Signal and System

• Signal: A signal is defined as the time history of some quantity, usually a voltage or current.
  – Deterministic Signals
  – Random Signals

• System: A system is a combination of devices and networks (subsystems) chosen to perform a desired function.
Signal Models

• **Deterministic Signals** are modeled as completely specified functions of time.

• Examples:
  
  - \( x(t) = A \cos(\omega_0 t), -\infty < t < \infty \)  
    - Sinusoidal
  
    • \( A \): magnitude
    • \( \omega_0 \): angular frequency
  
  - \( x(t) = \begin{cases} 
  1 & |t| < \frac{1}{2} \\
  0 & \text{otherwise} 
  \end{cases} \)  
    - unit rectangular pulse function
      - denoted by \( \Pi(t) \)

Signal Models…

• **Random signals** are signals that take on random values at any given time instant and must be modeled probabilistically.

• Example → Figure in the next slide
  
  a) A sinusoidal signal – deterministic
  
  b) Unit rectangular pulse signal – deterministic
  
  c) A random signal (its one sample function)
Signal Models…

- **Periodic signals** A signal $x(t)$ is periodic if
  
  $$x(t + T_0) = x(t), -\infty < t < \infty$$

  where the constant $T_0$ is a period. (deterministic)

- **Fundamental period** The smallest period is referred to as fundamental period.

- **Aperiodic signals** Any signal not satisfying

  $$x(t + T_0) = x(t) \quad \forall t$$

  is called aperiodic.
Signal Models…

• **Rotating phasor** A useful tool to deal with sinusoidal quantities.
  - A rotating phasor: \( \bar{x}(t) = Ae^{j(\omega_0 t + \theta)} \quad -\infty < t < \infty \)
  - Three parameters:
    • A: amplitude
    • \( \theta \): phase (in radians)
    • \( \omega_0 \): frequency (in radians per sec)

• Phasor: \( Ae^{j\theta} \) where \( e^{j\omega_0 t} \) is implicit.

\[
\bar{x}(t) = \bar{x}(t + T_0), \quad T_0 = \frac{2\pi}{\omega_0}
\]

\[\therefore \bar{x}(t + T_0) = Ae^{j[\omega_0 (t + T_0) + \theta]} \]

\[
= A\cos \left[ \omega_0 (t + \frac{2\pi}{\omega_0}) + \theta \right] + jA\sin \left[ \omega_0 (t + \frac{2\pi}{\omega_0}) + \theta \right]
\]

\[= A\cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta) \]

\[= Ae^{j(\omega_0 + \theta)} = \bar{x}(t) \]
Signal Classifications: Energy and Power

- Power and Energy

- \( p(t) = x^2(t) : \) power
  
  It is “normal” power (with \( 1\Omega \) impedance).

- Higher-energy signals are detected more reliably with fewer errors than lower energy signals.

Signal’s Energy and Power: Definition

Let \( x(t) \) be an arbitrary signal (possibly complex function).

- Its **total energy** is:
  
  \[
  E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt
  \]

- Its **average power** is:
  
  \[
  P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt
  \]
Energy signal vs. Power signal

• The function $x(t)$ is an **energy signal** iff
  \[ 0 < E < \infty \]
  (Hence $P = 0$, because of having non-zero and finite energy.)

• The function $x(t)$ is a **power signal** iff
  \[ 0 < P < \infty \]
  (Thus $E = \infty$, because of having non-zero and finite power.)

For a periodic signal $x_p(t)$

\[
P = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} |x_p(t)|^2 \, dt \quad T_0 \text{ is the period.}
\]

• No need to carry out the limiting operation to find $P$ for a periodic signal.

