CS 241-102, Spring 2015, Midterm Exam
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Exam #: _____
(Time: 2 Hours)

Name: Solution
First Last

1. Use algebraic method to prove each set equality.

7

(a) $A \cup \overline{A} \cap B = A \cup B$

$A \cup (\overline{A} \cap B)$

$= (A \cup \overline{A}) \cap (A \cup B)$ DIST

$= \overline{U} \cap (A \cup B)$

$= A \cup B$

	GRADE	
1		/ 15
2		/ 15
3		/ 15
4		/ 15
5		/ 15
6		/ 15
7		/ 10
SUM		/100

DIST

90's	12
80's	9
70's	7
60's	10
50's	3
40's	4
30's	2
20's	1

Tot: 48

High = 100
MEDIAN = 75
LOW = 20

8

(b) $A - (B \cup C) = (A - B) \cap (A - C)$

$A - (B \cup C)$

$= A \cap \overline{(B \cup C)}$

$= A \cap (\overline{B} \cap \overline{C})$ De Morgan

$= (A \cap \overline{B}) \cap (A \cap \overline{C})$, $A = A \cap A$

$= (A - B) \cap (A - C)$

2. (a) Use a truth table to show the following propositions are equivalent.

7

i. $A \to B$

ii. $\neg B \to \neg A$

iii. $\neg(A \wedge \neg B)$

A	B	$A \to B$ (1)	$\sim B$	$\sim A$	$\sim B \to \sim A$ (2)	$A \wedge \sim B$	$\sim(A \wedge \sim B)$ (3)
T	T	T	F	F	T	F	T
T	F	F	T	F	F	T	F
F	T	T	F	T	T	F	T
F	F	T	T	T	T	F	T

We see from the table that (1), (2), (3) are equivalent.

(b) Find the nagation of the following proposition. (Assume domain of x is positive integers, and $f(x)$ and $g(x)$ are Boolean functions of x.)

8

$$\sim\left(\ \forall x, \text{ if } f(x) \text{ then } g(x)\ \right)$$

$$\exists x, \sim(\text{if } f(x) \text{ then } g(x))$$

$$\exists x, \quad f(x) \wedge \sim g(x)$$

3. (a) Determine true/false value of each of the following propositions, where the domain of x and y is all integers. Provide your reasoning.

7

i. $\forall x \exists y, \quad (x + y = 0)$

True. For every x, there is a corresponding $y = -x$, so that $x + y = 0$.

ii. $\exists x \forall y, \quad (x + y = 0)$ NOTE: You must read from left to right.

FALSE. There is no single fixed x, so as for every y, $x+y=0$.

Counterexample: Given any x, pick $y = -x+1$, so $x+y \neq 0$

(b) Find the negation of each of the following propositions. Show the work step-by-step.

8

i. $\forall i \forall j \forall k$, if $(A[i,k] \wedge A[k,j])$ then $A[i,j]$.

$\exists i \exists j \exists k, \sim$ (if $(A[i,k] \wedge A[k,j])$ then $A[i,j]$)

$\exists i \exists j \exists k, (A[i,k] \wedge A[k,j] \wedge \sim A[i,j])$

ii. $\forall i \forall j$, if $(\exists k, (A[i,k] \wedge A[k,j]))$ then $A[i,j]$.

$\exists i \exists j, \sim$ (if $(\exists k, A[i,k] \wedge A[k,j])$ then $A[i,j]$)

$\exists i \exists j, (\exists k, A[i,k] \wedge A[k,j]) \wedge \sim A[i,j]$

$\exists i \exists j \exists k, (A[i,k] \wedge A[k,j] \wedge \sim A[i,j])$

Note that (i) and (ii) are the SAME negation.

4. (a) Given a rational number x and an irrational number y. Prove by **contradiction** that $x * y$ is irrational.

7

Suppose to the contrary that

$\boxed{xy \text{ is rational.}}$

So, $xy = i/j$ for some int i, j.

And we know x is rational. So, $x = \frac{k}{l}$ for some int k, l.

Then $y = \frac{xy}{x} = \frac{i/j}{k/l} = \frac{il}{kj}$ = Rational, which contradicts earlier fact.

Therefore, $\boxed{xy \text{ is irrational}}$.

(b) Suppose the domain of n is positive integers. Prove by **contrapositive method** the following statement is correct.

"If n^2 is not divisible by 4, then n is not divisible by 2."

8

if (n is DIVISIBLE by 2) then (n^2 is DIVISIBLE by 4).

To prove the latter, suppose n is divisible by 2.

Then, $n = 2k$ for some int k.

So, $n^2 = 4k^2$ which is divisible by 4.

5. (a) Prove by **simple induction** that any postage amount of n cents, $n \geq 14$, may be achieved by using only 8-cent stamps and 3-cent stamps. That is, prove that for every integer $n \geq 14$, there exist some non-negative integers A and B such that

$$n = 8A + 3B.$$

Base, $n = 14$: $14 = 8 * 1 + 3 * 2$. So, base is correct.

For any $n \geq 14$, suppose $\boxed{n = 8A + 3B.}$ for some non-Neg int A, B. (HYPO)

Then, we'll prove $n+1 = 8A' + 3B'$, for some Non-Neg int A', B'.

Case 1: $B \geq 5$

Then, $n+1 = 8(A+2) + 3(B-5)$.

Case 2: $\sim(B \geq 5)$. That is, $B \leq 4$. Since $n \geq 14 > 4 * 3$ Then, $A \geq 1$.

So, $n+1 = 8(A-1) + 3(B+3)$

(b) Next use **strong induction** to prove the above statement.

First prove 3 Base cases:

$$14 = 8 * 1 + 3 * 2$$
$$15 = 8 * 0 + 3 * 5$$
$$16 = 8 * 2 + 3 * 0$$

To prove for any $n \geq 17$, suppose true for all $m < n$.

In particular, $m = n - 3 \geq 14$, so true for $n-3$. So,

$n - 3 = 8A + 3B$, for some Non-Neg int A, B.

Then, we can ADD one 3-cent stamp to $n-3$ to get n.

That is,

$$n = 8A + 3(B+1).$$

Note: An alternative proof is to use 8 Base cases.

6. Consider the relation

$$R = \{(1,1),(1,2),(2,1),(3,1),(3,2),(3,3)\}.$$

Show the matrix of this relation.

2

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 \end{array}$$

By direct observation of the matrix, decide if the relation satisfies each property. Explain.

(a) Reflexive?

2 No, $(2,2) \notin R$. That is, $A[2,2] = 0$.

(b) Symmetric?

2 No, $A[1,3] = 0$ but $A[3,1] = 1$.

(c) Antisymmetric?

2 No, $A[1,2] = 1$ and $A[2,1] = 1$.

(d) Transitive? (Decide this by directly examining the ordered pairs. Do NOT use matrix multiplication for this part.)

3 No, $A[2,1] = 1 \wedge A[1,2] = 1$ but $A[2,2] = 0$.

(e) Is the relation a Partial Order?

2 No, Not (reflexive, antisymmetric, transitive).

(f) Is the relation an Equivalence Relation?

2 No, Not (reflexive, symmetric, transitive).

7. Use matrix multiplication to decide if each relation is transitive. Provide an explanation.

5

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $A \times A = A^2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ \boxed{0} & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \boxed{1} & 0 & 1 \end{bmatrix}$$

Not transitive because $A[3,1] = 0$ but $A^2[3,1] = 1$.

5

(b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} \boxed{0} & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Not transitive because

$A[1,1] = 0$ but $A^2[1,1] = 1$.