1. Prove the following polynomial is $\Theta(n^4)$.

$$P(n) = 5n^4 + 2n^3 - 10n^2 - 50n - 100$$

(a) Prove $O(n^4)$:

$$P(n) = 5n^4 + 2n^3 - 10n^2 - 50n - 100$$

$$\leq 5n^4 + 2n^3$$

$$\leq n^4 \left( 5 + \frac{2}{n} \right)$$

$$\leq n^4 \left( 5 + \frac{2}{100} \right)$$

$$\leq 5.02 \cdot n^4$$

(b) Prove $\Omega(n^4)$:

$$P(n) = 5n^4 + 2n^3 - 10n^2 - 50n - 100$$

$$\geq 5n^4 - 10n^2 - 50n - 100$$

$$\geq n^4 \left( 5 - \frac{10}{n^2} - \frac{50}{n^3} - \frac{100}{n^4} \right)$$

$$\geq n^4 \left( 5 - \frac{10}{(100)^2} - \frac{50}{(10)^3} - \frac{100}{(10)^4} \right)$$

$$\geq n^4 \left( 5 - 0.1 - 0.05 - 0.01 \right)$$

$$\geq 4.84 \cdot n^4$$
2. Find the exact number of times (in terms of $n$) the innermost statement ($X = X + 1$) is executed in the following code. That is, find the final value of $X$. Then express the total running time in terms of $O(\cdot)$.

\[
X = 0;
\text{for } i = 1 \text{ to } n - 3
\quad \text{for } j = 2i + 1 \text{ to } 3n + 5
\quad X = X + 1;
\]

\[
F = \sum_{i=1}^{n-3} \sum_{j=2i+1}^{3n+5} (i) \\
= \sum_{i=1}^{n-3} (3n+5 - 2i) \\
= (n-3)(3n+5) - 2 \sum_{i=1}^{n-3} i \\
= (n-3)(3n+5) - 2 (n-3) \frac{1+(n-3)}{2} \\
= (n-3)(3n+5) - (n-3)(n-2) \\
= (n-3)(3n+5 - n + 2) \\
= (n-3)(2n + 7) \\
F = 2n^2 + n - 21
\]

which is $O(n^2)$
3. Consider the following divide-and-conquer algorithm (recursive function). Parameter $i$ is the starting index of the array, and $n$ is the number of elements. The initial call is \texttt{COMPUTE}(A, 0, n).

\begin{verbatim}
int COMPUTE (int A[], int i, int n) {
    if (n == 1)
        \{ if (A[i] == 0) return 0 else return 1 \};
    n1 = \lfloor n/2 \rfloor; //Length of first half of array
    n2 = n - n1; //Length of second half of array
    C1 = COMPUTE (A, i, n1);
    C2 = COMPUTE (A, i + n1, n2);
    return (C1 + C2)
}
\end{verbatim}

(a) Figure out what the function does. (What does it compute?) Explain briefly.

\textit{It counts the number of \underline{non-zero} elements in the array.}

(b) Let $f(n)$ be the number of times the arithmetic operation ($C1 + C2$) is performed by this algorithm. Assume $n$ is a power of 2. Write a recurrence for $f(n)$. Guess the solution of the recurrence and prove the correctness by induction.

\[
f(n) = \begin{cases} 
0, & n = 1 \\
2f\left(\frac{n}{2}\right) + 1, & n \geq 2 
\end{cases}
\]

\textbf{Solution:} \boxed{f(n) = n - 1}

\textbf{Base, }\text{ }n = 1: \quad f(1) = 1 - 1 = 0, \text{ and from recurrence, } f(1) = 0, \text{ no base works.}

To prove for any $n \geq 2$, suppose $f\left(\frac{n}{2}\right) = \frac{n}{2} - 1$.

