1.2.3 The thermal energy balance for an arbitrary slice of cross-sectional area $A(x)$ between $x = a$ and $x = b$ is:

$$\int_a^b \frac{\partial}{\partial t}[e(x,t)A(x)]dx = \phi(a,t)A(a) - \phi(b,t)A(b) \tag{1}$$

where $Q(x,t) = 0$. We can rewrite equation 1 as:

$$\int_a^b \frac{\partial}{\partial t}[e(x,t)A(x)]dx = -\int_a^b \frac{\partial}{\partial x}(\phi(x,t)A(x))dx \tag{2}$$

or

$$\int_a^b [\frac{\partial}{\partial t}(e(x,t)A(x)) + \frac{\partial}{\partial x}(\phi(x,t)A(x))]dx = 0 \tag{3}$$

For an arbitrary slice, the integrand must be zero:

$$\frac{\partial}{\partial t}(e(x,t)A(x)) + \frac{\partial}{\partial x}(\phi(x,t)A(x)) = 0 \tag{4}$$

replacing $e(x,t) = \rho c u(x,t)$ and $\phi(x,t) = -K_o \partial u/\partial x$ in 4 gives:

$$\frac{\partial u(x,t)}{\partial t} = \frac{K_o \rho c}{A(x)} \frac{\partial}{\partial x}(A(x)\frac{\partial u}{\partial x}) \tag{5}$$

1.2.9(a) The thermal energy balance for an arbitrary slice between $x = a$ and $x = b$ is:

$$\int_a^b \frac{\partial}{\partial t}[e(x,t)A(x)]dx = \phi(a,t)A(a) - \phi(b,t)A(b) - \int_a^b [Q_s(x,t)P(x)]dx \tag{6}$$

where $A(x)$ is the cross sectional area, $P(x)$ is the perimeter, and $Q_s(x,t)$ is the heat loss through the lateral surface. We can rewrite equation 1 as:

$$\int_a^b \frac{\partial}{\partial t}[e(x,t)A(x)]dx = -\int_a^b \frac{\partial}{\partial x}(\phi(x,t)A(x))dx - \int_a^b [Q_s(x,t)P(x)]dx \tag{7}$$

or

$$\int_a^b [\frac{\partial}{\partial t}(e(x,t)A(x)) + \frac{\partial}{\partial x}(\phi(x,t)A(x)) + Q_s(x,t)P(x)]dx = 0 \tag{8}$$

For an arbitrary slice, the integrand must be zero:

$$\frac{\partial}{\partial t}(e(x,t)A(x)) + \frac{\partial}{\partial x}(\phi(x,t)A(x)) + Q_s(x,t)P(x) = 0 \tag{9}$$

replacing $e(x,t) = \rho(x)c(x)u(x,t)$ and $\phi(x,t) = -K_o \partial u/\partial x$ in 4 gives:

$$\frac{\partial}{\partial t}(\rho(x)c(x)u(x,t)A(x)) = \frac{\partial}{\partial x}(A(x)K_o\frac{\partial u}{\partial x}) - Q_s(x,t)P(x) \tag{10}$$

or

$$\rho c\frac{\partial u(x,t)}{\partial t} = K_o\frac{\partial^2 u}{\partial x^2} - Q_s(x,t)\frac{P}{A} \tag{11}$$

where ρ, c, K_o, A, and P are constants. 1.2.9(b) Replacing $Q_s(x,t) = h(x)[u(x,t) - u_B(x,t)]$ in 6 gives:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \frac{P}{c\rho A}h(x)[u(x,t) - u_B(x,t)] \tag{12}$$

where k is the thermal diffusivity and h is the proportionality coefficient. 1.2.9(c) For $Q_s(x,t) = 0$, we recover:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{13}$$

1.2.9(d) For $u_B(x,t) = 0$ and a circular cross section, we get:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \frac{2h}{c\rho r}u(x,t) \tag{14}$$

where h is constant. For uniform temperature, $\partial^2 u/\partial x^2 = 0$, therefore:

$$\frac{\partial u(t)}{\partial t} = -\frac{2h}{c\rho r}u(t) \tag{15}$$

1.2.9(e) With $u(t=0) = U_o$, solving 10 will give:

$$u(t) = U_o \exp(-\frac{2h}{c\rho r}t) \tag{16}$$