# Finite difference numerical methods for 1-D heat equation; Iterative methods, Jacobi, Gauss-Seidel, SOR.

# 1-D Heat Equation

Heat Equation

$$\rho\, c \frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2},$$

where $k/(\rho\, c)$ is the **diffusivity** considered constant here. Consider the following boundary and initial conditions, for $0 < x < L$:

$$U(0,t) = U_0 \qquad U(L,t) = U_L,$$

known as **Dirichlet** boundary conditions (we can also specify a heat flux, Neumann boundary conditions, or combination thereof, known as mixed or Robin boundary conditions), and

$$U(x,0) = f(x) \qquad 0 < x < L.$$

# Fourier Series Solution to the 1-D Heat Equation

A solution to the homogeneous Dirichlet boundary conditions problem is

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 \left(\frac{k}{\rho c}\right) t}$$

where

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

**Explicit Method** $O(\Delta t, \Delta x^2)$

$$u_i^{n+1} = \left(\frac{k\Delta t}{\rho c \Delta x^2}\right)\left(u_{i+1}^n + u_{i-1}^n\right) + \left(1 - \frac{2k\Delta t}{\rho c \Delta x^2}\right) u_i^n$$

with $\frac{2k\Delta t}{\rho c \Delta x^2} \leq 1$.

**Implicit Method** $O(\Delta t, \Delta x^2)$

$$u_i^{n} = -\left(\frac{k\Delta t}{\rho c \Delta x^2}\right)\left(u_{i+1}^{n+1} + u_{i-1}^{n+1}\right) + \left(1 + \frac{2k\Delta t}{\rho c \Delta x^2}\right) u_i^{n+1}$$

**Algorithm for Jacobi Iterations**

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1, j\neq i}^{N} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n$$

**Algorithm for Gauss-Seidel Iterations**

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{N} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n$$

**Algorithm for Successive Over Relaxation**

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\omega}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{N} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n$$

A sufficient condition for convergence is $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$.