Finite difference numerical methods for Poisson equation; Multigrid methods

Poisson Equation

$$\nabla^2 U(x, y, z) = -f(x, y, z),$$

where $\nabla^2 U(x, y, z)$ is the Laplacian of U.
Consider the following 1-D boundary value problem:
$0 < x < 1$

$$\frac{d^2 U}{dx^2} = -f$$

$$U(0) = U(1) = 0,$$

Discretized as

$$-2u_i + (u_{i+1} + u_{i-1}) = -\Delta x^2 f_i, \quad i = 0, 1, \ldots, N$$

with

$$u_0 = u_N = 0$$

HW 1

Write

$$-2u_i + (u_{i+1} + u_{i-1}) = -\Delta x^2 f_i, \quad i = 1, 2, \ldots, N-1$$

as a simple Jacobi iteration, i.e.:

$$u_i^n = \left(u_{i+1}^{n-1} + u_{i-1}^{n-1} + \Delta x^2 f_i\right)/2, \quad n > 0$$

with

$$u_0 = u_N = 0$$

Then write a program to find the solution.

Consider an intermediate u_i, called u_i^*:

$$u_i^* = \left(u_{i+1}^{n-1} + u_{i-1}^{n-1} + \Delta x^2 f_i\right)/2,$$

and write

$$u_i^n = (1-\omega)u_i^{n-1} + \omega u_i^*$$

we will then have th weighted Jacobi methode

$$u_i^n = (1-\omega)u_i^{n-1} + \omega\left(u_{i+1}^{n-1} + u_{i-1}^{n-1} + \Delta x^2 f_i\right)/2, \quad n > 0$$

Use the weighted Jacobi method and compare your results with the simple Jacobi method. Find the optimal ω:

The above discretization leads to the matrix equation $\mathbf{A}x = b$, where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}.$$

$$x = [u_1, \ldots, u_{N-1}]^T$$

$$b = \Delta x^2 [f_1, \ldots, f_{N-1}]^T$$

Write $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$$

where $\mathbf{D}$ is the diagonal of $\mathbf{A}$, and $-\mathbf{L}$ and $-\mathbf{U}$ are the strictly lower and upper triangular parts of $\mathbf{A}$, respectively.
Show that in matrix form, the Jacobi method then amounts to:

$$u^n = \mathbf{D}^{-1}\left(\mathbf{L} + \mathbf{U}\right) u^{n-1} + \mathbf{D}^{-1}\Delta x^2 f$$

In matrix form, write the weighted Jacobi method.
Use your matrix multiplication method to solve for u above and compare the performance with your simple Jacobi method code in HW1.

Consider the problem:
$0 < x < 1$

$$\frac{d^2U}{dx^2} = f$$

$$U(0) = U(1) = 0,$$

Find the exact solution for $f = -(k\pi)^2 \sin(k\pi x)$.
For $k = 1, 2, 3, 4, 5, 6$, use the Jacobi method with $N = 65$ and plot the norm of error as a function of the number of iterations for $n_{max} = 150$.
For $n = 1, 2, 5, 10, 20$, plot the error at grid points $i = 1$ to $i = 64$ for $k = 1, 2, 3, 4, 5, 6$.
Explain your results.

Write

$$-2u_i + (u_{i+1} + u_{i-1}) = 0, \quad i = 1, 2, \ldots, N-1$$

as a simple Jacobi iteration, i.e.:

$$u_i^n = \left(u_{i+1}^{n-1} + u_{i-1}^{n-1}\right)/2, \quad n > 0$$

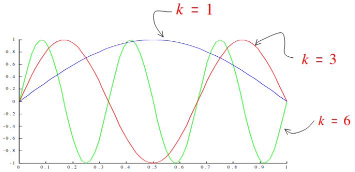
with

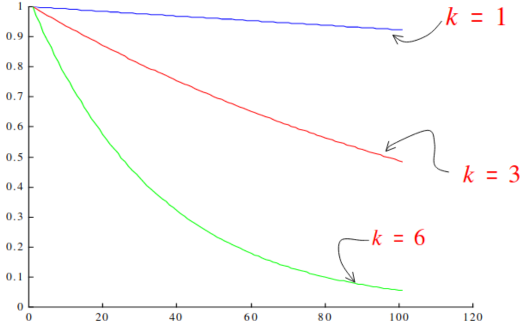
$$u_0 = u_N = 0$$

Use Fourier modes as initial iterates:

