1. Use algebraic method to prove each set equality.

(a) \( A \cup \overline{A} \cap B = A \cup B \)

\[
\begin{align*}
A \cup (\overline{A} \cap B) & = (A \cup \overline{A}) \cap (A \cup B) & \text{DIST} \\
& = \overline{\overline{U}} \cap (A \cup B) \\
& = A \cup B
\end{align*}
\]

(b) \( A - (B \cup C) = (A - B) \cap (A - C) \)

\[
\begin{align*}
A - (B \cup C) & = A \cap (\overline{B \cup C}) \\
& = A \cap (\overline{B} \cap \overline{C}) & \text{De Morgan} \\
& = (A \cap \overline{B}) \cap (A \cap \overline{C}) \\
& = (A - B) \cap (A - C)
\end{align*}
\]
2. (a) Use a truth table to show the following propositions are equivalent.
   
i. \( A \rightarrow B \)
   
ii. \( \neg B \rightarrow \neg A \)
   
iii. \( \neg(A \land \neg B) \)

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A \rightarrow B</th>
<th>\neg B</th>
<th>\neg A</th>
<th>\neg B \rightarrow \neg A</th>
<th>A \land \neg B</th>
<th>\neg(A \land \neg B)</th>
</tr>
</thead>
<tbody>
<tr>
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</table>

We see from the table that (1), (2), (3) are equivalent.

(b) Find the negation of the following proposition. (Assume domain of \( x \) is positive integers, and \( f(x) \) and \( g(x) \) are Boolean functions of \( x \).)

\[ \neg \left( \forall x, \text{ if } f(x) \text{ then } g(x) \right) \]

\[ \exists x, \neg \left( \forall \bar{x}, \text{ if } f(\bar{x}) \text{ then } g(\bar{x}) \right) \]

\[ \exists x, \; f(x) \land \neg g(x) \]
3. (a) Determine true/false value of each of the following propositions, where the domain of \( x \) and \( y \) is all integers. Provide your reasoning.

i. \( \forall x \exists y, \ (x + y = 0) \)

    True. For every \( x \), there is a corresponding \( y = -x \), so that \( x + y = 0 \).

ii. \( \exists x \forall y, \ (x + y = 0) \)  
    **Note:** You must read from left to right.

    FALSE. There is no single fixed \( x \),  
    so as for every \( y \), \( x + y = 0 \).
    
    **Counterexample:** Given any \( x \), pick \( y = -x + 1 \), so \( x + y \neq 0 \).

(b) Find the negation of each of the following propositions. Show the work step-by-step.

i. \( \forall i \forall j \forall k, \ (A[i, k] \land A[k, j]) \) then \( A[i, j] \).

\[
\exists i \exists j \exists k, \ \neg (\forall k \ (A[i, k] \land A[k, j]) \ \text{then} \ A[i, j])
\]

\[
\exists i \exists j \exists k, \ (A[i, k] \land A[k, j]) \land \neg A[i, j]
\]

ii. \( \forall i \forall j, \ (\exists k, (A[i, k] \land A[k, j])) \) then \( A[i, j] \).

\[
\exists i \exists j, \ \neg (\forall k \ (\exists k, (A[i, k] \land A[k, j])) \ \text{then} \ A[i, j])
\]

\[
\exists i \exists j, \ (\exists k, (A[i, k] \land A[k, j])) \land \neg A[i, j]
\]

\[
\exists i \exists j \exists k, \ (A[i, k] \land A[k, j]) \land \neg A[i, j]
\]

Note that (i) and (ii) are the same negation.
4. (a) Given a rational number $x$ and an irrational number $y$. Prove by contradiction that $x \cdot y$ is irrational.

Suppose to the contrary that $x \cdot y$ is rational.

So, $x \cdot y = \frac{i}{j}$ for some int $i, j$.

And we know $x$ is rational. So, $x = \frac{k}{l}$ for some int $k, l$.

