Chapter 8

OBJECTIVES

After studying this chapter, you will be able to:

- Understand the meaning of frequency response
- Discuss the concept of resonance
- Describe the various passive filter designs
- Create a Bode plot and understand the significance

KEY TERMS

- Phase
- Reactance
- Bode Plot
- Resonance
- Critical Frequency
- Decibels
8-1 Introduction to Frequency Analysis

If a sinusoidal signal, with amplitude of 1V, is passed through a resistive circuit, such as Figure 8-1, and the frequency is changed from a very low value (i.e. 1 Hz) to a higher value (i.e. 100 kHz), there is no observable change in the output due to this change in frequency.

** Insert Figure 8-1** (resistive network)

However, it was seen in Chapter 7 that if we add a capacitor to this circuit, as shown in Figure 8-2, there would be a significant change in output amplitude when the frequency is changed from 1 Hz to 1 kHz. When the frequency is increased by another order of magnitude, to 10 kHz, then the amplitude is reduced even more. Beyond 100 kHz, the amplitude is practically zero.

** Insert Figure 8-2**

If the input and output are compared at a certain frequency, such as 10 kHz, not only is the amplitude changed, but also it appears that the output is “shifted” to the right (see Figure 8-3).

** Insert Figure 8-3**

This change in amplitude and shifting will also result if a circuit involving resistors and inductors are created. Why? Remember, in chapter 5, capacitors and inductors are elements whose properties change with frequency. In chapter 7, this property, called
reactance was introduced. For a capacitor, the reactance is inversely proportional to the
frequency, while for an inductor, it is proportional to the frequency. These formulas are
\[ X_C = \frac{1}{\omega C} \quad ; \quad X_L = \omega L \] (8-1)

To understand what impact these formulas have on the frequency response of a circuit,
and therefore how to plot these changes with respect to frequency, the input will be a
sinusoid, whose frequency can vary from 0 to infinity. At very low frequencies, the
reactance is very high. In fact, at \( \omega = 0 \), the reactance of a capacitor is \( 1/0 \), or infinite.
This would be the same as saying the capacitor could be replaced with an open circuit.
Replacing the capacitor with an open circuit in Figure 8-2 would result in the circuit in
Figure 8-4.

** Insert Figure 8-4 **

Since the output is an open circuit, there is no current flowing through the resistor, and by
applying Kirchoff’s Voltage Law (KVL) the output voltage equals the input voltage.
This can be considered true not only at \( \omega = 0 \), but also at very low frequencies. What is
meant by very low frequencies is that these frequencies are significantly lower than a
specific frequency, which will be called the cutoff frequency.

When performing a frequency analysis on a circuit or system, the ratio that is most
commonly used is \( \frac{V_{\text{output}}}{V_{\text{input}}} \). Therefore, at very low frequencies, this ratio would be almost
one, since the voltage output is equal to the voltage input. At zero frequency, the ratio is exactly one.

At very high frequencies, which will be frequencies much higher than this cutoff frequency, the reactance of a capacitor approaches zero. In fact, at infinite frequency, the reactance is zero. Therefore, the voltage would approach zero, and the capacitor could be replaced by a short circuit. Figure 8-4 shows the RC circuit at very high frequencies.

**Insert Figure 8-5**

The ratio, \( \frac{V_{\text{output}}}{V_{\text{input}}} \), is also 0. What happens to this ratio between zero frequency and very high frequencies? Table 8-1 shows data taken from an experiment, where R = 1 k\( \Omega \) and C=1 \( \mu \)F. Figure 8-5 is a plot of relative amplitude, \( \frac{V_{\text{output}}}{V_{\text{input}}} \), vs. frequency.

**Insert Table 8-1 and Figure 8-6** (Table of RC and output/graph)

Note that the table and graph use \( f \), which is in units of Hz, as the frequency. In many cases, the filter plots will use the radian frequency, \( \omega \), with units of radians/s. This will produce a similar table and graph, with the exception that the frequencies will be different, since \( \omega = 2\pi f \). Also, since the relative amplitude, \( \frac{V_{\text{output}}}{V_{\text{input}}} \), can be greater or less than one (depending on if an amplifier is used), this ratio will also be called the gain of the circuit.
Example 8-1

Discuss how the circuit shown in Figure 8-7 would appear at frequencies significantly above and below the cutoff frequency.