• Energy and power classifications are mutually exclusive.
  1. An energy signal must have zero average power, \( \Rightarrow \) not a power signal.
  2. A power signal must have infinite energy, \( \Rightarrow \) not an energy signal

• There are some signals: neither energy nor power signals. (The ramp signal is such an example.)
Example 1

• Refer to a figure about unit step function.

\[ x(t) = Ae^{-\alpha t}u(t) \]

\[ \alpha > 0 \quad u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \]

\[ E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{0}^{\infty} \frac{A^2}{e^{2\alpha t}} \, dt \]

\[ = A^2 \frac{e^{2\alpha t}}{-2\alpha} \bigg|_{0}^{\infty} = A^2 \frac{e^{-2\alpha t}}{2\alpha} \bigg|_{0}^{\infty} = \frac{A^2}{2\alpha} \]

\[ \Rightarrow x(t) \text{ is an energy signal.} \]

• If \( \alpha \to 0, x(t) = Au(t) \), \( \Rightarrow \) infinite energy.

\[ P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} A^2 \, dt = \frac{1}{2} A^2 \]

• Final power.

• It is a power signal
Example 2

• Refer to figure shown before.

\[ x_p(t) = A \cos(\omega_0 t + \theta) \]

a sinusoidal signal

infinite energy

\[ P = \frac{1}{T_0} \int_{\theta_0}^{\theta_0 + T_0} A^2 \cos^2(\omega_0 t + \theta) \, dt \]

\[ = \frac{1}{T_0} \int_{\theta_0}^{\theta_0} \frac{A^2}{2} [1 + \cos 2(\omega_0 t + \theta)] \, dt \]

\[ = \frac{A^2}{2} \]

It is a power signal.

• A frequently used skill:

\[ \int_{\theta_0}^{\theta_0} \cos[2(\omega_0 t + \theta)] \, dt = 0 \]

\[ \therefore \frac{2\pi}{2\omega_0} = \frac{\pi}{\omega_0} = \frac{\pi}{2\pi} = \frac{T_0}{2} \]

• That is, period of \( \cos[2(\omega_0 t + \theta)] \) becomes half.

• Integration of a sinusoid within an integral number of periods is zero.
Energy Spectral Density (ESD)

1. **Derivation**
   
   Total Energy:
   
   $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$
   
   $$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} X(f)e^{j2\pi f t} \right] x^*(t) dt$$
   
   $$= \int_{-\infty}^{\infty} X(f) \left[ \int_{-\infty}^{\infty} x^*(t)e^{j2\pi f t} dt \right] df$$
   
   $$= \int_{-\infty}^{\infty} X(f) X^*(f) df = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \xi_x(f) df$$

2. **Definition**
   
   Energy spectral density of a signal $x(t)$.
   
   $$\xi_x(f) = |X(f)|^2$$
   
   $$E = \int_{-\infty}^{\infty} \xi_x(f) df = 2\int_{0}^{\infty} \xi_x(f) df$$
3. Unit

\[ X(f) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \]

\[ |X(f)|^2 : (\text{volts} - \text{seconds})^2 \]

\[ \therefore \quad \text{On a per ohm basis for total energy & average power} \]

\[ (\text{volts} - \text{seconds})^2 \Rightarrow \frac{(\text{volts})^2}{\text{ohm}} \cdot (\text{seconds})^2 \]

\[ = \frac{\text{watts} \cdot \text{seconds}}{\text{hertz}} \cdot \left( \frac{1}{\text{hertz}} = \text{second} \right) \]

\[ = \text{joules} / \text{hertz} \]

It is the Energy Density.

4. Comment 1. (Parseval’s Theorem)
(Rayleigh’s Energy Theorem)

\[ E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \]

Comment 2.
\[ \xi_s(f) \] is energy spectral density.

\[ \therefore \int_{-\infty}^{\infty} \xi_s(f) df = E \]
Power Spectral Density (PSD)

1. Consider $x(t)$ as a real-valued power signal, then its average power is

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t) dt$$

This is similar to definition given before (slide 10) except that the absolute value sign has been removed.

PSD…

2. If $x(t)$ is a periodic signal with period $T_0$, then its average power is

$$P_x = \frac{1}{T_0} \int_{\frac{T_0}{2}}^{\frac{T_0}{2}} x^2(t) dt$$

$$P_x = \frac{1}{T_0} \int_{\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot x^*(t) dt$$

$$= \sum_{n=-\infty}^{\infty} |X_n|^2$$

where $X_n$ is Fourier series (FS) coefficient of $x(t)$. 
PSD…

• This can be proved by using FS of x(t).
• Loosely speaking,

\[ x(t) = \sum_{n=-\infty}^{\infty} X_n \exp \left( \frac{j2\pi nt}{T_0} \right) \]

\[ x'(t) = \sum_{n=-\infty}^{\infty} X_n' \exp \left( -\frac{j2\pi n't}{T_0} \right) \]

All cross-product terms \( \Rightarrow 0 \), except as \( n = n' \),

due to orthogonality of complex exponential function, thus leading to \( \sum_n |X_n|^2 \).

