., $g$ is such that $|g| \leq C$ for some $C > 0$ independent of $t$. As a consequence, by parabolic regularity theory [34, Chapter 15] (see also [21, Proposition 2.1.1 and Theorem 3.1.1]) and recalling (5.13), for all $t \geq 1$ we have

$$
\begin{aligned}
(5.15) \qquad \|u(\cdot, t) - u_\infty\|_{H^1_{c^\dagger}(\Sigma)} &\leq C \left( \|w(\cdot, t - 1)\|_{L^2_{c^\dagger}(\Sigma)} + \int_{t-1}^{t} \frac{\|w(\cdot, s)\|_{L^2_{c^\dagger}(\Sigma)}}{\sqrt{t - s}} \, ds \right) \\
&\leq C e^{-\sigma t}
\end{aligned}
$$

for some $C > 0$ independent of $t$. In particular, from [20, Proposition 3.2], (5.15), the minimizing property of $u_\infty$, and hypothesis (H2) we get

$$
\begin{aligned}
\Phi_{c^\dagger}[u(\cdot, t)] &= \Phi_{c^\dagger}[u(\cdot, t)] - \Phi_{c^\dagger}[u_\infty] \\
&= \int_\Sigma e^{c^\dagger z} \left( \frac{1}{2} |\nabla w|^2 + V(u_\infty + w, y) - V(u_\infty, y) - V'(u_\infty, y)w \right) dx \\
&\leq C \|w\|_{H^1_{c^\dagger}(\Sigma)}^2 \leq C e^{-2\sigma t}
\end{aligned}
$$
(5.16)

for all $t \geq t_0$ with any $t_0 > 0$ and some $C = C(t_0) > 0$.

Let us now rewrite (5.14) in the form

$$
w_t = \mathcal{A}w + h(\cdot, t), \qquad h(\cdot, t) := f(u(\cdot, t)) - f(u_\infty).
$$
(5.17)

Recalling hypothesis (H2), (5.7), and (5.16), for all $t_2 \geq t_1 \geq t_0 > 0$ we have

$$
\begin{aligned}
\|h(\cdot, t_2) - h(\cdot, t_1)\|_{L^2_{c^\dagger}(\Sigma)} &= \|f(u(\cdot, t_2)) - f(u(\cdot, t_1))\|_{L^2_{c^\dagger}(\Sigma)} \\
&\leq C \|u(\cdot, t_2) - u(\cdot, t_1)\|_{L^2_{c^\dagger}(\Sigma)} \\
&\leq C \int_{t_1}^{t_2} \|u_t(\cdot, s)\|_{L^2_{c^\dagger}(\Sigma)} ds \\
&\leq C \sqrt{(t_2 - t_1) \Phi_{c^\dagger}[u(\cdot, t_1)]} \leq C \sqrt{t_2 - t_1}\, e^{-\sigma t_1}
\end{aligned}
$$

for some $C = C(t_0) > 0$. Then, reasoning as in [21, Theorem 4.3.1] with $t_1 = t - 1$ and $t_2 = t$ and using (5.15), we have

$$
\begin{aligned}
\|w(\cdot, t)\|_{H^2_{c^\dagger}(\Sigma)} &\leq C(\|\mathcal{A}w(\cdot, t)\|_{L^2_{c^\dagger}(\Sigma)} + \|w(\cdot, t)\|_{H^1_{c^\dagger}(\Sigma)}) \\
&\leq C \Bigg( \|w(\cdot, t-1)\|_{L^2_{c^\dagger}(\Sigma)} + \|w(\cdot, t)\|_{H^1_{c^\dagger}(\Sigma)} \\
&\qquad + \int_{t-1}^t \frac{\|h(\cdot, s) - h(\cdot, t)\|_{L^2_{c^\dagger}(\Sigma)}}{t - s} ds \Bigg) \\
&\leq C e^{-\sigma t}
\end{aligned}
$$
(5.18)

for all $t \geq t_0 + 1$ for any $t_0 > 0$ and some $C = C(t_0) > 0$. In writing (5.18), we used the same reasoning as in the standard estimate of the $H^2$-norm of a function in terms of the $L^2$-norm of the Laplacian to obtain the inequality in the first line. This gives (2.4) and concludes the proof of Theorem 1.    □

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