Since a capacitor could be replaced by an open circuit at very low frequencies, the output voltage is zero, and therefore the gain is zero. At very high frequencies, the capacitor can be replaced by a short circuit, the output voltage would equal the input voltage, and the gain would be one.

The problem with the graph shown in Figure 8-6 is that the region with significant information is from 0 to about 5 kHz. However, if the frequency axis were limited to that range, the changes in the output beyond 5 kHz would not be observed. What can be done?

One possibility would be to use a logarithmic plot for the x-axis, which is the frequency axis. Since the log of 10 is equal to 1, and the log of 1000 is equal to 3, the x-axis can be converted to powers of 10. When the frequency is plotted on the x-axis, the spacing between 1 and 10 Hz is the same as the spacing between 10 and 100 Hz. Figure 8-7 shows the same frequency plot when the x-axis is a logarithmic scale. Note that the changes in the gain for the range between 0 to 100 Hz are just as visible as the changes in the gain for the range between 100 and 1000 Hz. This is a better way of viewing the data, and will form the basis for the Bode plot in section 8-4.
What this figure shows is that the relative amplitude (or gain) changes with frequency, and goes from 0 to 1. This changing of gain with frequency is called filtering.

8-2 Types of Ideal Filters

The filter whose response is shown in Figure 8-7 has the characteristic that the relative amplitude decreases as the frequency increases. Another way of describing this filter action is to say that low frequencies are passed through the circuit, while high frequencies are attenuated. This type of filter is called a low pass filter. Since the gain of the circuit does not exceed 1, this circuit is also considered a passive filter. Passive filters are composed of resistors, capacitors, resistors and/or inductors.

When an active device, such as an operational amplifier, is used as part of the filter, the gain can be greater than one. This filter is called an active filter. One other advantage of an active filter is that you can construct any type of filter with just resistors and capacitors, thus eliminating the relatively large inductors.

The response that is shown in Figure 8-7 is a response of one type of filter – the low pass filter. Note that as the frequency increases, the relative amplitude gradually decreases. However, if an “ideal” low pass filter, with a maximum gain of one is drawn, such as in Figure 8-8, the gain is either one or zero. This filter separates the input voltage into two regions – one region where the input voltage is passed unchanged to the output, and
another region where the input voltage is reduced to zero, and therefore the output voltage is zero.

**Insert Figure 8-7**

The frequency where the transition between one and zero gain occurs is called the cutoff frequency. Note that in Figure 8-6, the frequency axis is expressed as the angular frequency, $\omega$, whose units are radians/s. This axis can also be modified to show units in Hz, by dividing $\omega$ by $2\pi$. The region where the signal is passed through (or the gain is one) is called the passband. The region where the signal is eliminated (or the gain is zero) is called the stopband.

This type of filter is also called a brickwall filter. Imagine a person running along the x-axis, from high frequencies to low frequencies. Eventually, that person would hit a brickwall. This filter is an ideal filter, and can’t be created with simple passive components.

There are actually four different types of filters. Figure 8-9 shows an ideal high pass filter. In this circuit, the passband exists above the cutoff frequency, not below the cutoff frequency as in the low pass filter.

**Insert Figure 8-9**

How can these filters be used? Imagine a radio station was playing a CD with two people singing. One person had a very high pitched voice, and one had a very deep voice.
Using the ideal high pass filter, the high pitched voice would be heard very clearly, and the deep voice would barely be heard.

There are two additional filters – the bandpass and the bandstop (or notch). The bandpass filter (see Figure 8-10) actually has two cutoff frequencies – a low and a high cutoff frequency. This type of filter has a passband that is the range of frequencies between the low and high cutoff, and two stopbands – one below the low cutoff frequency and one above the high cutoff frequency. The opposite type of filter would have a stopband between a low and high cutoff frequency, and two passbands - one below the low cutoff frequency and one above the high cutoff frequency. This type of filter is called a **bandstop** filter (see Figure 8-11). It is also known as a notch filter.

* **Insert Figure 8-10 and 8-11**

Both these types of filters are very useful. A notch filter could be used to eliminate certain frequencies which cause noise, such as a 60 Hz notch filter. A bandpass filter could eliminate both high frequency and low frequency noise, and only pass sounds that you want to hear.