PSD…

3. Define PSD of a periodic function \( x(t) \) as:

\[ PSD_x(f) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0) \]

• Recall that a periodic function has line spectra.

\[ X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - f_0n) \]
• Then

\[
P_x = \int_{-\infty}^{\infty} PSD_x(f) \, df
\]

\[
= \int_{-\infty}^{\infty} \left[ \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0) \right] \, df
\]

\[
= \sum_{n=-\infty}^{\infty} |X_n|^2 \int_{-\infty}^{\infty} \delta(f - nf_0) \, df
\]

\[
= \sum_{n=-\infty}^{\infty} |X_n|^2
\]

• Hence, \( PSD_x(f) \) thus defined is, indeed, PSD.

4. For a non-periodic function \( x(t) \), we first truncate \( x(t) \) as \( x_T(t) \) in \( \left[ -\frac{T}{2}, \frac{T}{2} \right] \) then find its FT: \( X_T(f) \).

It can be shown that

\[
PSD_x = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2
\]
Summary: ESD and PSD

- ESD: \( \xi_x(f) = |X(f)|^2 \)
- PSD:
  \[
  PSD_x(f) = \sum_{n=-\infty}^{\infty} X_n^2 \delta(f - nf_0), \quad x(t) \text{ periodic}
  \]
  \[
  PSD_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2, \quad x(t) \text{ non-periodic}
  \]
  \[
  X(f) = FT\{x(t)\}
  X_n = FS \text{ coefficient of } x(t)
  X_T(f) = FT\{x_T(t)\}
  \]

Autocorrelation

- Correlation function –
  Another approach to signal and systems.
- Autocorrelation –
  A measure of similarity (matching) of a signal with its delayed version.
Autocorrelation of an Energy Signal

• Definition

\[ R_x(\tau) = \Delta \int_{-\infty}^{\infty} x(t) x(t + \tau) dt, -\infty < \tau < \infty \]

• A measure of how closely the signal \( x(t) \) matches a copy of itself as the copy is shifted \( \tau \) units in time

• The larger the more correlated.

---

Properties:

1. \( R_x(\tau) = R_x(-\tau) \) symmetrical in \( \tau \) about \( \tau = 0 \)

2. \( R_x(\tau) \leq R_x(0) \forall \tau \) maximum value of \( R(\tau) \) occurs at \( \tau = 0 \)

3. \( R_x(\tau) \leftrightarrow \xi_x(f) = ESD_x \) an important FT pair

4. \[ R_x(0) = \int_{-\infty}^{\infty} x^2(t) dt = E_X \] (energy)
Autocorrelation of a Power Signal

• Definition

\[
R_x(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) x(t + \tau) \, dt, -\infty < \tau < \infty
\]

or \( \langle x(t) x(t + \tau) \rangle \)

• For a periodic signal \( x(t) \) with period \( T_0 \)

\[
R_x(\tau) = \frac{1}{T_0} \int_{T_0}^{2T_0} \int_{T_0}^{2T_0} x(t) x(t + \tau) \, dt, -\infty < \tau < \infty
\]

• Properties \( R_x(\tau) \) of a real-valued periodic signal \( x(t) \):

1. \( R_x(\tau) = R_x(-\tau) \) even symmetry

2. \( R_x(\tau) \leq R_x(0), \ \forall \ \tau \) maximum value

3. \( R_x(\tau) \leftrightarrow PSD_x(f) \) an important FT pair

4. \( R_x(0) = \frac{1}{T_0} \int_{T_0}^{2T_0} x^2(t) \, dt \) average power
Random Signals

