	90's	80's	70's	60's	50's	40's
DIST:	7	5	9	6	4	4

TOTAL = 35 STUDENTS

EXAM MEDIAN = 73

CS 241, Fall 2014, Exam 2 Exam #: ____ Name: Solution
Prof. David Nassimi (Time: 1.5 Hours) First Last

(12) PTS 1. Consider the relation $R = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,3)\}$.

(a) Show the matrix of this relation. By direct observation of the matrix (without matrix multiplication) determine if the relation satisfies the following properties. Provide a reasoning.

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 \end{array}$$

Reflexive?

YES, $(1,1), (2,2), (3,3)$ are all in R.

Symmetric?

No, $(3,1) \in R$ but $(1,3) \notin R$

Antisymmetric?

No, $(1,2) \in R$ and $(2,1) \in R$.

Transitive?

No, $(1,2) \in R$ and $(2,3) \in R$, but $(1,3) \notin R$

Partial Order?

No, Not antisymmetric and not transitive.

Equivalence Relation?

No, not symmetric and not transitive

(b) Now, use matrix multiplication to decide if the relation is transitive. Explain.

(8) PTS

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

	GRADE	
1		/20
2		/20
3		/20
4		/20
5		/20
SUM		/100

DIST:

90's	7
80's	5
70's	9
60's	6
50's	4
40's	4

Not transitive:

$A[1,3] = 0$ and $A^2[1,3] = 1$

For transitive,

$\forall\, i,j$, if $A[i,j] = 0$ then $A^2[i,j] = 0$

Note: It is not necessary to have $A = A^2$

2. Prove the following polynomial is $\Theta(n^4)$.

$$P(n) = 5n^4 - 2n^3 - 10n^2 + 50n + 100.$$

10 PTS (a) Prove $O(n^4)$:

$$P(n) = 5n^4 - 2n^3 - 10n^2 + 50n + 100$$

$$\leq 5n^4 + 50n + 100, \quad n \geq 0 \qquad \text{Drop Neg Terms}$$

$$\leq n^4 \left(5 + \frac{50}{n^3} + \frac{100}{n^4}\right)$$

$$\leq n^4 \left(5 + \frac{50}{1000} + \frac{100}{10{,}000}\right) \qquad n \geq 10$$

$$\leq 5.06\, n^4, \qquad n \geq 10$$

10 PTS (b) Prove $\Omega(n^4)$:

$$P(n) = 5n^4 - 2n^3 - 10n^2 + 50n + 100$$

$$\geq 5n^4 - 2n^3 - 10n^2, \quad n \geq 0 \qquad \text{Drop Pos Terms}$$

$$\geq n^4 \left(5 - \frac{2}{n} - \frac{10}{n^2}\right)$$

$$\geq n^4 (5 - 0.2 - 0.1) \qquad n \geq 10$$

$$\geq 4.7\, n^4, \qquad n \geq 10$$

3. The following program computes $(\log_2 n)$, assuming n is an integer power of 2.

(a) Trace the working of the algorithm for $n = 16$. (Show the values of m and k after each iteration of the while loop.)

(4)

```
int LOG (int n){
int m, k;
m = n;    k = 0;
while (m > 1)
  {m = m/2;    k = k + 1 }
ruturn (k)
}
```

Trace:

m	K
16	0
8	1
4	2
2	3
1	4

Return

$\boxed{K = 4}$

(b) Write a recursive version of this algorithm.

(8)

```
int LOG(int n)
{ if (n==1) return 0;
  return (1+ LOG(n/2))
}
```

(c) Let $f(n)$ be the number of division operations performed by your recursive algorithm to compute $(\log_2 n)$. Write a recurrence equation for $f(n)$. Guess the solution and prove it correct by induction.

(8)

$$f(n) = \begin{cases} 0, & n = 1 \\ 1 + f(\frac{n}{2}), & n \geq 2 \end{cases}$$

$\boxed{f(n) = \log n}$ CLAIM (GUESS)

Proof:

Base, $n = 1$: $f(1) = 0$, $f(1) = \log 1 = 0$, so base is correct.

For any $n \geq 2$, suppose CLAIM is correct for $n/2$. So,

$f(n/2) = \log \frac{n}{2} = \log n - 1$ (Hypothesis)

Then,

$f(n) = 1 + f(\frac{n}{2}) = 1 + (\log n - 1) = \log n$.

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4. Consider the following divide-and-conquer recursive algorithm. Parameter i is the starting index of array A, and n is the number of elements. The initial call is COMP$(A, 0, n)$.

Boolean COMP (int $A[\ ]$, int i, int n) {
Boolean $C1, C2$;
1. if $(n == 1)$ return FALSE ;
2. $C1 =$ COMP $(A, i, \lfloor n/2 \rfloor)$; —— first half
3. $C2 =$ COMP $(A, i + \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$; —— second half
4. return $(C1 \vee C2 \vee (A[i] \neq A[i + n - 1]))$;
}

(8)

(a) Figure out what the algorithm does. When does it return TRUE? Explain.

It returns true iff there is any pair not equal.
For $n=1$, returns false, since not unequal pair.
For $n \geq 2$, returns true iff
(1) A pair not equal in first half, OR
(2) " " " " " second half, OR
(3) An element in first half $\neq$ an element in second half

(b) Let $f(n)$ be the number of key comparisons (in step 4) performed by this algorithm for an array of size n. Consider the special case when n is a power of 2. Write a recurrence for $f(n)$. Find the solution of the recurrence by repeated substition.

(12)

$$f(n) = \begin{cases} c, & n = 1 \\ 2f(\frac{n}{2}) + 1, & n \geq 2 \end{cases}$$

$$f(n) = 1 + 2f(\tfrac{n}{2})$$
$$= 1 + 2(1 + 2f(\tfrac{n}{4}))$$
$$= 1 + 2 + 4f(\tfrac{n}{4})$$
$$= 1 + 2 + 4 + 8f(\tfrac{n}{8})$$
$$\vdots$$
$$= 1 + 2 + 4 + \cdots + 2^{k-1} + 2^k \cancel{f(\tfrac{n}{2^k})}, \quad \text{where } n = 2^k$$
$$f(1) = c$$
$$= 1 + 2 + 4 + \cdots + 2^{k-1}$$
$$= 2^k - 1 \quad \text{Geometric sum}$$
$$= n - 1$$

5. Find the exact solution of the following linear recurrence: $F_0 = 0, F_1 = 9$, and

$$F_n = 7F_{n-1} - 10F_{n-2}, \quad n \geq 2.$$

Try $F_n = r^n$

Then,

$$r^n - 7r^{n-1} + 10r^{n-2} = 0$$

$$r^{n-2}(r^2 - 7r + 10) = 0$$

$$r^2 - 7r + 10 = 0$$

$$(r-5)(r-2) = 0$$

$$r_1 = 5, \quad r_2 = 2$$

So, $\boxed{F_n = A\,5^n + B\,2^n}$

Use Base cases to find A, B:

$$\left.\begin{aligned} F_0 &= A + B = 0 \\ F_1 &= 5A + 2B = 9 \end{aligned}\right\} \quad \boxed{\begin{aligned} A &= 3 \\ B &= -3 \end{aligned}}$$

Therefore,

$$F_n = 3 \cdot 5^n - 3\,2^n$$