Modeling photon generation

C. J. McKinstrie

Bell Laboratories, Alcatel–Lucent, Holmdel, New Jersey 07733

mckinstrie@alcatel-lucent.com / 732-888-7275

Abstract

The photons required for a variety of quantum information experiments can be generated by parametric (four-wave mixing) processes in fibers. These processes are driven by one or two strong pumps and couple the evolution of two weak (signal and idler) sidebands. They are governed by the coupled-mode equations $d_z X = AX + BX^*$ and the associated input–output relations $X(z) = M(z)X(0) + N(z)X^*(0)$, where $X$ is the (sideband) amplitude vector, $A$ and $B$ are coefficient matrices (which depend on the fiber and pump parameters), and $M$ and $N$ are transfer matrices. Each of the sideband amplitudes could have one frequency and one polarization component, one frequency and two polarization components, or many frequency and one or two polarization components. For special cases in which the coefficient matrices commute, they are simultaneously diagonalizable. In these cases, the amplitudes of the normal modes obey the one-mode squeezing equations $d_z x_j = \alpha_j x_j + \beta_j x_j^*$, the solutions of which are known. (In-phase quadratures are stretched, whereas out-of-phase quadratures are squeezed.) Despite the fact that parametric processes have been studied for decades, there seems to be no simple solution for the general case, in which the coefficient matrices do not commute. The first goal of workshop is to catalog methods of attack for the general case and, if possible, to find the general solution. The second goal is to solve specific problems of current interest, which involve polarization dynamics, and to interpret the solutions.
Parametric devices based on four-wave mixing (FWM) in fibers can amplify, frequency convert, phase conjugate, regenerate and sample optical signals in classical communication systems [1]. They can also generate photon pairs for quantum information (communication and computation) experiments [2]. Three different types of FWM are illustrated in Fig. 1. Modulation interaction (MI) is the degenerate process in which two photons from the same pump are destroyed, and signal and idler (sideband) photons are created \((2\pi_p \rightarrow \pi_s + \pi_i)\), where \(\pi_j\) represents a photon with frequency \(\omega_j\). Inverse MI is the degenerate process in which two photons from different pumps are destroyed and two signal photons are created \((\pi_p + \pi_q \rightarrow 2\pi_s)\). Phase conjugation (PC) is the nondegenerate process in which two different pump photons are destroyed and two different sideband photons are created \((\pi_p + \pi_q \rightarrow \pi_s + \pi_i)\). MI and PC are reviewed in [3, 4, 5, 6].

![Figure 1](https://example.com/figure1.png)

**Figure 1**: Frequency diagrams for (a) modulation interaction, (b) inverse modulation interaction, and (c) outer-band and (d) inner-band phase conjugation. Long arrows denote pumps \((p\ and \ q)\), whereas short arrows denote sidebands \((s\ and \ i)\). Downward arrows denote modes that lose photons, whereas upward arrows denote modes that gain photons.

Parametric interactions of weak sidebands, driven by strong pumps, are governed by coupled-mode equations (CMEs) of the form

\[
d_z X = AX + BX^*,
\]

where \(z\) is distance, \(d_z = d/dz\), \(X = [x_j]\) is the \(n \times 1\) vector of sideband amplitudes (modes), \(A = [\alpha_{jk}]\) and \(B = [\beta_{jk}]\) are \(n \times n\) coefficient matrices, and \(\ast\) denotes a complex conjugate.
The entries of the amplitude vector could be the amplitudes of distinct monochromatic sidebands (continuous waves), or different frequency components of multichromatic sidebands (pulses), with one or two polarization components. For uniform media the coupling coefficients are constants, whereas for nonuniform media they vary with distance. Because Eq. (1) is linear in the amplitude vector, its (explicit or implicit) solution can be written in the input–output (IO) form

\[ X(z) = M(z)X(0) + N(z)X^*(0), \]

where \( M = [\mu_{jk}] \) and \( N = [\nu_{jk}] \) are transfer (Green) matrices.

For the aforementioned one- and two-mode interactions (scalar MI and PC of continuous waves in isotropic fibers), it is easy to solve the CMEs and interpret the IOEs. However, in some systems several modes interact simultaneously, or several two-mode interactions occur sequentially. For such systems, the CMEs and IOEs are complicated and two related questions arise: Under what conditions can we solve the CMEs explicitly and how can we interpret the (explicit or implicit) IOEs? The main goals of this project are to answer these questions. In these notes I will summarize some previous work briefly.

The quantum mechanical properties of parametric processes are not part of the project. However, the laws of quantum mechanics impose constraints on the coefficient and transfer matrices that are worth stating: \( A \) is anti-hermitian (so \( A = iJ \), where \( J \) is hermitian) and \( B \) is symmetric (so \( B = iK \), where \( K \) is also symmetric). In addition, \( MM^\dagger - NN^\dagger = I \) and \( MN^t - NM^t = 0 \). (Mini-project: Can these constraints also be deduced from the laws of classical Hamiltonian mechanics?)