- Random experiment: E
- Outcome of r.e.: ξ
- Sample space: S
- Definition of probability
- Axioms of probability
- Random event: A
- Random variable (r.v.): X(A)
  a function of A, a real number,
  (numerical attribute)

Random Signals

CDF

- **CDF** Cumulative Distribution Function

\[ F_X(x) = P(X \leq x) \]

- Properties:
  1. \( 0 \leq F_X(x) \leq 1 \)
  2. \( F_X(x_1) \leq F_X(x_2), \quad \text{if} \quad x_1 \leq x_2 \)
  3. \( F_X(-\infty) = 0 \)
  4. \( F_X(+\infty) = 1 \)
  5. \( F_X(x) \) continuous from the right
Random Signals

PDF

- **PDF**

  \[ f_X(x) = \frac{dF_X(x)}{dx} \]

  \[ P(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X \leq x_1) \]

  \[ = F_X(x_2) - F_X(x_1) \]

  \[ = \int_{x_1}^{x_2} f_X(x) \, dx \]

- **Properties**

  1. \( f_X(x) \geq 0 \)

  2. \[ \int_{-\infty}^{\infty} f_X(x) \, dx = F_X(\infty) - F_X(-\infty) = 1 \]

For discrete r.v.’s, instead of \( f_X(f) \), we often use \( P(X = x_i) \)

\[ \rightarrow \text{probability mass function (PMF)} \]

- **Strictly speaking**

\[
\begin{cases}
  f_X(x) = \sum_i P(X = x_i) \delta(x - x_i) & \text{pdf of discrete r.v.} \\
  \forall i; x_i < x \\
  F_X(x) = \sum_i P(X = x_i) u(x - x_i) & \text{cdf of discrete r.v.}
\end{cases}
\]
Ensemble Average (statistical average)

• Numerical attributes of r.v.
  
  – Mean value, $m_X$ (expected value)
    \[ m_X = E(X) = \int_{-\infty}^{\infty} xf_X(x)dx \]
  
  – $n^{th}$ moment
    \[ E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)dx \]

  
  \[
  \begin{align*}
  n = 1, & \text{ mean value of } X, \text{ mean} \\
  n = 2, & \text{ mean-square value of } X \\
  n = 3, & \text{ } \\
  n = 4, & \text{ } \\
  \end{align*}
  \]

very important (this and next slides)

Ensemble Average (statistical average)…

– Central moment
  \[ E[(X - m_X)^n] = \int_{-\infty}^{\infty} (x - m_X)^n f_X(x)dx \]

1\text{st} central moment:  0
2\text{nd} central moment:  \[ E[(X - m_X)^2] = \text{var}(X) = \sigma_X^2 \]

Important formula: \[ \sigma_X^2 = E[X^2] - m_X^2 \] variance
3\text{rd} central moment: skewness of pdf
4\text{th} central moment: kurtosis of pdf
Random Process (r.p.)

- $X(A, t)$  
  A: random event  
  t:  time

- a larger set
- a collection of r.v.’s
- or, a collection of random sample function
- often simplified as $X(t)$

Figure 8.1-1 from <Probability, Random Processes, and Estimation Theory for Engineers> by Stark & Woods 1994
• For a specified event $A_j$, $X(A_j, t) = X_j(t)$, a sample function

• For a specified time $t_k$, $X(A, t_k) = X_k$, a r.v.

• For specific $A_j$, $t_k$, $X(A_j, t_k) = a$ real value

Statistical Average of a r.p.

• A r.p. should be completely characterized by pdf of all r.v.’s. This is difficult.

• Hence, numerical characterization used often.

• A partial description:

  \{ 
  \begin{align*}
  & \text{mean function } (\text{1st order}) \\
  & \text{autocorrelation function } (\text{2nd order})
  \end{align*}
  \}

  \begin{align*}
  E[X(t_k)] &= \int_{-\infty}^{\infty} x f_{X_k}(x) dx = m_X(t_k) \\
  R_X(t_1, t_2) &= E[X(t_1) \cdot X(t_2)]
  \end{align*}
Stationarity

- **S.S.S. (Strict Sense Stationary)**
  A r.p. $X(t)$ is SSS if none of its statistics are affected by a shift in the time origin.

