1. Use both Venn diagram and algebraic method to prove each set equality.

(a) \[ A \cup (B - A) = A \cup B \]
\[ A \cup (B - A) = A \cup (B \cap \overline{A}) \]
\[ = (A \cup B) \cap (A \cup \overline{A}) \quad \text{DISTRIBUTIVE} \]
\[ = (A \cup B) \cap \top \]
\[ = A \cup B \]

(b) \[ (A \cap B) \cup (A \cap \overline{B}) = A \]
\[ (A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B}) \]
\[ = A \cap \top \]
\[ = A \]
2. (a) Express an equivalent expression for the following proposition, using only AND, OR, and NOT connectives. Then, use a truth table to show equivalence of the two expressions.

\[ \neg(P \rightarrow Q) \]
\[ P \land \neg Q \]

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \rightarrow Q</th>
<th>\neg(P \rightarrow Q)</th>
<th>\neg Q</th>
<th>P \land \neg Q</th>
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(b) Let the domain of \( x \) and \( y \) be all integers. Is the following proposition true or false? Give a reasoning.

\[ \forall x \exists y, \text{ if } (x > y) \text{ then } (x^2 < y^2) \]

**TRUE.** For any \( x \), we can pick \( y = -\lceil |x| - 1 \rceil \).

Then, \( x > y \) and \( x^2 < y^2 = x^2 + 2|\lceil x \rceil| + 1 \).

Examples: (Note: Examples by themselves are not proof.)

\[ x = 5, \quad y = -7 \]
\[ x = -5, \quad y = -6 \]

(c) Find negation of the above statement and simplify. Is the result true or false? Explain.

\[ \exists x \forall y, \quad \neg \left[ \forall y \left( (x > y) \implies (x^2 < y^2) \right) \right] \]
\[ \exists x \forall y, \quad \neg \left[ (x > y) \implies (x^2 < y^2) \right] \]
\[ \exists x \forall y, \quad (x > y) \land \neg (x^2 < y^2) \]
\[ \exists x \forall y, \quad (x > y) \land (x^2 \geq y^2) \]

**FALSE.** There is no \( x \) greater than every \( y \).

Also, it is false because it is negation of (b) which was true.
3. Use proof by contradiction or contrapositive proof, as appropriate, for each of the following.

(a) Prove that if \( n^3 \) is divisible by 8, then \( n \) is divisible by 2. Assume domain of \( n \) is positive integers.

We'll prove contrapositive equivalent form:

"If \( n \) is not divisible by 2,
then \( n^3 \) is not divisible by 8."

Suppose \( n \) is not divisible by 2.
So, \( n = 2k + 1 \) for some integer \( k \).
Then,
\[
 n^3 = (2k+1)^3 = (2k)^3 + 3(2k)^2 + 3(2k) + 1
   = 8k^3 + 12k^2 + 6k + 1
   = 2(4k^3 + 6k^2 + 3k) + 1
\]
Therefore, \( n^3 \) is ODD, so not divisible by 8.

(b) Let the domain of \( x \) and \( y \) be positive real numbers. Suppose \( x \) is rational and \( y \) is irrational. Prove that \( x + y \) is irrational.

We'll prove by contradiction.
Suppose \( (x+y) \) is rational.
Then, \( x + y = \frac{i}{j} \) for some int \( i, j \).
And, we know \( x \) is rational.
So, \( x = \frac{i'}{j'} \) for some int \( i', j' \).
Then,
\[
y = (x+y) - x = \frac{i}{j} - \frac{i'}{j'} = \frac{i'j - ij'}{jj'}
\]
So, \( y \) is rational.
This contradicts earlier fact that \( y \) is irrational.
Therefore, \( (x+y) \) is irrational.
4. Prove by simple induction that any postage amount of \( n \) cents, \( n \geq 12 \), may be achieved by using only 7-cent stamps and 3-cent stamps. That is, for every integer \( n \geq 12 \), there exist non-negative integers \( A \) and \( B \) such that,

\[
n = 7A + 3B.
\]

For BASE, \( n = 12 \), \( 12 = 7 \times 0 + 3 \times 4 \).

Now suppose for some \( n \geq 12 \),

\[
\boxed{n = 7A + 3B}
\]

We'll show how to achieve \( n+1 \).

Case 1: \( B \geq 2 \)

Then,

\[
n + 1 = 7(A+1) + 3(B-2)
\]

Case 2: \( n ( B \geq 2 ) \)

\[
B \leq 1
\]

Since \( n \geq 12 \), and \( B \leq 1 \), then \( A \geq 2 \).

So, \( A \geq 2 \).

Therefore,

\[
n + 1 = 7(A-2) + 3(B+5)
\]
5. Consider the relation \( R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\} \).

(a) Does the relation satisfy each of following properties? Explain.

Reflexive?
\[ \text{YES. } \forall x, (x, x) \in R. \]

Symmetric?
\[ \text{No, } (1, 2) \in R \text{ but } (2, 1) \notin R. \]

Antisymmetric?
\[ \text{YES. } \forall i \neq j, \text{ if } (i, j) \in R \text{ then } (j, i) \notin R. \]

Transitive?
\[ \text{No, } (1, 2) \in R \text{ and } (2, 3) \in R, \text{ but } (1, 3) \notin R. \]

Partial Order?
\[ \text{No, not transitive} \]

Equivalence Relation?
\[ \text{No, not symmetric and not transitive} \]

(b) Show the matrix of this relation. Use matrix multiplication to decide if the relation is transitive. Explain.

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
A^2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \times \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Not Transitive,
\[ A[1, 3] = 0 \text{ but } A^5[1, 3] = 1. \]