There is another term that is used when describing a filter – this term is the **order** of the filter. For a passive filter, the order of the filter is based on the number of independent capacitors or inductors in the circuit. Independent capacitors and inductors mean that they can’t be combined – they are neither in series nor in parallel. The filter of Figure 8-2, since it has only one capacitor, is called a first order filter.
When a circuit is built with a single inductor, capacitor, and a resistor, it is called a second order filter. This type of filter is a better approximation to the ideal filter than the first order, and depending on the positioning of each component, this type of filter can approximate any of the four types of filter described previously. This filter has a cutoff frequency and a quantity called the quality factor $Q$. This quality factor controls the shape of the frequency plot in the transition region between the passband and the stopband. For a simple RLC circuit, the quality factor can be calculated as

$$Q = \sqrt{\frac{L}{R^2C}} \quad (8-2)$$

When plotting a frequency plot of a second order filter, the various $Q$ values will cause a peaking at the cutoff frequencies. For example, looking at the frequency plot of an active, second order low pass filter, as $Q$ is increased, the “peaking” at the cutoff frequency also increases (see Figure 8-12)

**Insert Figure 8-12**

**8-2 Decibels**

The problem with measuring gain is that the numbers can be very large or very small, depending on whether the input is being attenuated or amplified. In the last section, plotting the logarithm of the frequency solved the problem of plotting a wide range of frequencies. A similar method of expressing large ratios for gain can be developed, and
is called the decibel (dB). For power, the decibel is defined as 10 times the logarithmic ratio of the power gain.

\[ dB = 10 \log \left( \frac{P_{output}}{P_{input}} \right) \]  

(8-3)

In many cases, voltage gain will be calculated rather than power gain. Since

\[ power = \frac{V^2}{R} \]  

(8-4)

then substituting equation 8-3 into 8-2 yields

\[ dB = 10 \log \left( \frac{V^2_{output}}{V^2_{input}} \right) = 10 \log \left( \frac{V_{output}}{V_{input}} \right)^2 \]  

(8-5)

Since one of the properties of logarithms is that \( \log x^2 = 2 \log x \), then

\[ dB = 20 \log \left( \frac{V_{output}}{V_{input}} \right) = 20 \log(gain) \]  

(8-5)

This is the equation which will be used in the Bode plot.

**Example 8-2**

The table for the low pass filter shown in Figure 8-2 is shown below. Note that the frequency is shown as \( w \), in radians/s. Create a new table showing the gain in decibels, and then plot this graph.

**Insert Table 8-2**

The first gain in the table, at \( w=0 \), is 1. To convert to decibels, first take the log of 1, which is zero. The next gain, at 10 radians/s, is 20 log (.995), which becomes -.04 dB.
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At 1000 radians/s, the gain is .707, and $20 \log (.707)$ is equal to $-3.01$ dB. The new table is shown in Table 8-3, and the new plot is shown in Figure 8-13.

**Insert Table 8-3 and Figure 8-13**

8-3 Bode Plots

In the 1930s at Bell Labs, H.W. Bode developed a means of plotting amplifier performance. A Bode plot actually consists of two plots – a plot of gain versus frequency and a plot of phase angle versus frequency. For both plots, the x-axis, or the frequency axis, is a log-frequency horizontal axis. This expands the ability to see a large range of frequencies.

The magnitude, or gain, versus frequency uses decibels (dB) for the y-axis. This is also to expand the ability to see a wide range of gains. The y-axis for the phase plots is in degrees.

Looking at Figure 8-13, which is the gain Bode plot of the RC circuit in Figure 8-2, the initial gain is 0 dB. This represents the maximum gain, $\frac{V_{\text{output}}}{V_{\text{input}}}$, of a passive circuit, which is 1. As the frequency increases, the gain in dB decreases, becoming increasingly negative. At a certain frequency, the gain is $-3$ dB, which is an important frequency for Bode plots. This frequency is called the critical frequency, or breakpoint frequency. It
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is also known as the **3dB point**. The 3dB point is that frequency at which the output power is equal to half of the maximum output power, or 3 dB down from the maximum. Another way to express this critical frequency is that it is the frequency at which the output voltage has an amplitude which is 0.7071 of the input voltage. Beyond this frequency, the gain decreases even further.