Then, $f(n) = 2f\left(\frac{n}{2}\right) + 1 = 2\left(\frac{n}{2} - 1\right) + 1 = n - 1$ \textit{which proves conclude.}

(c) Now consider the general case where $n$ is any integer. Write a recurrence for $f(n)$. Guess the solution again and prove it correct by induction.

\[
f(n) = \begin{cases} 
0, & n = 1 \\
\lfloor n/2 \rfloor + f\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 1, & n \geq 2 
\end{cases}
\]

\textbf{Solution:} \boxed{f(n) = n - 1}

\textbf{Base, }\text{ }n = 1: \text{ same as above.}

To prove for any $n \geq 2$, suppose $f(m) = m - 1$ for $m < n$.

Then, $f(n) = f\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + f\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 1 = \left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right) + \left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right) + 1 = n - 1$. 

4. (a) Consider the following recurrence equation. (Assume \( n \) is a power of 2.)

\[
T(n) = \begin{cases} 
8T(n/2) + n, & n \geq 2 \\
1, & n = 1.
\end{cases}
\]

By induction

Prove the solution is \( T(n) = An^3 + Bn \), and find the constants \( A, B \).

Base case: \( n = 1 \):
\[
T(1) = 1 = A + B \quad \Rightarrow \quad A + B = 1
\]

To prove for any \( n \geq 2 \), suppose true for \( n/2 \):
\[
T(n/2) = A \left( \frac{n}{2} \right)^3 + B \left( \frac{n}{2} \right)
\]

Then,
\[
T(n) = 8T(n/2) + n = 8 \left[ \frac{An^3}{8} + B \frac{n}{2} \right] + n
\]

\[
= An^3 + (4B + 1)n \quad \Rightarrow \quad 4B + 1 = B \quad \Rightarrow \quad B = -\frac{1}{3}, \quad A = \frac{4}{3}
\]

(b) Use repeated substitution to find the solution of the following recurrence. (Assume \( n \) is a power of 2.)

\[
T(n) = \begin{cases} 
2T(n/2) + n^2, & n \geq 2 \\
1, & n = 1.
\end{cases}
\]

\[
T(n) = 2T \left( \frac{n}{2} \right) + n^2
\]

\[
= n^2 + 2 \left[ \left( \frac{n}{2} \right)^2 + 2T \left( \frac{n}{4} \right) \right] = n^2 + \frac{n^2}{2} + 4T \left( \frac{n}{4} \right)
\]

\[
= n^2 + \frac{n^2}{2} + 4 \cdot \left[ \left( \frac{n}{4} \right)^2 + 2T \left( \frac{n}{8} \right) \right] = n^2 + \frac{n^2}{2} + \frac{n^2}{4} + 8T \left( \frac{n}{8} \right)
\]

\[
= n^2 \left( 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{k-1}} \right) + 2^k \cdot T \left( \frac{n}{2^k} \right)
\]

Geometric sum

\[
T(1) = 1
\]

\[
= n^2 \cdot \frac{1 - \frac{1}{2^k}}{1 - \frac{1}{2}} + 2^k = n^2 \cdot \frac{1 - \frac{1}{n}}{1 - \frac{1}{2}} + n
\]

\[
= \frac{2}{n^2 - n} + n^2
\]
5. Find the solution of the following linear recurrence.

\[ F_n = \begin{cases} 
0, & n = 0 \\
1, & n = 1 \\
8F_{n-1} - 15F_{n-2}, & n \geq 2 
\end{cases} \]  \hspace{1cm} (1)

**Try:** \[ F_n = r^n \]

\[ r^n - 8r^{n-1} + 15r^{n-2} = 0 \]

\[ r^{n-2} (r^2 - 8r + 15) = 0 \]

\[ (r-5)(r-3) = 0 \]

**Roots:** \[ r_1 = 5, \quad r_2 = 3 \]

\[ F_n = A5^n + B3^n \]

\[ F_0 = 0 = A + B \]
\[ F_1 = 1 = 5A + 3B \]

\[ \begin{align*}
A &= \frac{1}{2} \\
B &= -\frac{1}{2}
\end{align*} \]

\[ \therefore \quad F_n = \frac{1}{2} (5^n - 3^n) \]