1.2.9(a) The thermal energy balance is for an arbitrary slice between $x = a$ and $x = b$ is:

$$\int_a^b \frac{\partial}{\partial t}[e(x,t)A(x)]dx = \phi(a,t)A(a) - \phi(b,t)A(b) - \int_a^b [Q_s(x,t)P(x)]dx \quad (1)$$

where $A(x)$ is the cross sectional area, $P(x)$ is the perimeter, and $Q_s(x,t)$ is the heat loss through the lateral surface. We can rewrite equation 1 as:

$$\int_a^b \frac{\partial}{\partial t}[e(x,t)A(x)]dx = -\int_a^b \frac{\partial}{\partial x}(\phi(x,t)A(x))dx - \int_a^b [Q_s(x,t)P(x)]dx \quad (2)$$

or

$$\int_a^b [\frac{\partial}{\partial t}(e(x,t)A(x)) + \frac{\partial}{\partial x}(\phi(x,t)A(x)) + Q_s(x,t)P(x)]dx = 0 \quad (3)$$

For an arbitrary slice, the integrand must be zero:

$$\frac{\partial}{\partial t}(e(x,t)A(x)) + \frac{\partial}{\partial x}(\phi(x,t)A(x)) + Q_s(x,t)P(x) = 0 \quad (4)$$

replacing $e(x,t) = \rho(x)c(x)u(x,t)$ and $\phi(x,t) = -K_o \partial u/\partial x$ in 4 gives:

$$\frac{\partial}{\partial t}(\rho(x)c(x)u(x,t)A(x)) = \frac{\partial}{\partial x}(A(x)K_o\frac{\partial u}{\partial x}) - Q_s(x,t)P(x) \quad (5)$$

or

$$\rho c\frac{\partial u(x,t)}{\partial t} = K_o\frac{\partial^2 u}{\partial x^2} - Q_s(x,t)\frac{P}{A} \quad (6)$$

where ρ, c, K_o, A, and P are constants. 1.2.9(b) Replacing $Q_s(x,t) = h(x)[u(x,t) - u_B(x,t)]$ in 6 gives:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \frac{P}{c\rho A}h(x)[u(x,t) - u_B(x,t)] \quad (7)$$

where k is the thermal diffusivity and h is the proportionality coefficient. 1.2.9(c) For $Q_s(x,t) = 0$, we recover:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad (8)$$

1.2.9(d) For $u_B(x,t) = 0$ and a circular cross section, we get:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \frac{2h}{c\rho r}u(x,t) \quad (9)$$

where h is constant. For uniform temperature, $\partial^2 u/\partial x^2 = 0$, therefore:

$$\frac{\partial u(t)}{\partial t} = -\frac{2h}{c\rho r}u(t) \tag{10}$$

1.2.9(e) With $u(t=0) = U_o$, solving 10 will give:

$$u(t) = U_o \exp(-\frac{2h}{c\rho r}t) \tag{11}$$

1.3.2 The flux at $x = x_0$ must be continuous:

$$K_o^{x_o^+}\frac{\partial u}{\partial x}(x = x_0^+) = K_o^{x_o^-}\frac{\partial u}{\partial x}(x = x_0^-) \tag{12}$$

therefore, if $K_o^{x_o^+} = K_o^{x_o^-}$, then $\frac{\partial u}{\partial x}$ would be continuous across at $x = x_0$.

1.4.3 Equilibrium equations become:

$$k_1\frac{\partial^2 u_1}{\partial x^2} + 1 = 0 \quad 0 < x < 1 \tag{13}$$

$$k_2\frac{\partial^2 u_1}{\partial x^2} = 0 \quad 1 < x < 2 \tag{14}$$

with the following B.C's:

$$u_1(0) = u_2(2) = 0 \tag{15}$$

$$u_1(1) = u_2(1) \tag{16}$$

$$\frac{\partial u_1}{\partial x}(x = 1) = 2\frac{\partial u_2}{\partial x}(x = 1) \tag{17}$$

Integrating twice gives:

$$u_1(x) = -x^2/2 + c_1 x + c_2 \tag{18}$$

$$u_2(x) = c_3 x + c_4 \tag{19}$$

Imposing the B.C.'s gives $c_1 = 2/3, c_2 = 0, c_3 = 1/6, c_4 = 1/3$.