Math 707, **Final Project**, SPRING 2002
DUE: 11 AM, May 9, 2002.

General Comments

A final project is a typed report (done with, e.g., the typesetting program called LaTeX). The report must contain properly labeled graphs with descriptive legends in order to support your answers to the various questions posed in the description of the final project. Your project must include an appendix that contains the programs you constructed in order to answer the questions. Any mathematics required to answer the questions must be typed.

Hand written final projects and/or incomprehensible graphs are unacceptable.

Project description

- This part of the project is concerned with the nonlinear parabolic IBVP,

$$\begin{aligned} U_t + U^2 U_x &= U + \nu U_{xx}; \quad x \in (0, 2\pi),\ t > 0 \\ U(x,0) &= \sin kx; \quad x \in (0, 2\pi) \\ U(0,t) &= U(2\pi, t); \quad t \geq 0, \end{aligned} \tag{1}$$

  derived in *SIAM Journal on Applied Mathematics, Vol. 60, No. 5, pp. 1543-1564* as the equation governing the modulational stability of Taylor vortices in a high Reynolds number flow through a curved channel of small gap. Solve (1) by modifying the program of HW4 accordingly. In addition, have your program compute the energy, $E(t) = ||U^2(\cdot, t)||_2$, every 100 timesteps using Parseval's identity.

  Perform a linear stability analysis of the $U = 0$ solution of the PDE to determine for what wavenumbers of a perturbation this state is unstable and for what wavenumbers of a perturbation this state is stable. Armed with the linear stability analysis, set k in the unstable region and compute the solution at $t = 10$ for $\nu = 0.2$. Graph $U(x, 10)$, and the energy $E(t)$ for $t \in (0, 10)$. Then, set k in the stable region and compute the solution at $t = 10$ for $\nu = 0.2$. Again, graph $U(x, 10)$, and the energy $E(t)$ for $t \in (0, 10)$. Explain how, using the graphs of the energy, one can conclude that initial conditions with $0 < k < \frac{1}{\sqrt{\nu}}$ lead to solutions which persist in time, while initial conditions with $k > \frac{1}{\sqrt{\nu}}$ lead to solutions which die out in time (hence, the $U = 0$ solution of the PDE is unstable to perturbations). Here, your are verifying the linear stability analysis.

- This part of the project is concerned with the HW2 nonlinear reaction-diffusion system of PDE's discussed in *Phys. Rev. E 54, 4860 (1996)*. Implement the Chebyshev pseudospectral method to solve the parabolic system of equations (with Neumann boundary conditions) which we encountered in HW2. As for HW2, use Euler to discretize time (the timestep here will be $\Delta t = O(N^{-4})$). You can use the FFT or the matrix approach to compute second order partial derivatives as described in class, remembering that homogeneous Neumann boundary conditions have to be imposed around the rectangular domain. For an initial condition as in HW2 reproduce with your pseudospectral code the Finite Difference solution and compare the computational resources (e.g., number of collocation points) needed to do this. NOTE: a) the Finite Difference solution was computed over a domain of size $L_x \times L_y$ hence you must first map this rectangle to the square $[-1, 1] \times [-1, 1]$, b) The Finite Difference solution is computed on a grid of equi-spaced points in each direction while the pseudospectral method computes the solution on the Chebyshev collocation points, hence the comparison of results can only be done graphically.

- This part of the project is concerned with the nonlinear parabolic IBVP,

$$\begin{aligned} U_t &= \nu U_{xx} + U - U^3; \quad x \in (-1, 1), \ t > 0 \\ U(x, 0) &= \sin kx; \quad x \in (-1, 1) \qquad (2) \\ U(-1, t) &= U(1, t) = 0; \quad t \geq 0. \end{aligned}$$

This PDE has three basic states, $U = 0, \pm 1$, of which $U = 0$ is unstable. Perturbations of the zero state evolve to either one of the stable states (you can verify this with a linear stability analysis of a constant state $U = U_c$). In the presence of boundary conditions (homogeneous Dirichlet here) the long-time solution should be one of the stable states ($U = \pm 1$) inside the domain $(-1, 1)$ with a boundary layer at each end ($x = \pm 1$) to connect it to the boundary condition.

- Use the Chebyshev matrix or FFT approaches to differentiation with Euler time-integration in order to compute long-time solutions of the initial boundary value problem in (2) (sometimes called Fisher's equation, Ginsburg-Landau, or Allen-Cahn equation). The timestep here will also be of the form $\Delta t = O(N^{-4})$. Pick ν to be 1 and 0.1, and use enough collocation points to produce smooth solutions (no spurious oscillations). Display these long-time solutions, and explain why one should prefer the Chebyshev method over *any* Finite Difference method that uses an equispaced grid.
- Implement the Crank-Nicholson time integration scheme for the diffusive term in (2) with Euler for the $U - U^3$ term. You will have to use the matrix approach to do Chebyshev pseudospectral differentiation, and the program *factor.f* (available from the course webpage), to compute the required inverse matrix. Compute and display the same long-time solutions as for the previous part. Did the implicitness help in speeding up your computation (by allowing a timestep larger than $O(N^{-4})$) with fixed number of collocation points ?