- **W.S.S. (Wide Sense Stationary)**
  A r.p. $X(t)$ is WSS if its mean and autocorrelation function do not vary with a shift in the time origin.

For a W.S.S. random process,

\[
E[X(t)] = m_x \quad \text{a constant}
\]

\[
R_x(t_1, t_2) = R_x(t_1 - t_2) \quad \text{only a function of } t_1 - t_2
\]

- **Relation between SSS and WSS**

  \[
  \text{SSS} \quad \rightarrow \quad \text{WSS}\n  \]

  \[
  \text{WSS} \quad \leftrightarrow \quad \text{not necessarily}
  \]
Autocorrelation of a WSS r.p.

- Let $\tau = t_1 - t_2$
  \[ R_x(\tau) = E[X(t)X(t + \tau)], \quad -\infty < \tau < \infty \]
- $R_x(\tau)$ gives us an idea of the frequency response
- If $R_x(\tau)$ changes slowly as $\tau$ changes, then $X(t)$ has more low frequency components; otherwise, it has more high frequency comp.
- Four similar properties of $R_x(\tau)$ to that of autocorrelation of an energy (power) signal.

Time Average & Ergodicity

- To compute $m_X$ and $R_x(\tau)$ needs to have $f_X(x)$ for all $x$, sometimes not possible, \(\Rightarrow\) prefer the time average
- Time average over a single sample function of the r.p.:
  \[
  \langle X(t) \rangle \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)dt \\
  \langle X(t)X(t + \tau) \rangle \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)X(t + \tau)dt
  \]
  \[
  \left\{ \begin{array}{l}
  \text{always possible}
  \end{array} \right. 
  \]
• An ergodic r.p.: 
  \[ m_X(t) = \langle X(t) \rangle \]
  \[ R_X(\tau) = \langle X(t)X(t+\tau) \rangle \]

  \[ \rightarrow \]
  \[ \text{ergodicity} \quad \nleftrightarrow \quad \text{SSS} \]
  \[ \text{not necessarily} \]

• A r.p. is ergodic in the mean, if
  \[ m_X = \langle X(t) \rangle \]

• A r.p. is ergodic in the autocorrelation function if
  \[ R_X(\tau) = \langle X(t)X(t+\tau) \rangle \]

**A reasonable assumption** in the analysis of most communication signals (in the absence of transient effects) is that random waveforms are ergodic in the mean and autocorrelation function.

• For ergodic r.p., time averages = statistical (ensemble) averages
some observations:
• $m_x = \text{dc levels of the signal}$
• $m_x^2 = \text{normalized power of dc component}$
• $E[X^2] = \text{total average normalized power}$
• $\sqrt{E[X^2]} = \text{root-mean-square (rms) value of } X$
• $\sigma_x^2 = \text{average normalized power in ac component}$
• If $m_x = 0$, $\sigma_x^2 = \text{mean-square value of } X \text{ or total power}$
• $\sigma_x = \text{rms of ac component}$
• If $m_x = 0$, $\sigma_x = \text{rms of the signal}$

PSD of a Random Process

• A r.p. $X(t)$ can generally be considered as a power signal.

• $PSD_X(f)$:
  $$PSD = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2$$

  $x(t) \rightarrow x_T(t) \quad \text{as} \quad t \in \left(-\frac{T}{2}, \frac{T}{2}\right) \leftrightarrow X_T(f)$

• Useful in communication systems: Indicates the distribution of a signal’s power in frequency domain.
**PSD of a Random Process…**

- **Properties:**
  1. $PSD_x( f ) \geq 0$ nonnegative real-valued
  2. $PSD_x( f ) = PSD_x(- f )$ if $X(t)$ is real-valued
  3. $R_x(\tau) \leftrightarrow PSD_x( f )$
  4. $P_x = \int_{-\infty}^{\infty} PSD_x( f ) df$

