Poker Hands
Conditional Probabilities
Given that the first card is an ace

\[ C(52,5) = \frac{(52*51*50*49*48)}{(5*4*3*2*1)} = 2,598,960 \]
\[ C(51,4) = \frac{(51*50*49*48)}{(4*3*2*1)} = 249,900 \]

(A) Four-of-a-Kind (unconditional probability)

Num {4-of-kind} = C(13,1)*C(4,4)*C(48,1) = 13*48 = 624
Prob {4-of-kind} = 624 / C(52,5) = 624 / 2598960 = 1 / 4165

Four-of-a-Kind, given that the first card is an ACE

Ways to get it:
  1) Get 3 more aces, plus one of another kind; OR
  2) Get 4 of another kind.

Num {4-of-kind | first ace} = C(3,3)*C(48,1) + C(12,1)*C(4,4) = 48 + 12 = 60
Prob {4-of-kind | first ace} = 60 / C(51,4) = 60 / 249900 = 1 / 4165  (Prob didn’t change)

(B) Full-House

Num{FH} = C(13,1)*C(4,3)*C(12,1)*C(4,2) = 13*4*12*4 = 3744
Prob{FH} = 3744/C(52,5) = 3744 / 2598960 = 1 / 694

Full-House, Given that the first card is an ACE

Ways to get it:
  1) Get 2 more aces, plus 2 of another kind; OR
  2) Get one more aces, plus 3 of another kind.

Num {FH | First Ace} = C(3,2)*C(12,1)*C(4,2) + C(3,1)*C(12,1)*C(4,3)
= 3*12*6 + 3*12*4 = 360
Prob {FH | First Ace} = 360 / C(51,4) = 360 / 249900 = 1 / 694  (Prob didn’t change)

But some probabilities do change.
For example, try:

Straight Flush Given that the first card is an ace.

Next, let us look at the probability of various poker hands, given that the first two cards are both aces.
Poker Hands
Given that the first 2 cards are both ACES

Total possible hands, given that the first 2 cards are both aces:

\[ U = \binom{50}{3} = \frac{(50*49*48)}{(3*2*1)} = 19,600 \]

**Four-of-a-Kind, given that the first 2 cards are both ACES**

Num (4-of-kind | first 2 aces) = \( \binom{2}{2} \) * \( \binom{48}{1} \) = 48

Prob (4-of-kind | first 2 aces) = \( \frac{48}{\binom{50}{3}} \) = \( \frac{48}{19600} \) = \( \frac{1}{408} \)

Note that this probability is higher than the unconditional probability of 4-of-a-kind, which was 1/4165. Why? Obviously, if you already have two aces, you have a better chance of ending up with 4-of-a-kind.

**Full-House, Given that the first two cards are 2 ACES**

Ways to get it:

1) Get 1 more ace, plus 2 of another kind; OR
2) Get 3 of another kind.

Num (FH | First 2 Aces) = \( \binom{2}{1} \) * \( \binom{12}{2} \) * \( \binom{4}{2} \) + \( \binom{12}{1} \) * \( \binom{4}{3} \) = 2*12*6 + 12*4 = 192

Prob (FH | First 2 Aces) = 192 / \( \binom{50}{3} \) = 192 / 19600 = \( \frac{1}{102} \)

**Three-of-a-kind, given that first 2 cards are ACES**

Num = \( \binom{2}{1} \) * \( \binom{12}{2} \) * \( \binom{4}{1} \) * \( \binom{4}{1} \) = 2*[\( (12*11)/2 \)]*4*4 = 12*11*16 = 2,112

**Two Pairs, given that first 2 cards are ACES**

Num = \( \binom{12}{1} \) * \( \binom{4}{2} \) * \( \binom{44}{1} \) = 12*6*44 = 3,168

**One Pair, given that first 2 cards are ACES**

Num = \( \binom{12}{3} \) * \( \binom{4}{1} \) * \( \binom{4}{1} \) * \( \binom{4}{1} \) = [\( (12*11*10)/(3*2) \)]*4*4*4 = 220*4^3 = 14,080

**Total:**

\[ 48 + 192 + 2112 + 3168 + 14080 = 19,600 \]