Fast Computation of Loss Distributions for Credit Portfolios

MPI Workshop
NJIT, June 2011

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Agenda

• Risk Analysis of Credit Portfolios
• Monte Carlo Simulation for Portfolio Risk Analysis
• An Importance Sampling Method for Credit Portfolio Simulations
• Simple Example
Disclaimer

• The models and mathematical analysis presented here are exclusively part of a preliminary quantitative research effort intended to better understand the computational accuracy of financial risk measures estimated with Monte Carlo simulations.

• This presentation does not comment on Standard & Poor’s credit ratings or any current or future criteria or models used in the ratings process for credit portfolios or structured finance products.
S&P Overview

- Standard and Poor’s is a leading provider of independent credit analysis, and a key source for financial market intelligence.

- Best known for S&P 500 Index and Credit Ratings

- Wholly owned subsidiary of McGraw-Hill

- S&P Ratings Services and McGraw-Hill Financial have about 10,000 employees worldwide with over $2.6 Billion in revenue

- Brands include Compustat, CUSIP, CapIQ, ClariFl, Risk Solutions, GICS, S&P Ratings Direct, IMAKE
Credit Portfolio Risk Analysis

\[ \Pi(t) = \sum_{i=1}^{N} \omega_i \ V_i(t) \]

\[ R_{\Pi}(T_0, T_H) = \sum_{i=1}^{N} \hat{\omega}_i \ R_i(T_0, T_H) \]

\[ EL = E\left( R_{\text{ref}} - R_{\Pi} \right) \]

\[ UL = \sigma\left( R_{\text{ref}} - R_{\Pi} \right) \]

\[ P\left( R_{\Pi} < R^*(\alpha) \right) = \alpha \]
Simple Example: Default/No Default

\[ \hat{\omega}_i = \frac{1}{N} \]

\[ V_i(T_0) = 1 \]

\[ V_i(T_H) = \begin{cases} 1 & \text{if no default} \\ 1 - LGD & \text{if default} \end{cases} \]

\[ R_\Pi = -\frac{LGD}{N} \sum_{i=1}^{N} X_i \quad X_i = 1 \text{ if default} \]

\[ EL = LGD \times PD \]

Loss Distribution depends on probability distribution of joint defaults. Default dependence determines credit risk of portfolios.
Example Loss Density Function for a Large Credit Portfolio

Typical Portfolio Loss Probability Density Function

UL = .0077, EC(99.9%) = .074
Dependence Model

• Gaussian Copula – Standard Model
  – Normalized ‘Asset Return’ to Horizon modeled as a standard Normal random variable
  – Default (or downgrade) occurs when return is below a threshold
  – Value at Horizon is a function of the Asset Return
  – Multi-variate Normal distribution with correlation matrix P described the joint credit migration behavior

• Factor Models

\[ z_i = \sqrt{\rho_i} \beta_i^T \varepsilon_F + \sqrt{1 - \rho_i} \varepsilon_{I,i} \]

\[ \rho_{ij} = \sqrt{\rho_i \rho_j} \beta_i^T \beta_j \]
In matrix notation, the factor model is

$$z = \Gamma^{1/2} B \hat{\delta}_F + [I - \Gamma]^{1/2} \hat{\delta}_I$$

The correlation matrix can be represented as

$$C = \Gamma^{1/2} B B^T \Gamma^{1/2} + I - \Gamma$$
Credit Migration

A credit state described by a rating \( R_{g}(T) \) can change with probability given by a ratings transition matrix. The asset return draw can be used to sample from the distribution of ratings at Horizon.

\[
\sum_{j=1}^{m-1} \Pr(R_{g}(T_{0}) \rightarrow R_{g_{j}}) \leq \Phi(\varepsilon) < \sum_{j=1}^{m} \Pr(R_{g}(T_{0}) \rightarrow R_{g_{j}}) \Rightarrow \]

\[
R_{g}(T_{H}) = R_{m}
\]

Correlated asset returns give correlated credit migration. Value is often modeled as a function of credit rating.
Monte Carlo Simulation for Portfolio Risk Analysis
Standard Monte Carlo Approach to Credit Portfolios

- Compute value of each instrument, and thus value of portfolio, at T0 based on credit state.