Then $y = \frac{x \cdot y}{x} = \frac{i/j}{k/l} = \frac{i \cdot l}{k \cdot j}$ which is rational,

Therefore, $x \cdot y$ is irrational.

(b) Suppose the domain of $n$ is positive integers. Prove by contrapositive method the following statement is correct.

"If $n^2$ is not divisible by 4, then $n$ is not divisible by 2."

If $(n \text{ is divisible by } 2)$ then $(n^2 \text{ is divisible by } 4)$.

To prove the latter, suppose $n$ is divisible by 2.

Then, $n = 2k$ for some int $k$.

So, $n^2 = 4k^2$ which is divisible by 4.
5. (a) Prove by simple induction that any postage amount of \( n \) cents, \( n \geq 14 \), may be achieved by using only 8-cent stamps and 3-cent stamps. That is, prove that for every integer \( n \geq 14 \), there exist some non-negative integers \( A \) and \( B \) such that

\[
n = 8A + 3B.
\]

- **Base, \( n = 14 \):**
  \[
  14 = 8 \times 1 + 3 \times 2.
  \]
  So, base is correct.

- **For any \( n \geq 14 \), suppose \( n = 8A + 3B \).**

- Then, we'll prove \( n+1 = 8A' + 3B' \), for some non-negative int \( A', B' \).

  - **Case 1:** \( B \geq 5 \)
    \[
    n+1 = 8(A+2) + 3(B-5).
    \]

  - **Case 2:** \( n(B \geq 5) \). That is, \( B \leq 4 \). Since \( n \geq 14 > 4 \times 3 \)

    - **Thus,** \( A \geq 1 \).

  So,

  \[
  n+1 = 8(A-1) + 3(B+3)
  \]

(b) Next use strong induction to prove the above statement.

First prove 3 base cases:

\[
14 = 8 \times 1 + 3 \times 2 \\
15 = 8 \times 0 + 3 \times 5 \\
16 = 8 \times 2 + 3 \times 0
\]

To prove for any \( n \geq 17 \), suppose true for all \( m < n \).

In particular, \( n = n-3 \geq 14 \), not true for \( n-3 \). So,

\[
n-3 = 8A + 3B \] for some non-negative int \( A, B \).

Thus, we can add one 3-cent stamp to \( n-3 \) to get \( n \).

That is,

\[
n = 8A + 3(B+1).
\]

Note: An alternative proof is to use 8 base cases.
6. Consider the relation

\[ R = \{(1,1), (1,2), (2,1), (3,1), (3,2), (3,3)\} \]

Show the matrix of this relation.

\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
\]

By direct observation of the matrix, decide if the relation satisfies each property. Explain.

(a) Reflexive?

\[ \text{No, } (2,2) \notin R. \] That is, \( A[2,2] = 0 \).

(b) Symmetric?

\[ \text{No, } A[1,3] = 0 \text{ but } A[3,1] = 1. \]

(c) Antisymmetric?

\[ \text{No, } A[1,2] = 1 \text{ and } A[2,1] = 1. \]

(d) Transitive? (Decide this by directly examining the ordered pairs. Do NOT use matrix multiplication for this part.)

\[ \text{No, } A[2,1] = 1 \land A[1,2] = 1 \text{ but } A[2,2] = 0. \]

(e) Is the relation a Partial Order?

\[ \text{No, } \text{not (reflexive, antisymmetric, transitive).} \]

(f) Is the relation an Equivalence Relation?

\[ \text{No, } \text{not (reflexive, symmetric, transitive).} \]
7. Use matrix multiplication to decide if each relation is transitive. Provide an explanation.

(a) \( A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \) \( A \times A = A^2 \)

\[
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}
\]

Not transitive, because \( A_{3,1} = 0 \) but \( A^2_{3,1} = 1 \).

(b) \( A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \)

\[
\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

Not transitive, because \( A_{1,1} = 0 \) but \( A^2_{1,1} = 1 \).