Where does it end? Example 8-3 will show how to find the limit in decibels.

**Example 8-3**

At very high frequencies, the ratio $\frac{V_{\text{output}}}{V_{\text{input}}}$ approaches zero. The question is what would that value be in dB?

To calculate that value, consider starting from a gain of 1. The dB equivalent, by calculating $20 \log \left( \frac{V_{\text{output}}}{V_{\text{input}}} \right)$, is 0 dB. As frequency increases, the gain decreases. At a value of 0.1, the decibel equivalent is $20 \log (.1)$, which is the same as $20 \log (10^{-1})$, or –20 dB. Table 8-4 shows the results of continuing these calculations.

**Insert Table 8-4**

Looking at this table, as the ratio $\frac{V_{\text{output}}}{V_{\text{input}}}$ approaches zero, the gain in dB approach minus infinity.
8-4 Phase and the Derivation of Bode Plots

The Bode plot can be derived theoretically from the circuit, by using the complex impedance formulas for capacitors and inductors (see Chapter 7 for detail), and can also be derived experimentally. To derive the Bode plot theoretically, first redraw the circuit with the impedance values next to each component.

- For a resistor, it is just the value of the resistor.
- For a capacitor, it is \( \bar{Z}_C = \frac{1}{j\omega C} \)
- For an inductor, it is \( \bar{Z}_L = j\omega L \)

Note that there is a term j in the impedance expression for both the capacitor and inductor. Remember that j is equal to \( \sqrt{-1} \), and therefore the impedance is a complex number. That is why there is a bar above the Z for both \( Z_C \) and \( Z_L \).

Phase

Appendix B3 explains in detail complex numbers, and how to work with two forms – rectangular and polar. What will happen when complex numbers are used is that the Bode plot will not only consist of a magnitude plot versus frequency, but also will have a phase plot versus frequency. Remember, from Chapter 7, phase is expressed in either degrees or radians, and can be viewed as the relative position of where a sinusoid (since Bode plots use sinusoids) begins. Looking at Figure 8-14, the sine wave labeled A would have a phase angle of 0°, while the sine wave labeled B would have a phase angle of −90°.
These two values will actually be the limits for phase for the circuit in the next section. At each frequency, from almost 0 Hz to a very large number, the phase will change, and how this can be calculated will also be shown in the next section. However, one question is how to measure phase vs. frequency experimentally, since the magnitude vs. frequency has already been discussed.

To measure phase experimentally, consider Figure 8-15. The distance between peaks, for waveform A, is the period of the waveform, and would be in units of time (s, ms, or µs). Since a sine wave goes through 360° in one cycle, or one period, then the distance in time for the period is also the same as 360°. If the distance between the peaks of the two waveforms, A and B, is called \( t_1 \), then the ratio

\[
\frac{t_1}{T} = \frac{\theta}{360°}
\]

(8-6)

can be formed, where \( \theta \) is the phase angle between the two waveforms. Therefore,

\[
\theta = 360° \left( \frac{t_1}{T} \right)
\]

(8-7)

Thus, both the output and the input of a circuit can be viewed on an oscilloscope, and the ratio of the magnitudes (in dB) can be determined, and the phase between the waveforms can be determined, for each frequency change. This would provide an experimental method to obtain a Bode plot.
Note that in Figure 8-15, waveform A reaches its peak before waveform B. Another way of stating this is to say that A leads B by \( \theta \) degrees, or the phase difference for B is negative.

**Example 8-4**

For the waveforms shown in Figure 8-16, determine the phase angle \( \theta \), between the output waveform, B, and the input waveform, A.

**Insert Figure 8-16**

From Figure 8-16, the period of the input waveform, A, is 20 ms, and the distance between A and B is 2 ms. Since the peak of A occurs before the peak of B, the phase angle will be negative (B lags A). The angle is found by multiplying the ratio 2/20 by 360\(^\circ\), which results in a phase angle of 36\(^\circ\). Therefore, \( \theta = -36^\circ \).

The gain, \( \frac{V_{\text{output}}(\omega)}{V_{\text{input}}(\omega)} \), is a function of frequency, and can be derived using standard circuit theorems.

**LOW PASS FILTERS - MAGNITUDE**

Consider the RC circuit shown in Figure 8-14. Derive the Bode plot (both magnitude and phase).