**Noise in Communication Systems**

- **Noise:** Unwanted electrical signals always present in electrical system

  **Man-made Noise:** switching transients, spark-plug ignition

  **Natural Noise:** thermal (always exists)

  Gaussian noise (central limit theorem (slide 54))
Normalized Gaussian pdf

- \( m = 0 \)  
- \( \sigma^2 = 1 \)  

\[
f(n) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[ -\frac{1}{2} \left( \frac{n}{\sigma} \right)^2 \right]
\]

Gaussian Noise

- A random signal \( z = a + n \)
  - \( a \): deterministic component of the signal
  - \( n \): additive random noise

\[
f(z) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[ -\frac{1}{2} \left( \frac{z-a}{\sigma} \right)^2 \right]
\]

- Gaussian distribution is extremely important in practice due to the central limit theorem
- The theorem: The probability distribution of the sum of \( j \) statistically independent r.v.’s approaches the Gaussian distribution as \( j \to \infty \), no matter what the individual distribution functions may be.
White Noise

- **Def.**

  

  - “White” comes from the fact that white light contains equal amounts of all frequencies within the visible band of electromagnetic radiation

- **Autocorrelation function**

  \[ R_n(\tau) = FT^{-1}\{PSD_x(f)\} = FT^{-1}\left\{ \frac{N_0}{2} \right\} = \frac{N_0}{2} \delta(\tau) \]

  - \( n(t) \) is totally decorrelated from its time-shifted version for any \( \tau > 0 \)

- **Meaning:** Any two different samples of a white noise are uncorrelated no matter how close together in time they are taken.

- **Average power**

  \[ P_n = \int_{-\infty}^{\infty} \frac{N_0}{2} df = \infty \]
• In practice, as long as the bandwidth of the noise is appreciably larger than that of the system, the noise can be considered to have an infinite bandwidth.

• Thermal noise:
  Additive white Gaussian Noise (AWGN)
  Very important in both theory and practice.

---

**Signal Transmission Through Linear Systems**

- x(t), h(t), y(t) --- time domain
- X(f), H(f), Y(f) --- frequency domain
- h(t): unit impulse response of the LTI system
- H(f): transfer function
• **h(t): unit impulse response**
  - \( h(t) = y(t) \) when \( x(t) = \delta(t) \)
  - \( y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \)

• **Causality**
  - A system is causal if there is no output prior to the time, \( t = 0 \), when the input is applied.
  - A system is causal if \( h(t) = 0, \forall t < 0 \)
    \[ \Rightarrow y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau \]

• **H(f): Frequency transfer function.**
  - \( H(f) = FT\{h(t)\} \)
  - \( Y(f) = FT\{y(t)\} = FT\{x(t) * h(t)\} = X(f) \cdot H(f) \)
  - \( H(f) = \frac{Y(f)}{X(f)} \)
Frequency Response

- $H(f)$: Complex in general
  \[ H(f) = |H(f)|e^{\theta(f)} \]

- Amplitude frequency response: $|H(f)|$
- Phase frequency response: $\theta(f)$

- The transfer function of a LTI system can be measured by using a sinusoidal testing signal (that is swept over the frequency of interest) since the spectrum of sinusoid is a line at the testing frequency.

- For sinusoidal input, i.e., $x(t) = A \cos(2\pi f_0 t + \phi)$
  \[ y(t) = A|H(f_0)|\cos(2\pi f_0 t + \phi + \theta(f_0)) \]
Random Process & Linear Systems

- If $x(t)$: a r.p.
  $h(t)$: LTI system
  then $y(t)$: output, another r.p.
  i.e., every sample function of the input r.p.
  $\rightarrow$ a corresponding sample function

- $PSD_x(f)$: PSD of $x(t)$
  $PSD_y(f)$: PSD of $y(t)$

$$PSD_y(f) = PSD_x(f) \cdot |H(f)|^2$$

- The relation between the power spectral density (PSD) at the input, $PSD_x(f)$, and the output, $PSD_y(f)$

$$PSD_y(f) = |H(f)|^2 \cdot PSD_x(f)$$

- The power transfer function of the LTI system

$$G_h(f) = \frac{PSD_y(f)}{PSD_x(f)} = |H(f)|^2$$

$$\therefore PSD_y(f) = \lim_{T \to \infty} \frac{1}{T} |Y_T(f)|^2$$

$$Y(f) = X(f)H(f)$$

$$PSD_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2$$
• If $x(t)$ is a Gaussian r.p., $h(t)$ is LTI, then $y(t)$ is also Gaussian.