- **Simulation Loop:**
  - Sample Systemic Factors $z$ and idiosyncratic risk for each instrument modeled as independent standard Normal random variables
  - Compute correlated Asset Returns for each instrument
  - Compute value of each instrument at Horizon as a function of its credit state at Horizon as determined by its Asset Return
  - Compute value of the portfolio at Horizon
  - Record as sample from Return/Loss Distribution

- **Compute risk statistics such as UL and Economic Capital based on samples from the Loss Distribution.**
An Importance Sampling Method for Credit Portfolio Simulation
Importance Sampling

- Density function $\phi_{\theta_0}(z)$ with parameter dependence on $\theta_0$

$$\mu = E_{\theta_0}[h] = \int h(z)\phi_{\theta_0}(z)dz$$

$$W_{\theta}[z] = \frac{\phi_{\theta_0}(z)}{\phi_{\theta}(z)}$$

$$\mu = E_{\theta}[W_{\theta}h] = E_{\theta_0}[h]$$

$$m_h(\theta) = E_{\theta}[(W_{\theta}h)^2] = E_{\theta_0}[W_{\theta}h^2]$$

$$\sigma^2_h(\theta) = m_h(\theta) - \mu^2$$
Monte Carlo with Importance Sampling

\[ \mu_h(M, \theta) = \frac{1}{M} \sum_{i=1}^{M} W_\theta(z_i)h(z_i) \]

\[ E \left( \left( \mu - \mu_h(M, \theta) \right)^2 \right) = \frac{\sigma_{h}^{2}(\theta)}{M} \]

For credit portfolio problem

\[ h(z) = \begin{cases} 
1 & \text{if } L(z) > L^* \\
0 & \text{if } L(z) \leq L^* 
\end{cases} \]
Gaussian Covariance Importance Sampling

\[
C(\theta, Q) = C + \sum_{i=1}^{N} \frac{\theta_i}{1 - \theta_i} q_i q_i^T
\]

\[
Q^T C^{-1} Q = I
\]

\[-\infty < \theta_i < 1\]
Gaussian Covariance Importance Sampling

- Original idea: increase number of samples in tail of loss distribution by increasing volatility of asset returns by scaling up correlation matrix volatility. Scaling in all directions (i.e. multiply matrix by a scale up factor) doesn’t work – creates too many extremely unlikely samples.

- Better idea: Scale only in one direction – direction of largest eigenvalue.

\[ q_1 = \text{largest eigenvector} \]

\[ \theta_1 = 0.5 \]

- Works pretty well for many credit portfolios (30 times variance reduction), but this doesn’t consider the function \( h(z) \).

- Can the direction(s) and scale factor be chosen better?
Optimal Scaling in Gaussian Covariance IS

\[ W(z, \theta, Q) = \frac{\phi_C(z)}{\phi_{C(\theta,Q)}(z)} \]

\[ W(z, \theta, Q) = \prod_{i=1}^{N} f_i(z, \theta_i, q_i) \]

\[ f_i(z, \theta_i, q_i) = \frac{1}{\sqrt{1-\theta_i}} \exp\left(-\frac{\theta_i}{2}(q_i^T C^{-1} z)^2\right) \]

\[ \Delta_\theta W > 0 \rightarrow \Delta_\theta m_h(\theta) > 0 \]
If \( z \) is a sample from the original correlation matrix (e.g. obtained using the factor model formulation), a sample from the Importance Sampling distribution can be obtain as follows, and the weight function can also be easily computed:

\[
\begin{align*}
  z_\theta &= z + \sum_{i=1}^{N} \frac{1 - \sqrt{1 - \theta_i}}{\sqrt{1 - \theta_i}} \left( q_i^T C^{-1} z \right) q_i \\
  \left( q_i^T C^{-1} z_\theta \right)^2 &= \frac{1}{1 - \theta_i} \left( q_i^T C^{-1} z \right)^2
\end{align*}
\]
Open Questions

• For a given set of directions Q, find an algorithm for determining the optimal scaling factors $\theta$.

• For a single direction, find an algorithm for determining the option direction and scaling factor.

• For multiple directions, determine the optimal number of directions to use and how to find them.

• When the original correlation matrix is taken to be the identity matrix, a large number of directions may be considered. How can the optimal scaling be determined for this case? Is it better to use a small number of directions in the correlated case or a large number in the uncorrelated case?
Simple Example
Simple Example: Discrete First Passage Time

- Consider standard Brownian Motion $Z(t)$ on the time interval $[0,T]$ taking $T = 1$ with $Z(0) = 0$. Let $b > 0$ be a barrier. For equally spaced time steps over the interval, define:

$$Z = \begin{bmatrix} Z(t_1) & Z(t_2) & \cdots & Z(T = t_N) \end{bmatrix}$$

$$h(Z) = \begin{cases} 
1 & \text{if } \max(Z) > b \\
0 & \text{if } \max(Z) < b 
\end{cases}$$

$$Z \sim N(Z \mid 0, C)$$

$$C_{i,j} = \Delta t \min(i, j)$$
Simple Example (First Passage): One Step
Simple Example (First Passage): 100 Steps
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