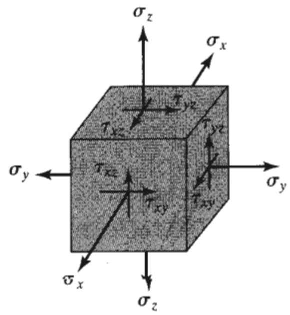
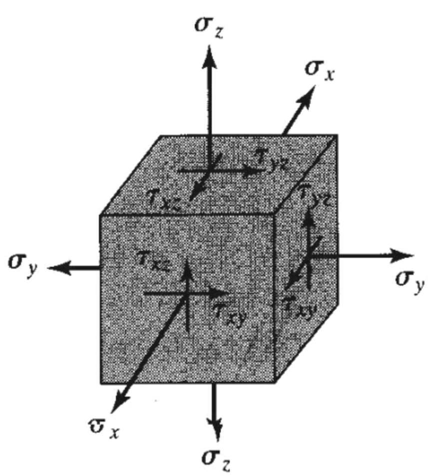
1-21 Stress in three dimensions

In three dimensions, in each orthogonal direction X, Y & Z, there could be one normal and two shear stresses. Thus the most generalized state stress at a point in 3D is as shown below. It is also conveniently described by a stress tensor as follows:

$$\text{Stress Tensor} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



It can be proved that if we orient the orthogonal axis system XYZ to a specific orientation, denoted by the directions 1, 2 & 3, the shear stress from all faces will vanish and there will be only normal stresses. These normal stresses are called **principal normal stresses, S_1, S_2 & S_3**.



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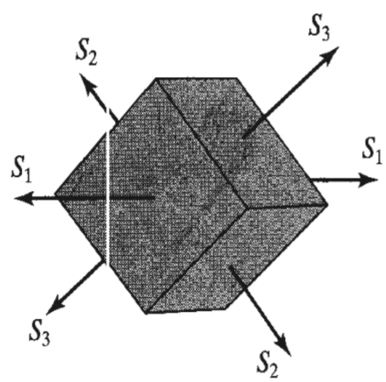


The values of the three principal normal stresses (S_1, S_2 & S_3) can be found from the three real roots of S of the following cubic equation:

$$\begin{aligned} S^3 &- (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})S^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{yz}^2 - \tau_{zx}^2 - \tau_{xy}^2)S \\ &- (\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{yz}\tau_{zx}\tau_{xy} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2) = 0 \end{aligned} \quad \text{..................(1)}$$

The values of S_1, S_2 & S_3 should include the maximum and minimum normal stresses.

Once we know the values of these three principal stresses, then we can consider two of them at a time (a 2 D problem) and find the true maximum shear stress τ_{max}.

$$\tau_{max} = \frac{Max\ of\ \left(|S_1 - S_2|, |S_2 - S_3|, |S_3 - S_1|\right)}{2} \quad \text{..(2)}$$

We can use two at a time, as a 2D stress, because of the fact that a stress does not have any affect in its orthogonal plane.

These S_1, S_2 & S_3 could be positive, zero or negative. Thus to determine the true τ_{max} we need to find the largest difference between two S's and then divide it by 2.

Example 1: Following example shows an application of the above method .

Let us assume the stress condition at a point in a part is given by the following tensor:

$$\begin{bmatrix} \sigma_{xx} = 50 & \tau_{xy} = 20 & \tau_{zx} = 0 \\ \tau_{xy} = 20 & \sigma_{yy} = 40 & \tau_{yz} = 0 \\ \tau_{zx} = 0 & \tau_{zy} = 0 & \sigma_{zz} = 0 \end{bmatrix}$$

It is a 2D stress condition, because shear stresses in zx and yz planes are zero and the normal stress in z plane is also zero. $\tau_{zx}=\tau_{yz}=\sigma_{zz}=0$.

Putting values in equation (1) we get:

$S^3 - (50+40)S^2 + (50\times 40 - 20^2)S = 0$; Products with zero terms are dropped.

$S^3 - 90S^2 + 1600S = 0$; this is still a cubic equation.

$S(S^2 - 90S + 1600) = 0$; that is $S_1 = 0$ and $S^2 - 90S + 1600 = 0$

$S^2 - 90S + 1600 = 0$ is a quadratic equation and has two roots

$$S_2, S_3 = \frac{90 \pm \sqrt{90^2 - 4\times 1600}}{2} = 65.6,\ 24.4$$

Thus the three principal stresses are 0, 65.6 & 24.4
Consequently true maximum shear stress

$$\tau_{\max} = \frac{Max\ of\ \left(|S_1 - S_2|, |S_2 - S_3|, |S_3 - S_1|\right)}{2}$$

$\tau_{max} = (65.6-0)/2 = 32.8$

[Notice that (65.6-24.4)/2, or (24.2-0)/2 does not provide true max shear stress τ_{max}]

Use of equation (1) and (2) to find the principal normal stresses for 2D stress situation is fairly easy, because we know one of the principal normal stress is zero and we only solve one quadratic equation to obtain the two roots.

Note: We could have used Mohr circle to find the two principal normal stresses S_2 & S_3, knowing that $S_1 = 0$ as no stress in Z direction.

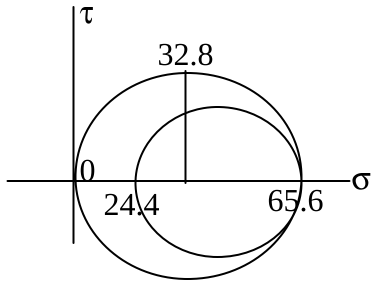
Center of the circle (average stress) = $(\sigma_{xx}+\sigma_{yy})/2 = (50+40)/2 = 45$

Radius of the circle = $\sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{50-40}{2}\right)^2 + 20^2} = \sqrt{5^2+20^2} = 20.6$

S_2=45+20.6 = 65.6

S_3=45-20.6 = 24.4

S_1 = 0



When stress is truly 3D

When there are either normal or shear stress in yz or zx plane, equation (1) becomes a cubic equation with a non-zero constant term (the last term in the expression). Unfortunately, there is no direct formula to determine the roots of such a cubic equation. You may try one of the following methods:

(1) Use the web tool in http://www.1728.com/cubic.htm.
(2) Use Module 1-5 in your textbook.
(3) Matlab software has a "root" function which can determine roots of a cubic equation.
(4) You can graph the value of the left hand side of the equation for many values of S in your graphing calculator and can find the three values of S which will cause the value of the expression to be zero.