**Insert Figure 8-17**

First, redraw this circuit, placing the impedances next to each element (see Figure 8-15).
This circuit can be analyzed using the voltage division method, where the impedances are used rather than resistance. Therefore, the output voltage would be

$$
\bar{V}_{out} = \frac{Z_c}{R + Z_c} \bar{V}_{in}
$$

(8-8)

The reason the input and output voltages have a bar on top of the letter is that they are also complex, having both a magnitude and a phase. Since $Z_c = \frac{1}{j \omega C}$, equation 8-8 can be modified as shown in equation 8-9.

$$
\frac{\bar{V}_{out}}{\bar{V}_{in}} = \frac{1}{j \omega C} \frac{1}{R + \frac{1}{j \omega C}}
$$

(8-9)

This can be simplified to

$$
\frac{\bar{V}_{out}}{\bar{V}_{in}} = \frac{1}{1 + j \omega CR} = \frac{1}{1 + j \frac{\omega}{\omega_0}}
$$

(8-10)

where $\omega_0 = \frac{1}{RC}$

What does $\omega_0$ represent? For this simple circuit, it represents the critical frequency.

At this point, the gain is complex. Since equation 8-10 shows a division of two complex numbers, from Appendix B3 the polar form (magnitude and phase) needs to be used. Is the numerator, with a value of 1, a complex number? If the answer was no, you are wrong. The number 1 can also be written as a magnitude of 1, and a phase angle of 0.

For the denominator, remember that to calculate the magnitude of a complex number, the real portion is squared, added to the imaginary portion squared, and then the square root
is taken. Therefore, the magnitude of the gain, which will be represented by $G(\omega)$, becomes

$$G(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

(8-11)

In the previous section, gain was expressed in decibels. To convert the results in equation 8-11, take 20 times the log of the results, or

$$G(\omega) = 20 \log \left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}\right)$$

(8-12)

where $\omega_0 = \frac{1}{RC}$

What to do now. The “simplest” way would seem to just plug in numbers. The cutoff frequency $w_0$ is found by multiplying the value of the resistor (in ohms) by the value of the capacitor (in farads).

**Example 8-5**

If the values for the circuit in Figure 8-17 were $R = 1 \, k\Omega$ and $C = 1 \, \mu F$, calculate the cutoff frequency. How would this change if $R = 10 \, k\Omega$?

The value of $R$ can also be expressed as $10^3 \, \Omega$, while the value of $C$ can be expressed as $10^{-6} \, F$. For most standard values of circuits, using powers of ten can be very helpful.

The cutoff frequency would be
\[
\omega_0 = \frac{1}{RC} = \frac{1}{10^7 \times 10^{-6}} = \frac{1}{10^{-3}} = 1000 \text{ radians/sec}
\]

Since \( \omega = 2\pi f \), the answer can also be calculated in terms of Hz. An alternative answer would be \( f_0 = 159 \text{ Hz} \).

If the value of the resistor were to increase by a factor of 10, to 10 k\( \Omega \), then from the equation for \( \omega_0 \), the frequency would decrease by a factor of 10. Therefore, the answer to the second part would be 100 radians/s, or 15.9 Hz.

To obtain \( G_{\text{dB}}(\omega) \), as stated before, various values of \( \omega \) can be used to calculate this gain as a function of frequency. The question is what values of the frequency should be used?

The answer is to use several values that are significantly below the cutoff value, several values that are just below and above the cutoff value, and several values that are significantly above the cutoff value. What will be obtained is a magnitude plot very similar in shape to Figure 8-13, with the cutoff frequency calculated as \( \omega_0 \). However, there is a way of approximating the shape of the magnitude plot, without calculating all the values.

From equation 8-12,

\[
G(\omega) = 20 \log \left( \frac{1}{\sqrt{1 + \left( \frac{\omega}{\omega_0} \right)^2}} \right) = -20 \log \left( \sqrt{1 + \left( \frac{\omega}{\omega_0} \right)^2} \right)
\]

since \( 20 \log 1 \) is equal to zero.
Consider three different ranges of frequency – frequencies much below the cutoff frequency, a frequency at the cutoff frequency, and frequencies much higher than the cutoff frequency.