• If the input to a LTI system is periodic with spectrum given by

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - n \cdot f_0)$$

where \( \{X_n\} \) is the complex exponential FS coefficients of the input signal, then the output signal’s spectrum is

$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(nf_0) \delta(f - n \cdot f_0)$$

---

**RC Low-Pass Filter**

• Example 3-1

\[ x(t) = R \ i(t) + y(t) \]

\[ \therefore i(t) = C \frac{dy(t)}{dt} \implies RC \frac{dy(t)}{dt} + y(t) = x(t) \]

From the FT property of the differentiation,

\[ \implies RC(\ j2\pi f \ )Y(f) + Y(f) = X(f) \]
Distortionless Transmission

• What is an ideal transmission line?

• In time domain
  – Some time delay is allowed \((y(t) \text{ vs. } x(t))\)
  – A scale change in magnitude is allowed

\[ y(t) = K x(t-t_0) \quad K: \text{ scale change} \quad t_0: \text{ time delay} \]
• In frequency domain

\[ Y(f) = K \cdot X(f) e^{-j2\pi ft_0} \]

i.e., \[ H(f) = K e^{-j2\pi ft_0} \]

- \[ |H(f)| = K \] constant magnitude change \( \forall f \)
- \[ \theta(f) = \angle H(f) = -2\pi ft_0 \]

\[ t_0 = \frac{-\theta(f)}{2\pi f} \]

\( t_0 \) needs to be fixed.

→ Phase shift must be proportional to frequency in order for the time delay of all components to be identical, i.e.,

phase delay \( \theta \propto f \)

“Equalization”: phase or amplitude correction network

---

**Ideal Filters**

- One cannot build the ideal network described above since it implies an infinite bandwidth capability (Sklar’s, page 33).

- An approximation to the ideal infinite-bandwidth network is to use a truncating network that passes all freq. components between \( f_l \) and \( f_u \) without distortion, where \( f_l \) and \( f_u \) are the lower and upper cutoff frequency, respectively.

- Ideal BPF
  - LPF
  - HPF
• Take a look at ILPF

$$H(f) = |H(f)|e^{-j\theta(f)}$$

$$|H(f)| = \begin{cases} 1 & \text{for } |f| < f_u \\ 0 & \text{for } |f| \geq f_u \end{cases}$$

$$e^{-j\theta(f)} = e^{-j2\pi f_0}$$
$$h(t) = FT^{-1}\{H(f)\} = \int_{-\infty}^{\infty} H(f)e^{-j2\pi ft} df$$

$$= \int_{-\frac{f_u}{2}}^{\frac{f_u}{2}} e^{-j2\pi f t_0} e^{j2\pi ft} df$$

$$= \int_{-\frac{f_u}{2}}^{\frac{f_u}{2}} e^{j2\pi f(t-t_0)} df$$

$$= 2f_u \frac{\sin [2\pi f_u (t-t_0)]}{2\pi f_u (t-t_0)}$$

$$= 2f_u \text{Sinc} [2\pi f_u (t-t_0)]$$

**Unit Impulse Response of ILPF**
Effect of an ILPF on White Noise

- Example 1.2.

\[
PSD_n(f) = \frac{N_0}{2}, \quad PSD_Y(f) = ?, \quad R_Y(\tau) = ?
\]

- Solution:

\[
PSD_Y(f) = PSD_n(f) |H(f)|^2
\]

\[
= \begin{cases} 
\frac{N_0}{2} & \text{for } |f| < f_u \\
0 & \text{otherwise}
\end{cases}
\]

\[
R_Y(\tau) = FT^{-1} \{ PSD_Y(f) \}
\]

\[
= 2 \cdot f_u \cdot \frac{N_0}{2} \cdot \text{sinc} \left( \frac{2f_u}{\text{area of } PSD_Y} \right) \cdot \text{width of } PSD_Y(f) \cdot \frac{1}{2f_u} \cdot \text{1st zero crossing}
\]

After LPF, is it white noise or not?