Recall that every complex matrix \( M \) has the singular value decomposition (SVD), sometimes called the Schmidt decomposition, \( M = UDV^\dagger \), where \( U \) and \( V \) are unitary and \( D \) is diagonal [7, 8]. The \( n \) columns of \( U \) are the eigenvectors of \( MM^\dagger \), the columns of \( V \) are the eigenvectors of \( M^\dagger M \), and the entries of \( D \) are the (common) non-negative eigenvalues of \( MM^\dagger \) and \( M^\dagger M \). The aforementioned constraints on the transfer matrices ensure that they have the simultaneous SVDs \( M = U\mu V^\dagger \) and \( N = U\nu V^\dagger \), where \( D_{\mu} = \text{diag}(\mu_j) \), \( D_{\nu} = \text{diag}(\nu_j) \) and their entries (Schmidt coefficients) satisfy the auxiliary
equations $\mu_j^2 - \nu_j^2 = 1$. Hence, Eq. (2) can be rewritten in the form

$$X(z) = U D_\mu V^\dagger X(0) + U D_\nu V^\dagger X^*(0),$$

(3)

where $U$, $V$, $D_\mu$ and $D_\nu$ depend implicitly on $z$ [9]. It follows from Eq. (3) that the columns of $V$ define input (Schmidt) modes, the columns of $U$ define output modes, and the mode amplitudes ($y_j$) obey the (one-mode squeezing) equations

$$y_j(z) = \mu_j(z)y_j(0) + \nu_j(z)y_j^*(0).$$

(4)

Equations (3) and (4) are remarkable. They tell us that every parametric process, no matter how complicated, can be decomposed into independent squeezing transformations, the properties of which are known (in-phase quadratures are stretched, whereas out-of-phase quadratures are squeezed). We only need to determine the input and output modes, and the associated squeezing parameters (analytically or numerically). Unfortunately, the decomposition theorem does not tell us how to calculate the transfer matrices, upon which the decomposition depends. Notice that both sets of (input and output) modes (usually) depend on $z$, as do the coefficients (squeezing parameters). Notice also that decompositions do not concatenate, in the sense that the input modes associated with two successive transformations are not (necessarily) equal to the input modes of the first transformation, and the output modes are not equal to the output modes of the second transformation.

One possible way to determine the transfer matrix is to use the adjoint method. Equation (1) and its conjugate can be written in the form

$$d_z Y = iLY,$$

(5)

where the $2n \times 1$ mode vector $Y = [X^t, X^\dagger]^t$ and the $2n \times 2n$ coefficient matrix

$$L = \begin{bmatrix} \ J & \ K \\ -K^* & -J^* \end{bmatrix}. $$

(6)

In general, $L$ is not hermitian (self-adjoint). However,

$$L = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \ J & \ K \\ K^* & J^* \end{bmatrix} = \begin{bmatrix} \ J & -K \\ -K^* & J^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

(7)
so $L$ is simply related to hermitian matrices. (Mini-project: What does this relationship imply about the eigenvalues and eigenvectors of $L$?)

The eigenvectors of $L$ are defined by the equation $LE_j = \lambda_j E_j$, whereas the adjoint eigenvectors are defined by the adjoint equation $L^\dagger F_j = \lambda_j^* F_j$. Neither set of vectors is orthonormal by itself. However, $F_j^\dagger E_k = \delta_{jk}$. The eigenvectors $E_j$ are both input and output modes, and the mode amplitudes ($y_j$) satisfy the simple evolution equations $d_y y_j = i\lambda_j y_j$, so $y_j(z) = e^{i\lambda_j z} y_j(0)$. Hence, the solution of Eq. (5) can be written in the IO form

$$Y(z) = T(z) Y(0), \quad T(z) = \sum_j E_j e^{i\lambda_j z} F_j^\dagger. \quad (8)$$

The transfer matrix $T$ can be written in the alternative form $ED\lambda F^\dagger$, where $E_j$ is the $j$th column of $E$, $e^{i\lambda_j z}$ is the $j$th entry of the diagonal matrix $D\lambda$ and $F_j$ is the $j$th column of $F$. Notice that the input and output matrices (modes) are constants. Notice also that adjoint decompositions concatenate, in the sense that $T(z_2)T(z_1) = T(z_1 + z_2)$: The input and output modes of the individual and combined transformations are the same. In both regards, the adjoint decomposition is simpler and more informative that the Schmidt decomposition. If the transfer matrix is written in the block-matrix form

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad (9)$$

then Eqs. (2) and (9) imply that $T_{11} = M$, $T_{12} = N$, $T_{21} = T_{12}^*$ and $T_{22} = T_{11}^*$. Although Eqs. (3) and (8) are both formal solutions to Eq. (1), they are based on different input and output matrices: The $n \times n$ matrices $U$ and $V$ are unitary (so their $n \times 1$ columns are self-orthonormal), whereas the $2n \times 2n$ matrices $E$ and $F$ are not unitary (their $2n \times 1$ columns are cross-orthonormal). (Mini-project: Is there a simple relation between the Schmidt and adjoint decompositions? If so, under what conditions?)

An alternative method of analysis is to rewrite the complex CME (1) as a real CME, in which the variables are the mode quadratures (real and imaginary parts of the mode amplitudes). In this equation, the coefficient matrix is not symmetric. There is an interesting relation between the Schmidt decompositions of the real and complex CMEs [10]. It might be easier to analyze the properties of the real CME.
References