For frequencies much below the cutoff frequency, the ratio \( \frac{\omega}{\omega_0} \) is very small, and can be considered zero. Therefore, equation 8-13 becomes

\[
G(\omega) = -20 \log \left( \frac{\omega}{\omega_0} \right)^2 = -20 \log(1) = 0 \text{ dB} \quad (8-14)
\]

Therefore, for frequencies much below the cutoff frequency, the gain is 0 dB. If the cutoff frequency was 1000 radians/s, then this would be true for frequencies below at least 100 radians/s.

Note that since there is a ratio \( \frac{\omega}{\omega_0} \), this would be the same as \( \frac{2\pi f}{2\pi f_0} \) or \( \frac{f}{f_0} \). Thus, if the cutoff frequency was converted to Hz, the three ranges that are discussed here can also be thought of in Hz.

What happens when the cutoff frequency is reached? The ratio \( \frac{\omega}{\omega_0} \) becomes 1, and equation 8-13 becomes

\[
G(\omega) = -20 \log \left( 1 + \left( \frac{\omega}{\omega_0} \right)^2 \right) = -20 \log \left( \sqrt{2} \right) \quad (8-15)
\]

This value is approximately –3 dB, and therefore many texts will call the cutoff frequency the 3 dB point.
Finally, what happens at frequencies much above the cutoff frequency. At these frequencies, the ratio $\frac{\omega}{\omega_0}$ becomes much larger than 1, and the expression

$$\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

becomes $\frac{\omega}{\omega_0}$. Therefore, equation 8-13 becomes

$$G(\omega) = -20 \log \left( \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \right) = -20 \log \left( \frac{\omega}{\omega_0} \right)$$

(8-16)

When you plot this equation on axes that consist of dB for the y-axis and log $\omega$ as the x-axis, the results in equation 8-16 represent a straight line. The slope of this line can be determined by consider a frequency, $\omega$, that is 10 times larger than $\omega_0$. The ratio becomes 10, and $-20 \log(10)$ becomes $-20$ dB. Thus, for a change of 10 times the frequency in the x-axis, the gain decreases by a factor of 20 dB. The change in frequency by a factor of 10 is also called a **decade**. Therefore, the slope of the line, above the cutoff frequency, is $-20$ dB/decade.

Another way of describing certain ratios is to describe what happens when the frequency is doubled. This is also called an octave, and for the same circuit in Figure 8-17, above the cutoff frequency, the change in gain can also be written as $-6$ dB/octave.

For example, if certain values of the resistor and capacitor were chosen so that the circuit in Figure 8-17 had a cutoff frequency of 200 Hz (note that the cutoff frequency is expressed here in Hz rather than radians/s), then the gain at 200 Hz would be $-3$ dB. What would be gain be at 400 Hz? Since 400 Hz is twice 200 Hz, then it is one octave,
and the gain would be reduced by 6 dB. Therefore, the gain at 400 Hz would be \(-3\) dB + \(-6\) dB), or \(-9\) dB. What about at 2000 Hz? Since that exactly ten times 200 Hz, it is an decade, and the gain change is \(-20\) dB. Therefore, the actual gain at 2000 Hz would be \(-23\) dB.

**Example 8-6**

If the cutoff frequency for an RC circuit, as in Figure 8-14, was 2 kHz, what is the gain at 8 kHz, and at 40 kHz?

First, look at the ratio of 8 (from 8 kHz) to 2 (from 2 kHz) – it is four. Then think of ratios that are powers of either two (octaves) or ten (decades). If 2 is doubled, it becomes 4. If 4 is doubled, it becomes 8. Therefore, 8 kHz is 2 octaves above 2 kHz. Since 2 kHz is the cutoff frequency, the gain at that frequency is \(-3\) dB. Since the gain changes by \(-6\) dB/octave, at 8 kHz it would have changed by \(-12\) dB. Since the gain was originally \(-3\) dB at 2 kHz, then the actual gain at 8 kHz is \(-15\) dB.

The same process occurs in finding the gain for the frequency 40 kHz. If the cutoff frequency is multiplied by 10, it becomes 20 kHz. How to get to 40 kHz? Multiply by 2. Therefore, 40 kHz is a decade and an octave above 2 kHz, the cutoff frequency. The gain, above the cutoff frequency, changes by \(-20\) dB/decade, or \(-6\) dB/octave, the total change from the cutoff frequency is \(-26\) dB. Since the cutoff frequency gain started at \(-3\) dB, the actual gain at 40 kHz would be \(-29\) dB.
What has been reviewed so far is the magnitude plot, calculated theoretically. The phase plot will now be calculated.