Not a white noise anymore
Realizable Filters

- **RC LPF:**

  ![RC LPF Circuit Diagram]

- **RC filter (frequency analysis of sinusoidal circuits)**

  \[
  \frac{V_{\text{out}}}{V_{\text{input}}} = \frac{1}{j 2 \pi f_c} \frac{1}{R + \frac{1}{j 2 \pi f_c}} = \frac{1}{1 + j 2 \pi f R_c}
  \]

\[
|H(f)| = \frac{1}{\sqrt{1 + (2\pi f R_c)^2}}
\]

\[
\theta(f) = -\tan^{-1}(2\pi f R_c)
\]

![Amplitude and Phase Responses](image-url)
• Consider R = 1, i.e., the normalized case.

\[ P = \frac{V^2}{R} = V^2 \]

\[ \frac{V_2}{V_1} = \frac{\sqrt{2}}{2} \Rightarrow \frac{V_2^2}{V_1^2} = \frac{1}{2}, \quad \frac{P_2}{P_1} = \frac{1}{2} \]

\[ \frac{V_2}{V_1} = 0.707 \Leftrightarrow \frac{P_2}{P_1} = \frac{1}{2} \quad \text{half-power} \]

• No. of dB

\[ 10\log_{10} \left( \frac{P_2}{P_1} \right) = 10\log_{10} \left( \frac{1}{2} \right) = -10 \cdot 0.3010 = -3\text{dB} \]

• Hence, half-power \( \Leftrightarrow -3\text{dB} \)

Effect of an RC filter on white noise

• Example 1.3

\[ G_n = \frac{N_0}{2} \]

\[ G_s(f) = G_n(f) \cdot |H(f)|^2 = \frac{N_0}{2} \cdot \frac{1}{1 + (2\pi f Rc)^2} \]

\[ R_s(\tau) = F^{-1} \{ G_s( f ) \} = \frac{N_0}{4Rc} \exp\left( \frac{-|\tau|}{Rc} \right) \text{ exponential function} \]

When input is white noise, output of the RC filter: Not white noise anymore
Several Useful Realizable Filters

- Butterworth filter (most flat one in passband)
- Chebychev filter Ripple in passband smaller variation in stopband
- For pass-band and stop-band refer to the figure on slide 82

Butterworth Low Pass Filter

\[ |H_n(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_u}\right)^{2n}}}, n \geq 1 \]

\( f_u \), or \( f_c \) called corner frequency

\( n \rightarrow \infty, H_n(f) \rightarrow ILPF \)
Butterworth Amplitude Response

Signals, Circuits & Spectra

• Input signal $x(t)$, its spectrum $|X(f)|$
  \[ \text{rect}(t) \quad |\text{sinc}(f)| \quad \text{e.g.} \]
• Circuit, RC circuit, $|H(f)|$, $\theta(f)$
• Output $y(t)$, $|Y(f)|$
• Case 1: Output bandwidth is constrained by input signal bandwidth,
  i.e., $H(f)$ is wideband.
• Case 2: Output bandwidth is constrained by filter bandwidth,
  i.e., $H(f)$ is narrowband.
Bandwidth of Digital Data

- Baseband vs. Bandpass
  - Double sideband (DSB) modulation
    - $x(t)$ signal (low-pass or baseband signal)
      - $|X(f)|$ spectrum, $0 - f_m$: baseband bandwidth
    - DSB modulated signal
      - $x_c(t) = x(t) \cos(2\pi f_c t), f_c \geq f_m$
      - Baseband signal $x(t) \rightarrow \otimes \rightarrow x_c(t)$ DSB modulated signal
        $\uparrow$
      - $\cos(2\pi f_c t)$ local oscillator (LO) carrier
\[ X_c(f) = \frac{1}{2} \left[ X(f - f_c) + X(f + f_c) \right] \]

\[
\therefore F\{\cos(2\pi f_c t)\} = \frac{1}{2} \left[ \delta(f - f_c) + \delta(f + f_c) \right]
\]

and convolution of a normal factor with a \(\delta\) function.