1. Prove the following polynomial is $\Theta(n^4)$.

\[ P(n) = 5n^4 - 2n^3 - 10n^2 + 2n + 5 \]

(a) Prove $O(n^4)$:

\[ P(n) = 5n^4 - 2n^3 - 10n^2 + 2n + 5 \]
\[ \leq 5n^4 + 2n + 5, \quad n \geq 0 \]
\[ \leq n^4 \left( 5 + \frac{2}{n^3} + \frac{5}{n^4} \right), \quad n \geq 10 \]
\[ \leq n^4 \left( 5 + 0.002 + 0.0005 \right) \leq 5.0025n^4 \]

(b) Prove $\Omega(n^4)$:

\[ P(n) = 5n^4 - 2n^3 - 10n^2 + 2n + 5 \]
\[ \geq 5n^4 - 2n^3 - 10n^2 \]
\[ \geq n^4 \left( 5 - \frac{2}{n} - \frac{10}{n^2} \right) \]
\[ \geq n^4 \left( 5 - \frac{2}{10} - \frac{10}{10^2} \right) \]
\[ \geq 4.7n^4 \]
2. Find the exact number of times (in terms of $n$) the innermost statement ($X = X + 1$) is executed in the following code. That is, find the final value of $X$. Then express the total running time in terms of $O(\cdot)$.

\[
X = 0; \\
\text{for } i = 1 \text{ to } n \\
\quad \text{for } j = 1 \text{ to } 3n - i \\
\quad \quad X = X + 1;
\]

\[
F = \sum_{i=1}^{n} \sum_{j=1}^{3n-i} (1) = \sum_{i=1}^{n} (3n-i^2)
\]

\[
= (n)(3n) - \sum_{i=1}^{n} i^2
\]

\[
= 3n^2 - n \frac{(n+1)}{2}
\]

\[
= 3n^2 - \frac{n^3}{2} - \frac{n}{2}
\]

\[
F = 2.5n^2 - 0.5n
\]

The total time is $O(n^2)$. 

3. Consider the following recursive algorithm. Input is an integer array $A[0..n-1]$.

```c
int COMP (int A[], int n) {
    int T;
    if (n == 0) return 0;
    T = COMP (A, n - 1)
    if (A[n - 1] < 0)
        T = T + 1;
    return T;
}
```

(a) What does this algorithm do? What does it compute? Be specific, and provide a brief reasoning.

IT counts the number of negative values in the array.

(“It may be proved by induction.”)

(b) Figure out the return value of the algorithm when $n = 8$ and the input array is:

\[ A = [2, 0, -1, -2, 0, 1, 2, 5] \]

It returns 2.

(c) Let $f(n)$ be the worst-case number of ADDITIONS performed by this algorithm for an array of size $n$. Write a recurrence for $f(n)$. Guess the solution of the recurrence and prove the correctness by induction.

\[
f(n) = \begin{cases} 
0, & n = 0 \\
f(n-1) + 1, & n \geq 1 
\end{cases}
\]

Solution: \[ f(n) = n, \quad n \geq 0 \]

Induction:

Base, $n = 0$: \[ f(0) = 0 = 0 \] and $f(0) = 0$ from recur.

So base correct.

To prove for any $n \geq 1$, suppose true for $n-1$:

Then\[ f(n) = f(n-1) + 1 = (n-1) + 1 = n. \]
4. (a) Consider the following recurrence equation. (Assume $n$ is a power of 2.)

$$T(n) = \begin{cases} 
2T(n/2) + n^2, & n \geq 2 \\
1, & n = 1.
\end{cases}$$

Prove the solution is $T(n) = An^2 + Bn$, and find the constants $A, B$.

Base, $n = 1$: $T(1) = A + B = 1$

For any $n \geq 2$, suppose $T(n/2) = A \left(\frac{n}{2}\right)^2 + B \left(\frac{n}{2}\right)$.

Then,

$$T(n) = 2T(n/2) + n^2 = 2 \left[A \left(\frac{n}{2}\right)^2 + B \left(\frac{n}{2}\right)\right] + n^2$$

$$= \left(\frac{A}{2} + 1\right) n^2 + Bn \overset{?}{=} An^2 + Bn$$

Need $\left(\frac{A}{2} + 1\right) = A$, so $A = 2$, $B = 1 - A$ (B = -1)

(b) Use repeated substitution to find the solution of the following recurrence.

$$F(n) = \begin{cases} 
3F(n-1) + 1, & n \geq 2 \\
1, & n = 1.
\end{cases}$$

$$F(n) = 1 + 3F(n-1)$$

$$= 1 + 3 \left(1 + 3F(n-2)\right) = 1 + 3 + 3^2 F(n-2)$$

$$= 1 + 3 + 3^2 + \cdots + 3^{K-1} + 3^K F(n-K)$$

$$= 1 + 3 + 3^2 + \cdots + 3^{m-2} + 3^{m-1} F\left(n-(n-1)\right)$$

$$= 1 + 3 + 3^2 + \cdots + 3^{n-2} + 3^{n-1} \overset{\text{Geom sum}}{=} \frac{3^{n-1}}{3-1} = \frac{1}{2} \left(\frac{n}{3} + 1\right)$$
5. Find the solution of the following linear recurrence.

\[ F_n = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ 7F_{n-1} - 10F_{n-2}, & n \geq 2 \end{cases} \]  \hspace{1cm} (1)

\[ F_n - 7F_{n-1} + 10F_{n-2} = 0 \]

Let \( F_n = r^n \)

\[ r^n - 7r^{n-1} + 10r^{n-2} = 0 \]

\[ r^{n-2} (r^2 - 7r + 10) = 0 \]

\[ r^2 - 7r + 10 = 0 \]

\[ (r-5)(r-2) = 0 \]

\[ r_1 = 5, \quad r_2 = 2 \]

\[ F_n = A \cdot 5^n + B \cdot 2^n \]

To find \( A \) and \( B \), use BASE CASES:

\[ \begin{align*}
F_0 &= 0 = A + B \\
F_1 &= 1 = 5A + 2B
\end{align*} \]

\[ \begin{align*}
A &= \frac{1}{3} \\
B &= -\frac{1}{3}
\end{align*} \]

\[ F_n = \frac{1}{3} (5^n - 2^n) \]