**LOW PASS FILTERS - PHASE**

To calculate the phase angle, consider the equation 8-10

\[
\frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j\frac{\omega}{\omega_0}}
\]

where \( \omega_0 = \frac{1}{RC} \)

How to calculate the phase angle, \( \theta(\omega) \), which is a function of frequency? From Appendix B3, if the gain is in the polar form (magnitude and phase), then phase can be calculated. The formula for calculating phase from a complex number (rectangular form) is

\[
\theta = \tan^{-1}\left(\frac{b}{a}\right), \text{ where the complex number is } a + jb
\]

(8-18)

The numerator has a value of 1, which can be written as 1\(\angle\)0

Degree. The phase of the denominator can be written as \( \tan^{-1}\left(\frac{\omega}{\omega_0}\right) = \tan^{-1}\left(\frac{\omega}{\omega_0}\right) \). To calculate the final phase, subtract the phase of the denominator from the phase of the numerator. Therefore,

\[
\theta(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)
\]

(8-19)

Before plotting this as a continuous function of \( \omega \), again consider Consider three different ranges of frequency – frequencies much below the cutoff frequency, a frequency at the cutoff frequency, and frequencies much higher than the cutoff frequency.
For frequencies much below the cutoff frequency, the ratio $\frac{\omega}{\omega_0}$ is very small, and can be considered zero. Therefore, the phase from equation 8-19 becomes $0^\circ$.

What happens when the cutoff frequency is reached? The ratio $\frac{\omega}{\omega_0}$ becomes 1, and the phase, from equation 8-19 becomes $-\tan^{-1}(1)$ or $-45^\circ$.

Finally, what happens at frequencies much above the cutoff frequency. At these frequencies, the ratio $\frac{\omega}{\omega_0}$ becomes very large, and the phase approaches $-90^\circ$.

Unfortunately, for phase, there is no simple slope (such as $-20$ dB/decade) that can be used to calculate the phase at various frequencies. The only alternative is to plot the phase versus frequency. Figure 8-19 shows the plot of $\theta(\omega)$ for $\frac{\omega}{\omega_0}$ ranging from 0 to 10.

**Insert Figure 8-19**

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**Example 8-7**

For an RC circuit, as in Figure 8-14, the resistor is 1 kΩ, and the capacitor is 1 µF. Approximately what is the phase at 320 Hz?
First, the cutoff frequency needs to be calculated. Since \( \omega = \frac{1}{RC} \), then \( \omega = 1000 \) radians/sec. Also, \( f = \frac{\omega}{2\pi} \), and the cutoff frequency in Hz is equal to \( \frac{1000}{2\pi} \), or 318 Hz.

Since 320 Hz is almost the same value as the cutoff frequency, the phase would be \(-45^\circ\).

**High Pass Filter**

If the resistor and capacitor from Figure 8-17 were reversed, as shown in Figure 8-20, a different filter would result. This filter is called a high pass filter.

**Insert Figure 8-20**

A similar analysis to the low pass filter can be performed. First, redraw this circuit, placing the impedances next to each element (see Figure 8-21).

**Insert Figure 8-21**

Then, using voltage division

\[
\bar{V}_{\text{out}} = \frac{\bar{Z}_c}{R + \bar{Z}_c} \bar{V}_{\text{in}} \tag{8-20}
\]

Substituting \( \bar{Z}_c = \frac{1}{j\omega C} \), equation 8-20 can be modified

\[
\frac{\bar{V}_{\text{out}}}{\bar{V}_{\text{in}}} = \frac{R}{R + \frac{1}{j\omega C}} \tag{8-21}
\]

This can be simplified to
\[
\frac{V_{out}}{V_{in}} = \frac{j\omega CR}{1 + j\omega CR} = \frac{j\frac{\omega}{\omega_0}}{1 + j\frac{\omega}{\omega_0}}
\]  

(8-22)

where \(\omega_0 = \frac{1}{RC}\)