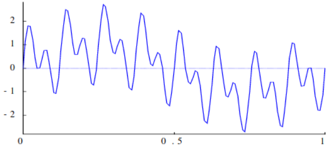
$$(u_i^0)_k = \sin(ik\pi/N)$$

Multigrid method: Introduction

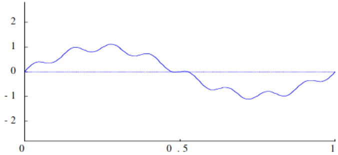




- Initial error: $v_{kj} = \sin\left(\frac{2j\pi}{N}\right) + \frac{1}{2}\sin\left(\frac{16}{N}\right.$

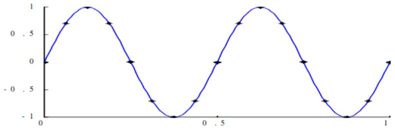


- Error after 35 iteration sweeps:

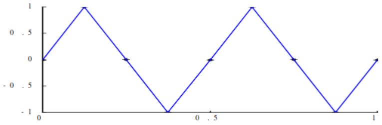


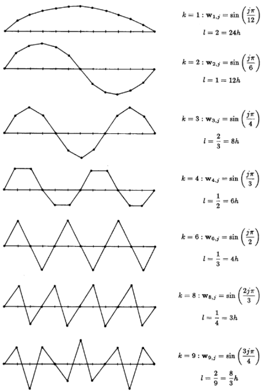
Multigrid method: Introduction

- A smooth function:



- Can be represented by linear interpolation from a coarser grid:

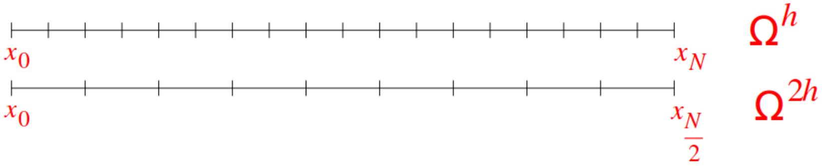




Multigrid method

Error appears to be relatively higher in frequency on the coarse grid, making modes of the error eliminated more effectively using Jacobi iteration, for example.

Coarse gird can be used to improve the initial guess for the fine grid.



Multigrid method

- Smoothing: reducing high frequency errors, for example using a few iterations of the Jacobi method.
- Restriction: downsampling the residual error to a coarser grid.
- Interpolation or prolongation: interpolating a correction computed on a coarser grid into a finer grid.

- Values at points on the coarse grid map unchanged to the fine grid.
- Values at fine grid points NOT on the coarse grid are the averages of their coarse grid neighbors.

$$\frac{1}{2}\begin{bmatrix} 1 & & \\ 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & 1 \\ & & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{2h} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}_h = \mathbf{v}^h$$

Multigrid method: Full weighting restriction - 1D

- Mapping from the fine grid to the coarse grid:

$$\frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}_h = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{2h} = \mathbf{v}^{2h}.$$

Full Multigrid method

- Apply the Jacobi iteration m times on Ω_h with initial guess v^h.
- Compute the fine grid h solution and restrict to the coarse gird $2h$.
- Compute the coarse grid solution.
- Interpolate the coarse grid solution to the fine grid.
- Apply the Jacobi iteration m times on Ω_h with initial guess v^h.

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Explain your results.