1. Prove that
\[ \sum_{i=1}^{n} \log i = \Theta(n \log n). \]

(a) Prove \( O() \):

\[
\begin{array}{|c|}
\hline
\text{GRADE} \\
\hline
1 & /20 \\
2 & /20 \\
3 & /20 \\
4 & /20 \\
5 & /20 \\
\text{SUM} & /100 \\
\hline
\end{array}
\]

(b) Prove \( \Omega() \):
2. The following program computes \((\log_2 n)\), assuming \(n\) is an integer power of 2.

```c
int LOG (int n)
{
    int m, k;
    m = n;   k = 0;
    while (m > 1)
    {
        m = m/2;   k = k + 1
    }
    return (k)
}
```

(a) Write a recursive version of this algorithm.

(b) Let \(f(n)\) be the number of division operations performed by your recursive algorithm to compute \((\log_2 n)\). Write a recurrence equation for \(f(n)\). Find the solution by repeated substitution.
3. Consider the following divide-and-conquer recursive algorithm. Parameter \( i \) is the starting index of the array, and \( j \) is the ending index. The initial call for an array of size \( n \) is \( \text{FIND}(A, 0, n - 1) \).

Boolean \( \text{FIND} \) (int \( A[] \), int \( i \), int \( j \)) {
    if \( (i == j) \) return TRUE;
    \( m = \lfloor (i + j)/2 \rfloor \);
    \( C1 = \text{FIND}(A, i, m) \);
    \( C2 = \text{FIND}(A, m + 1, j) \);
    \( C3 = (A[m] \leq A[m + 1]) \)
    return (\( C1 \land C2 \land C3 \))
}

(a) Figure out what the algorithm does. When does the algorithm return TRUE? Be precise. Explain briefly how the algorithm works.

(b) Let \( f(n) \) be the number of key comparisons performed by this algorithm for an array of size \( n \). Assuming \( n \) is a power of 2, write a recurrence equation for \( f(n) \). Guess the solution of the recurrence and prove the correctness of it by induction.
4. Consider the following recurrence equation, where \( n \) is a power of 2.

\[
T(n) = \begin{cases} 
2T(n/2) + n, & n \geq 2 \\
0, & n = 1.
\end{cases}
\]

Prove by induction that the solution is as follows, and find the constant \( A \).

\[
T(n) = An \log n.
\]
5. Consider the relation $R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$.

(a) Does the relation satisfy each of following properties? Explain.

- Reflexive?

- Symmetric?

- Antisymmetric?

- Transitive? For this part, you must decide it by directly examining the ordered pairs, and not using matrix multiplication.

- Partial Order?

- Equivalence Relation?

(b) Show the matrix of this relation. Use matrix multiplication to decide if the relation is transitive. Explain.