Next the gain and phase, as a function of frequency, needs to be derived. The magnitude of the gain, \(G(\omega)\), in decibels, becomes

\[
G(\omega) = 20\log \left( \frac{\omega}{\omega_0} \right) \left[ 1 + \left( \frac{\omega}{\omega_0} \right)^2 \right]^{-1/2}
\]  

(8-23)

Another way to write equation 8-23, since \(\log(x) = \log(x) - \log(y)\), is

\[
G(\omega) = 20\log \left( \frac{\omega}{\omega_0} \right) - 20\log \left[ 1 + \left( \frac{\omega}{\omega_0} \right)^2 \right]
\]  

(8-24)

While this equation can be used to create a gain plot, first consider the three different ranges of frequency – frequencies much below the cutoff frequency, a frequency at the cutoff frequency, and frequencies much higher than the cutoff frequency.

For frequencies much below the cutoff frequency, the ratio \(\frac{\omega}{\omega_0}\) is very small, and can be considered zero. Therefore, equation 8-24 becomes

\[
G(\omega) = 20\log(0) - 20\log(1) = 20\log(0)
\]  

(8-25)

What is \(\log(0)\)? If a calculator is used, an error might result. To understand what this value is, first consider \(20\log(1)\), which is 0. What is the value of \(20\log(0.1)\), which is the
same as \(20\log(10^{-1})\)? It is \(-20\) dB, since \(\log(x^n) = n\log(x)\). What about \(20\log(0.01)\)? It is \(-40\) dB.

What is happening is that for \(20\log(x)\), as \(x\) becomes very small, \(20\log(x)\) becomes a very large negative number. In fact, as \(x\) approaches 0, \(20\log(x)\) approaches \(-\infty\) dB.

Thus, at very low frequencies, for this filter, the gain is a very large negative number, which means that the output is attenuated greatly.

What happens when the cutoff frequency is reached? The ratio \(\frac{\omega}{\omega_0}\) becomes 1, and equation 8-24 becomes

\[
G(\omega) = 20\log\left(1 - 20\log\left(\sqrt{1 + (1)^2}\right)\right) = -20\log\left(\sqrt{2}\right)
\]  

(8-26)

This value is approximately \(-3\) dB.

Finally, what happens at frequencies much above the cutoff frequency? At these frequencies, the ratio \(\frac{\omega}{\omega_0}\) becomes much larger than 1, and the expression

\[
\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}
\]

becomes \(\frac{\omega}{\omega_0}\). Therefore, equation 8-24 becomes

\[
G(\omega) = 20\log\left(\frac{\omega}{\omega_0}\right) - 20\log\left(\sqrt{\frac{\omega}{\omega_0}}\right)^2 = 0\ dB
\]   

(8-27)

Therefore, for this filter, the magnitude plot starts at a very negative dB value, and eventually goes to 0 dB, as shown in Figure 8-22. Since this filter “passes” high frequency signals (minimally attenuates the input) and significantly attenuates low frequency signals, this filter is known as a high pass filter.
To find the phase response for this filter, consider the complex gain from equation 8-22.

\[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{j\omega CR}{1 + j\omega CR} = \frac{j\frac{\omega}{\omega_0}}{1 + j\frac{\omega}{\omega_0}} \]

where \( \omega_0 = \frac{1}{RC} \)

As was done with the low pass filter, before plotting this as a continuous function of \( \omega \), consider the three different ranges of frequency – frequencies much below the cutoff frequency, a frequency at the cutoff frequency, and frequencies much higher than the cutoff frequency.

For frequencies much below the cutoff frequency, the ratio \( \frac{\omega}{\omega_0} \) is very small, and can be considered zero. Therefore, the phase from equation 8-19 becomes 0°.

What happens when the cutoff frequency is reached? The ratio \( \frac{\omega}{\omega_0} \) becomes 1. The phase angle for the numerator is 90°, since the numerator expression would be j. The phase angle for the denominator would be 45°. Therefore, the total phase angle is 90° – 45°, or 45°.

Finally, at frequencies much above the cutoff frequency, the ratio \( \frac{\omega}{\omega_0} \) becomes very large, and the phase approaches 90°.
The actual shape of the phase angle- frequency plot is similar to the low pass filter plot, except it is flipped around (see Figure 8-23).

** Insert Figure 8-23**